ON STRONG IFP NEAR-RINGS

P. Dheena¹, B. Elavarasan²

¹Department of Mathematics
Annamalai University
Annamalainagar, 608 002, INDIA

²Department of Mathematics
Karunya University
Coimbatore, 641 114, Tamilnadu, INDIA

Abstract: In this paper, we introduce the notion of strong IFP and weak IFP near-rings. Weak IFP near-ring is a generalization of IFP near-ring. We study the basic properties of right weak IFP near-rings. We show that if \( N \) is a 2-primal near-ring and if \( N \) is strong IFP, then \( N \) is left weakly regular if and only if every prime ideal of \( N \) is maximal.

AMS Subject Classification: 16Y30
Key Words: regular, IFP near-ring, reduced, 2-primal and weakly regular

1. Introduction

Throughout this paper, \( N \) denotes a zero-symmetric near-ring not necessarily with identity unless otherwise stated. Let \( P(N) \) denote the prime radical and \( N(N) \) the set of nilpotent elements of the near-ring \( N \). For \( X \subseteq N \), \( l(X) \) (resp. \( r(X) \)) and \( < x > \) denote the left (resp. right) annihilator of \( X \) and the ideal of \( N \) generated by \( x \) respectively.

For any subsets \( A, B \) of \( N \), we denote \( (A : B) = \{n \in N/nB \subseteq A\} \). It is trivial to check that if \( A \) is left ideal of \( N \) and \( B \) is a \( N \)-subgroup of \( N \), then \( (A : B) \) is an ideal of \( N \) by [8, Corollary 1.43].
A near-ring $N$ is said to be reduced if $N(N) = 0$. A near-ring $N$ is said to be regular if for any $a \in N$, there exists $x \in N$ such that $a = axa$.

Recall that a near-ring $N$ is said to be 2-primal if $P(N) = N(N)$. A near-ring $N$ is subdirectly irreducible if $N$ has nonzero intersection of nonzero ideals. A near-ring $N$ is said to be strong IFP if $xy \in P(N)$ implies $xNy = 0$ for $x, y \in N$. A near-ring $N$ is said to be IFP if $ab = 0$ implies $anb = 0$ for all $n \in N$ and $a, b \in N$. Clearly every strong IFP near-ring is a IFP near-ring. If $N$ is reduced, then the notions of IFP and strong IFP coincide.

A near-ring $N$ is said to be left weak IFP if $ab = 0$ for $a(\neq 0), b \in N$ implies $a'Nb = 0$ for some $a'(\neq 0) \in < a >$. The right weak IFP can be defined symmetrically. A near-ring $N$ is said to be weak IFP if $ab = 0$ for any nonzero elements $a, b \in N$ implies $a'Nb' = 0$ for some $a'(\neq 0) \in < a >$ and $b'(\neq 0) \in < b >$.

Clearly IFP near-ring is a weak IFP near-ring, but the converse need not be true as the following example shows.

**Example 1.1.** Let $N = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where $F = \{0,1\}$ is the field under addition and multiplication modulo 2. Then $N$ is a weak IFP near-ring but not IFP near-ring, since if $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $xy = 0$ and $xNy \neq 0$. Here $N$ is neither left weak IFP nor right weak IFP.

Clearly every strong IFP near-rings are IFP near-rings, however IFP near-ring need not be strong IFP as can be seen by the following example.

**Example 1.2.** Let $(N, +)$ (where $N = \{0, a, b, c\}$) be the klein’s four group. Define multiplication in $N$ as follows

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>a</td>
</tr>
</tbody>
</table>

Then $(N, +, .)$ is a near-ring (see Pilz[8], P-408, Scheme-11) which is a IFP near-ring but not a strong IFP near-ring, since $ab \in P(N)$, but $aNb \neq 0$.

Clearly every reduced near-ring is a 2-primal and strong IFP near-ring, but the converse need not be true as the following example shows.

**Example 1.3.** Let $(N, +)$ (where $N = \{0, a, b, c\}$) be the klein’s four group. Define multiplication in $N$ as follows
ON STRONG IFP NEAR-RINGS

Then \((N, +, .)\) is a near-ring (see Pilz [8], P-408, Scheme-12) which is a 2-primal and strong IFP near-ring, but not reduced.

G.F. Birkenmeier, J.Y. Kim and J.K. Park [2] have shown that a reduced ring \(R\) is weakly regular if and only if every prime ideal of \(R\) is maximal. We extend this result to strong IFP near-rings which are 2-primal. For basic terminology in near-ring we refer to Pilz [8].

2. Main Results

**Lemma 2.1.** Let \(N\) be a near-ring with identity. If \(N\) is left weak IFP, then for any \(x, y \in N\) with \(xy = 1\) implies \(yx = 1\).

Proof. Let \(N\) be a left weak IFP near-ring and \(xy = 1\). Suppose \(yx \neq 1\). Then \((1 - yx)yx = 0\). Since \(N\) is left weak IFP, we have \(x' Nyx = 0\) for some \(x' (\neq 0) \in < 1 - yx >\). Now, \(x' Ny = x' Nxy = 0\). Then \(x' = x' xy \in x' Ny = 0\), a contradiction.

**Proposition 2.2.** Let \(N\) be a regular near-ring. Then the following conditions are equivalent:

i) \(N\) is a right weak IFP near-ring

ii) If \(x(\neq 0) \in r(a)\), then \(r(a)\) contains a non-zero ideal \(I\) with \(I \subseteq < x >\)

iii) If \(x(\neq 0) \in r(a)\), \(r(aN)\) contains a non-zero ideal \(I\) with \(I \subseteq < x >\)

iv) If \(x(\neq 0) \in r(a)\), \(i \in r(aN)\) for some \(i(\neq 0) \in < x >\)

Proof. i) \(\Rightarrow\) ii) Let \(x(\neq 0) \in r(a)\). Then \(aNx' = 0\) for some non-zero element \(x' \in < x >\). For any \(n \in N\), \(x' na = (x' na)t(x' na) = x' n(atx')na = 0\) for some \(t \in N\). Thus \(x' Na = 0\) and so \(x' > Na = 0\). Let \(y \in < x' >\). Then by regularity of \(N\), we have \(ay = 0\). Thus \(a < x' >= 0\).

ii) \(\Rightarrow\) iii) Let \(x(\neq 0) \in r(a)\). Then \(aI = 0\) and so \(aNJ = 0\).

iii) \(\Rightarrow\) iv) It is obvious.

iv) \(\Rightarrow\) i) Let \(ab = 0\) for \(a(\neq 0), b \in N\). Then \(b' \in r(aN)\) for some \(b'(\neq 0) \in < b >\). Thus \(aNb' = 0\).

We now give an example to show that Proposition 2.2 is not true if \(N\) is not a regular near-ring.
Example 2.3. Consider the dihedral group $N=\{0, a, 2a, 3a, b, a+b, 2a+b, 3a+b\}$ with addition and multiplication defined as in Pilz ([9, P-339, Scheme-2]).

\[
\begin{array}{cccccccc}
  \cdot & 0 & a & 2a & 3a & b & a+b & 2a+b & 3a+b \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 2a & 3a & b & a+b & 2a+b & 3a+b \\
2a & 0 & 2a & 0 & 2a & 0 & 0 & 0 & 0 \\
3a & 0 & 3a & 2a & a & b & a+b & 2a+b & 3a+b \\
b & 0 & b & 2a & b & b & 0 & 2a+b & 0 \\
a+b & 0 & a+b & 0 & 3a+b & 0 & a+b & 0 & 3a+b \\
2a+b & 0 & 2a+b & 2a & 2a+b & b & 0 & 2a+b & 0 \\
3a+b & 0 & 3a+b & 0 & a+b & 0 & a+b & 0 & 3a+b \\
\end{array}
\]

Then $(N, +, \cdot)$ is a near-ring. Clearly $(N, +, \cdot)$ is a right weak IFP and $a+b \in r(b)$, but $r(b)$ does not contains a non-zero ideal $I$ with $I \subseteq <a+b>$. 

Proposition 2.4. Let $N$ be a near-ring with identity. Then the following conditions are equivalent:

i) $N$ is a left weak IFP near-ring

ii) If $x(\neq 0) \in l(a)$, then $l(a)$ contains a non-zero ideal $I$ with $I \subseteq <x>$

iii) If $x(\neq 0) \in l(a)$, then $l(\langle a \rangle)$ contains a non-zero ideal $I$ with $I \subseteq <x>$

iv) If $x(\neq 0) \in l(a)$, then $i \in l(\langle a \rangle)$ for some $i(\neq 0) \in <x>$

Proof. Proof is as similar in Proposition 2.2.

Lemma 2.5. Let $N$ be a regular near-ring. If $N$ is subdirectly irreducible, then the following conditions are equivalent:

i) $N$ is a right weak IFP near-ring

ii) if $x(\neq 0) \in r(S)$, then $r(S)$ contains a non-zero ideal $I$ of $N$ with $I \subseteq <x>$ for any subset $S$ of $N$.

Proof. i) $\Rightarrow$ ii) Let $N$ be a right weak IFP near-ring and $x \in r(S)$ for any subset $S$ of $N$. For any $s_i \in S$, by Proposition 2.2, we have $r(s_i)$ contains a non-zero ideal $I_i$ of $N$ with $I_i \subseteq <x>$ and so $0 \neq \cap I_i \subseteq <x>$ with $S(\cap I_i) = 0$.

ii) $\Rightarrow$ i) It is trivial.

Proposition 2.6. Let $N$ be a regular near-ring. If $N$ is subdirectly irreducible, then the following conditions are equivalent:

i) $N$ is a right weak IFP near-ring

ii) $N$ is a reduced near-ring

iii) $N$ is a strong IFP near-ring

iv) $N$ is a IFP near-ring.
Proof. i) ⇒ ii) Let $a(\neq 0) \in N$ such that $a^2 = 0$. Since $N$ is regular, we have $a = axa$ for some $x \in N$. Set $e = ax$. Let $S = \{n - ne/n \in N\}$. Then $r(S)$ contains a non-zero ideal $J$ with $J \subseteq e$ and so $nj = nej$ for all $n \in N$ and for all $j \in J$. Let $j(\neq 0) \in J$. Then there exists $y \in N$ such that $j = jyj = j(yj) = je(yj) = ja(xyj) = (ja)e(xyj) = ja^2x^2yj = 0$, a contradiction.

ii) ⇒ iii) It follows from Proposition 2.94 of [8].

iii) ⇒ iv) and iv) ⇒ i) are trivial.

Hereafter $N$ denote a zero-symmetric near-ring with left identity. Following G. F. Birkenmeier and N. J. Groenewald [1], a near-ring $N$ is said to be left (resp. right) weakly $\pi$-regular if $x^n \in <x^n > x^n$ (resp. $x^n \in x^n < x^n >$) for all $x \in N$ and for some natural number $n = n(x)$. A near-ring $N$ is called weakly $\pi$-regular if $N$ is both left and right weakly $\pi$-regular. A weakly $\pi$-regular near-ring is called weakly regular when $n = 1$.

A near-ring $N$ is said to be left (resp. right) pseudo $\pi$-regular if $x^n \in < x > x^n$ (resp. $x^n \in x^n < x >$) for all $x \in N$ and for some natural number $n = n(x)$. A near-ring $N$ is called pseudo $\pi$-regular if $N$ is both left and right pseudo $\pi$-regular.

**Proposition 2.7.** Let $P$ be a completely prime ideal of $N$. If $N/P(N)$ is a left weakly $\pi$-regular near-ring, then $P$ is a maximal ideal of $N$.

Proof. Let $P$ be a completely prime ideal of $N$ and $N/P(N)$ be a left weakly $\pi$-regular near-ring. Suppose $M$ is an ideal of $N$ such that $P \subset M$. Let $a \in M \setminus P$. Then $P + < a > \subseteq M$. Since $N/P(N) = \bar{N}$ is a left weakly $\pi$-regular, we have $\bar{N}a^n = < \bar{a}^n > \bar{a}^n$ for some positive integer $n$. So $\bar{N}a^n = \bar{M}a^n$. Hence $\bar{a}^n = \bar{b}a^n$ for some $\bar{b} \in \bar{M}$ and so $(1 - b)a^n \in P$ which implies $N = M$. □

**Corollary 2.8.** (5 Theorem 2.3) Let $P$ be a completely prime ideal of a ring $R$. If $P/P(R)$ is a left weakly $\pi$-regular near-ring, then $P$ is a maximal ideal of $R$.

G.F.Birkenmeier, J.Y.Kim and J.K.Park [2] have shown that a reduced ring $R$ is weakly regular if and only if $R$ is right weakly regular and if and only if every prime ideal of $R$ is maximal. We shall prove this result under generalized conditions.

**Proposition 2.9.** Let $N$ be a 2-primal near-ring. If $N$ is strong IFP, then the following conditions are equivalent:

i) $N$ is left weakly regular

ii) $N$ is left weakly $\pi$-regular

iii) $N/P(N)$ is left weakly $\pi$-regular
iv) $N/P(N)$ is left pseudo $\pi$-regular
v) Every prime ideal of $N$ is maximal.

Proof. i) \implies ii), ii) \implies iii) and iii) \implies iv) Proofs are trivial.
iv) \implies v) It follows from Corollary 3.10 of [1].
v) \implies i) Suppose $N$ is not a left weakly regular. Then there exists an element $a \in N$ such that $a \notin < a > a$. Let $T$ be a union of all prime ideals of $N$, such that each of them contain $a$. Let $S = N \setminus T$. Then $S$ is a multiplicative closed subset of $N$ by Theorem 5 of [4]. Let $F$ be the multiplicative closed system generated by $\{a\} \cup S$. Suppose $0 \notin F$. Then there exists a proper prime ideal $M$ of $N$ with $M \cap F = \phi$ by Proposition 2.81 of [8]. Since $a \notin M$, we have $M + < a > = N$ and so there exists $b \in M$ and $c \in < a >$ such that $b + c = 1$. Clearly $b \notin T$, which implies $b \in F \cap M = \phi$, a contradiction. Thus $0 \in F$.

So $0 = a^{n_1} s_1 a^{n_2} ... a^{n_t} s_t$ where $s_i \in S$ and $n_1, n_2, ..., n_t$ are positive integers. For any prime ideal $P$, we have $a^{n_1} s_1 a^{n_2} ... a^{n_t} s_t \in P$. Since $P$ is completely prime, we have $a \in P$ or $s_i \in P$ for some $i$. Let $s = s_1 s_2 ... s_t$. Then for any prime ideal $P$; either $a \in P$ or $s \in P$. Then $sa \in P(N)$. Since $N$ is a strong IFP near-ring, we have $sNa = 0$. Then $< s > N a = 0$. Observe that a prime ideal can not contains both $a$ and $s$; otherwise a prime ideal would contain both of them, which contradicts the definitions of $S$ and $T$ which implies $< s > + < a > = N$ and so $N$ is a left weakly regular near-ring.

Corollary 2.10 (2, Theorem 8). Let $R$ be a reduced ring. Then the following conditions are equivalent:
i) $R$ is weakly regular
ii) $R$ is right weakly $\pi$-regular
iii) Every prime ideal of $R$ is maximal

Proof. The proof is an immediate consequence of Proposition 2.9 and Theorem 12 of [3].

Corollary 2.11 (2, Corollary 9). Let $R$ be a 2-primal ring. Then the following conditions are equivalent:
i) $R/P(R)$ is weakly regular
ii) $R/P(R)$ is right weakly regular
iii) Every prime ideal of $R$ is maximal.

References


