**A GENERALIZED SCHEME FOR THE NUMERICAL
SOLUTION OF INITIAL VALUE PROBLEMS IN ORDINARY
DIFFERENTIAL EQUATIONS BY THE RECURSIVE
FORMULATION OF TAU METHOD**

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Abstract: The generalization of the recursive form of the tau method for both overdetermined and non-overdetermined ordinary differential equations of the initial value type is the main thrust of the work reported here. This will facilitate an automation of this variant of the method and consequently an efficient utilization of the technique.

Results from the numerical experiment confirm the validity and effectiveness of the derived scheme.

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1. Introduction

The tau method of Lanczos, first proposed 1938 to solves the linear ordinary

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differential equations

$$Ly(x) \equiv \sum_{r=0}^m \left(\sum_{k=0}^{N_r} P_{rk} x^k \right) y^r(x) = \sum_{r=0}^{\sigma} f_r x^r, \quad a \leq x \leq b, \quad (1.1)$$

$$L^* y(x_{rk}) \equiv \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, \quad k = 1(1)m,$$

by seeking an approximant

$$y_n(x) = \sum_{r=0}^n a_r x^r, \quad r < +\infty,$$

of $y(x)$ which is the exact solution of the corresponding perturbed system

$$Ly_n(x) = \sum_{r=0}^{\sigma} f_r x^r + H_n(x) \quad (1.2)$$

$$L^* y_n(x_{rk}) = \alpha_k, \quad k = 1(1)m$$

(see [1]-[6], [8]). In (1.1) L is the differential operator, $y^r(x)$ is the derivative of order r of $y(x)$, $P_{rk}(x)$, σ , α_k , x_{rk} , a and b are real constants and

$$H_n(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} T_{n-m+i+1}(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} x^r \quad (1.3)$$

is the perturbation term. The parameters $\tau_i, i = 1(1)(m+s)$ in (1.3) are to be determined.

$$T_n(x) \equiv \sum_{r=0}^n C_r^{(n)} x^r = \text{Cos} \left[n \text{Cos}^{-1} \left(\frac{2x - a - b}{b - a} \right) \right] \quad (1.4)$$

is the chebyshev polynomial valid in the interval $[a, b]$. The number s , called the overdetermination number of (1.1), is given by

$$s = \max \{ N_r - r > 0 \mid 0 \leq r \leq m \} \quad (1.5)$$

In 1956 (see [11]), Lanczos introduced the use of canonical polynomials $\{Q_r(x)\}$, $r = 0, 1, \dots$ into the tau method. The difficulty involved in their construction was removed by Ortiz in 1969 (see [13]), when he reported a recursive generation of the polynomials, $Q_r(x)$ defined by

$$LQ_r(x) = x^r \quad (1.6)$$

2. Generalization of Lanczos-Ortiz Canonical Polynomials

For the purpose of generalizing the tau approximant of the solution of $y(x)$ to equation(1.1), we shall briefly review in this section the work of Yisa (2012), see [14].

We consider canonical polynomials for initial value problem (1.1), for the cases $m = 1, 2, \dots$, $s = 1, 2, \dots$ and then generalize $Q_n(x) \forall m$ and s .

Case $m = 1, s = 1$: From(1.1), we have

$$Ly(x) \equiv (P_{0,0} + P_{0,1}x)y(x) + (P_{1,0} + P_{1,1}x + P_{1,2}x^2)y'(x) = \sum_{r=0}^{\sigma} f_r x^r$$

$$\Rightarrow Lx^r = (P_{0,0} + rP_{1,2})x^{r+1} + (P_{0,0} + rP_{1,1})x^r + rP_{1,0}x^{r-1}$$

Since $LQ_r(x) = x^r$, we have

$$Q_{r+1}(x) = \frac{x^r - rP_{1,0}Q_{r-1}(x) - (P_{0,0} + rP_{1,1})Q_r(x)}{P_{0,1} + rP_{1,2}}, \quad r \geq 0. \quad (2.1)$$

Case $m = 1, s = 2$:

$$(P_{0,0} + P_{0,1}x + P_{0,2}x^2)y(x) + (P_{1,0} + P_{1,1}x + P_{1,2}x^2 + P_{1,3}x^3)y'(x) = \sum_{r=0}^{\sigma} f_r x^r$$

$$\Rightarrow Q_{r+2}(x) = \frac{x^r - rP_{1,0}Q_{r-1}(x) - (P_{0,0} + rP_{1,1})Q_r(x) - (P_{0,1} + rP_{1,2})Q_{r+1}(x)}{P_{0,2} + rP_{1,3}}.$$

Case $m = 1, s = 3$: We have

$$Q_{r+3}(x) = \frac{x^r - rP_{1,0}Q_{r-1}(x) - (P_{0,0} + rP_{1,1})Q_r(x) - (P_{0,1} + rP_{1,2})Q_{r+1}(x) - (P_{0,2} + rP_{1,3})Q_{r+2}(x)}{P_{0,2} + rP_{1,4}}.$$

Generally, for case $m = 1, s = s$ we have

$$Q_n(x) = \frac{x^{n-s} - [\sum_{j=0}^s (P_{0,j-1} + (n-s)P_{1,j})Q_{n-s+j-1}(x)]}{P_{0,s} + (n-s)P_{1,s+1}},$$

$$n = r + s, \quad r \geq 0, \quad (2.2)$$

where $P_{0,-1} = 0$ and $P_{0,s} + (n-s)P_{1,s+1} \neq 0$.

Case $m = 2, s = 1$: From (1.1), we have

$$\begin{aligned} (P_{2,0} + P_{2,1}x + P_{2,2}x^2 + P_{2,3}x^3) y''(x) + (P_{1,0} + P_{1,1}x + P_{1,2}x^2) y'(x) \\ + (P_{0,0} + P_{0,1}x) y(x) = \sum_{r=0}^{\sigma} f_r x^r, \end{aligned}$$

$$Q_{r+1}(x) = \frac{x^r - [(P_{0,0} + rP_{1,1} + r(r-1)P_{2,2})Q_r(x) + (rP_{1,0} + r(r-1)P_{2,1})Q_{r-1}(x) + r(r-1)P_{2,0}Q_{r-2}(x)]}{P_{0,1} + rP_{1,2} + r(r-1)P_{2,3}}.$$

Case $m = 2, s = 2$:

For this case we obtained

$$\begin{aligned} Q_{r+2}(x) = \frac{1}{P_{0,2} + rP_{1,3} + r(r-1)P_{2,4}} (x^r - [(r(r-1)P_{2,3} + rP_{1,2} + P_{0,1})Q_{r+1}(x) \\ + (P_{0,0} + rP_{1,1} + r(r-1)P_{2,2})Q_r(x) \\ + (rP_{1,0} + r(r-1)P_{2,1})Q_{r-1}(x) + r(r-1)P_{2,0}Q_{r-2}(x)]). \end{aligned}$$

Generally, for case $m = 2, s = s$ We have

$$Q_n(x) = \frac{x^{n-s} - [\sum_{j=0}^s (P_{0,j-1} + rP_{1,j} + r(r-1)P_{2,j+1}) Q_{r+j-1}(x)]}{P_{0,s} + rP_{1,s+1} + r(r-1)P_{2,s+2}}, \quad n = r + s, r \geq 0 \quad (2.3)$$

where $P_{0,-1} = 0$ and $P_{0,s} + rP_{1,s+1} + r(r-1)P_{2,s+2} \neq 0$.

Case $m = m, s = s$: Continuing the process above, then derived the general canonical polynomial for both overdetermined and non-overdetermined ordinary differential equations (see Yisa [14]).

$$\begin{aligned} Q_n(x) = \frac{1}{\sum_{k=0}^m k! \binom{n-s}{k} P_{k,k+s}} \left\{ x^{n-s} - \left[\sum_{k=1}^m \left(\sum_{j=k}^m j! \binom{n-s}{j} P_{j,j-k} \right) Q_{n-s-k}(x) \right. \right. \\ \left. \left. + \sum_{k=0}^{s-1} \left(\sum_{j=0}^m j! \binom{n-s}{j} P_{j,j+k} \right) Q_{n-s+k}(x) \right] \right\}. \quad (2.4) \end{aligned}$$

Theorem 1. (see Yisa [14]) If

$$Ly(x) \equiv \sum_{r=0}^m \left(\sum_{k=0}^{N_r} P_{rk} x^k \right) y^r(x) = \sum_{r=0}^{\sigma} f_r x^r = f(x)$$

where $P_k(x)$ is a k -th degree polynomials, $y^r(x)$ is the r -th derivative of $y(x)$ and $f(x)$ is a polynomial of degree σ , is an m -th order ordinary differential equation, then the canonical polynomial associated with its differential operator

$$L = \sum_{r=0}^m \left(\sum_{k=0}^{N_r} P_{rk} x^k \right) \frac{d^r}{dx^r}$$

is

$$Q_n(x) = \frac{1}{\sum_{k=0}^m k! \binom{n-s}{k} P_{k,k+s}} \left\{ x^{n-s} - \left[\sum_{k=1}^m \left(\sum_{j=k}^m j! \binom{n-s}{j} P_{j,j-k} \right) Q_{n-s-k}(x) + \sum_{k=0}^{s-1} \left(\sum_{j=0}^m j! \binom{n-s}{j} P_{j,j+k} \right) Q_{n-s+k}(x) \right] \right\}.$$

3. Derivation of Recursive Formulation of the Tau Approximant

In this section, the tau approximant for the recursive form using the generalized canonical polynomial is considered for varying orders and degrees. From the perturbed equation(1.2)

$$Ly_n(x) = \sum_{r=0}^{\sigma} f_r x^r + H_n(x),$$

$$L^* y_n(x_{rk}) = \alpha_k, k = 1(1)m,$$

where

$$H_n(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} T_{n-m+i+1}(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} x^r$$

is the perturbation term.

Case $m = 1, s = 1$: We have

$$Ly_n(x) = \sum_{r=0}^{\sigma} f_r x^r + \sum_{i=0}^1 \tau_{i+1} \sum_{r=0}^{n+i} C_r^{(n+i)} x^r$$

$$\begin{aligned}
&= \sum_{r=0}^{\sigma} f_r L Q_r(x) + \sum_{i=0}^1 \tau_{i+1} \sum_{r=0}^{n+i} C_r^{(n+i)} L Q_r(x) \\
&= L \left\{ \sum_{r=0}^{\sigma} f_r Q_r(x) + \sum_{i=0}^1 \tau_{i+1} \sum_{r=0}^{n+i} C_r^{(n+i)} Q_r(x) \right\} \\
\Rightarrow y_n(x) &= \sum_{r=0}^{\sigma} f_r Q_r(x) + \tau_1 \sum_{r=0}^n C_r^{(n)} Q_r(x) + \tau_2 \sum_{r=0}^{n+1} C_r^{(n+1)} Q_r(x).
\end{aligned}$$

Now, inserting (2.4); expand the resulting equation and equate the coefficient of undetermined canonical polynomial (i.e $Q_0(x)$) to zero, to obtain

$$\tau_1 \sum_{r=0}^n C_r^{(n)} P_r + \tau_2 \sum_{r=0}^{n+1} C_r^{(n+1)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0. \quad (3.1)$$

Take $P_0 = 1$. Equation (3.1) is the coefficient of undetermined canonical polynomial (i.e $Q_0(x)$) and for ease generalization we assume $Q_0(x) = 1$, to obtain

$$y_n(x) = \sum_{r=1}^{\sigma} f_r q_r(x) + \sum_{i=0}^1 \tau_{i+1} \sum_{r=1}^{n+i} C_r^{(n+i)} q_r(x), \quad (3.2)$$

where $q_n = Q_n(x) - P_n$ and

$$P_n = \frac{-P_{0,0}P_{n-1} - (n-1)P_{1,0}P_{n-2} - (n-1)P_{1,1}P_{n-1}}{P_{0,1} + (n-1)P_{1,2}}. \quad (3.3)$$

Case $m = 1, s = 2$: From (1.2), we have

$$\begin{aligned}
y_n(x) &= \sum_{r=0}^{\sigma} f_r Q_r(x) + \tau_1 \sum_{r=0}^n C_r^{(n)} Q_r(x) + \tau_2 \sum_{r=0}^{n+1} C_r^{(n+1)} Q_r(x) \\
&\quad + \tau_3 \sum_{r=0}^{n+2} C_r^{(n+2)} Q_r(x).
\end{aligned}$$

Again, insert (2.4); expand the resulting equation and equate the coefficient of undetermined canonical polynomials (i.e $Q_0(x), Q_1(x)$) to zero, to obtain

$$\tau_1 \sum_{r=0}^n C_r^{(n)} P_r + \tau_2 \sum_{r=0}^{n+1} C_r^{(n+1)} P_r + \tau_3 \sum_{r=0}^{n+2} C_r^{(n+2)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0. \quad (3.4)$$

Take $P_0 = 1$, $P_1 = 0$ for coefficient of $Q_0(x)$, for coefficient of $Q_1(x)$ take $P_0 = 0$, $P_1 = 1$ and for $y_n(x)$ we assume $Q_0(x) = Q_1(x) = 1$, and take $P_0 = P_1 = 1$, to obtain

$$y_n(x) = \sum_{r=2}^{\sigma} f_r q_r(x) + \sum_{i=0}^2 \tau_{i+1} \sum_{r=2}^{n+i} C_r^{(n+i)} q_r(x), \quad (3.5)$$

where

$$P_n = \frac{-(n-2)P_{1,0}P_{n-3} - (P_{0,0} + (n-2)P_{1,1})P_{n-2} - (P_{0,1} + (n-2)P_{1,2})P_{n-1}}{P_{0,2} + (n-2)P_{1,3}},$$

and $q_n = Q_n(x) - P_n$.

Case $m = 2$, $s = 1$: Insert (2.4) in (1.2); expand the resulting equation and equate the coefficient of undetermined canonical polynomial (i.e $Q_0(x)$) to zero, to obtain

$$\tau_1 \sum_{r=0}^{n-1} C_r^{(n-1)} P_r + \tau_2 \sum_{r=0}^n C_r^{(n)} P_r + \tau_3 \sum_{r=0}^{n+1} C_r^{(n+1)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0. \quad (3.6)$$

Take $P_0 = 1$ for the coefficient of $Q_0(x)$, and for $y_n(x)$ we assume $Q_0(x) = P_0 = 1$, then we obtain the solution

$$y_n(x) = \sum_{r=1}^{\sigma} f_r q_r(x) + \sum_{i=0}^2 \tau_{i+1} \sum_{r=1}^{n-1+i} C_r^{(n-1+i)} q_r(x) \quad (3.7)$$

and

$$y'_n(x) = \sum_{r=1}^{\sigma} f_r Q'_r(x) + \sum_{i=0}^2 \tau_{i+1} \sum_{r=1}^{n+i} C_r^{(n-1+i)} Q'_r(x). \quad (3.8)$$

Case $m = 2$, $s = 2$: Insert (2.4) in (1.2); expand the resulting equation and equate the coefficient of undetermined canonical polynomials (i.e $Q_0(x)$ and $Q_1(x)$) to zero to obtain

$$\begin{aligned} \tau_1 \sum_{r=0}^{n-1} C_r^{(n-1)} P_r + \tau_2 \sum_{r=0}^n C_r^{(n)} P_r + \tau_3 \sum_{r=0}^{n+1} C_r^{(n+1)} P_r + \tau_4 \sum_{r=0}^n C_r^{(n+2)} P_r \\ + \sum_{r=0}^{\sigma} f_r P_r = 0 \end{aligned}$$

for the coefficient of $Q_0(x)$, take $P_0 = 1$ and $P_1 = 0$, and for the coefficient of $Q_1(x)$, take $P_1 = 1$, $P_0 = 0$ and for $y_n(x)$ we assume $Q_0(x) = Q_1(x) = P_0 = P_1 = 1$, then we obtain the solution

$$y_n(x) = \sum_{r=2}^{\sigma} f_r q_r(x) + \sum_{i=0}^3 \tau_{i+1} \sum_{r=1}^{n-1+i} C_r^{(n-1+i)} q_r(x) \quad (3.7)$$

and

$$y'_n(x) = \sum_{r=2}^{\sigma} f_r Q'_r(x) + \sum_{i=0}^3 \tau_{i+1} \sum_{r=1}^{n+i} C_r^{(n-1+i)} Q'_r(x).$$

Observing the trend in the tau approximant of the first and second order ordinary differential equations presented above, we derived the general formula

$$y_n(x) = \sum_{r=s}^{\sigma} f_r q_r(x) + \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)} q_r(x). \quad (3.9)$$

Assume $Q_r(x) = P_r = 1, r = 0(1)(s-1)$

$$y_n^\lambda(x) = \sum_{r=s}^{\sigma} f_r Q_r^\lambda(x) + \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)} Q_r^\lambda(x) = \alpha_\lambda, \quad \lambda = 0(1)(m-1), \quad (3.10)$$

$$\sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} P_r + \sum_{r=s}^{\sigma} f_r P_r = 0, \quad (3.11)$$

where $q_n(x) = Q_n(x) - P_n$ and equation (3.11) is the coefficient of undetermined canonical polynomials.

$$P_n = \frac{-1}{\sum_{k=0}^m k! \binom{n-s}{k} P_{k,k+s}} \left\{ \sum_{k=1}^m \left(\sum_{j=k}^m j! \binom{n-s}{j} P_{j,j-k} \right) P_{n-s-k} + \sum_{k=0}^{s-1} \left(\sum_{j=0}^m j! \binom{n-s}{j} P_{j,j+k} \right) P_{n-s+k} \right\}. \quad (3.12)$$

Take $P_r = 1$, when equating the coefficient of $Q_r(x)$ to zero, otherwise $P_r = 0$, $r = 0(1)(s-1)$.

The formula also valid for non-overdetermined differential equation, by replacing $s = 0$ in (2.4), $P_r = 0, \forall r$ in (3.9) and neglect (3.11) and (3.12).

4. Numerical Experiments

In this section we present some numerical experiments to illustrate the numerical results presented in this paper. The exact errors are defined as $\varepsilon_n = \max_{a \leq x \leq b} |y(x_k) - y_n(x_k)|$, where $\{x_k\} = (0.01k)$, $k = 0(1)100$.

Problem 1.

$$Ly(x) = y'(x) + 2xy(x) = 4x, y(0) = 3,$$

with analytical solution $y(x) = 2 + \exp(-x^2)$.

Comparing with (1.2), $P_{0,0} = 0$, $P_{0,1} = 2$, $P_{1,0} = 1$, $P_{1,1} = P_{1,2} = 0$, $f_0 = 0$, $f_1 = 4$, $\sigma = 1$, $m = 1$, $s = 1$.

For degree 5 (i.e $n = 5$), applying equations (3.9) and (3.11), we have

$$y_5(x) = 3 + \frac{164x}{3360543} - \frac{3356272x^2}{3360543} - \frac{29952x^3}{1120181} + \frac{2100992x^4}{3360543} - \frac{779264x^5}{3360543}.$$

For degree 6 (i.e $n = 6$), we have

$$y_6(x) = 3 - \frac{35308x}{160301129} - \frac{159018016x^2}{160301129} - \frac{10087168x^3}{160301129} + \frac{112491776x^4}{160301129} - \frac{48468992x^5}{160301129} + \frac{3788800x^6}{160301129}.$$

For degree 7 (i.e $n = 7$), we have

$$y_7(x) = 3 - \frac{560x}{56926267} - \frac{56902112x^2}{56926267} - \frac{210560x^3}{56926267} + \frac{29105920x^4}{56926267} - \frac{403456x^5}{56926267} - \frac{11161600x^6}{56926267} + \frac{3588096x^7}{56926267}.$$

Problem 2.

$$Ly(x) = y'(x) - x^2y(x) = 0, \quad y(0) = 1,$$

with analytical solution $y(x) = \exp(\frac{1}{3}x^3)$.

For degree 5, 6 and 7 approximate solution, we have

$$y_5(x) = 1 + \frac{42887122x}{26442910625} - \frac{1012427712x^2}{26442910625} + \frac{13794322304x^3}{26442910625} - \frac{9264926976x^4}{26442910625} + \frac{6901011968x^5}{26442910625},$$

$$y_6(x) = 1 - \frac{5102898720x}{20423863454407} + \frac{172910611296x^2}{20423863454407} + \frac{795917236736x^3}{2917694779201} \\ + \frac{3494579851008x^4}{20423863454407} - \frac{4385551601664x^5}{20423863454407} + \frac{3231878008832x^6}{20423863454407},$$

$$y_7(x) = 1 + \frac{54229310046x}{1148232742907190} - \frac{2553117942456x^2}{1148232742907190} + \frac{81717617906288x^3}{229646548581439} \\ - \frac{108270560916864x^4}{1148232742907190} + \frac{44581151086848x^5}{229646548581439} - \frac{165543248190464x^6}{1148232742907190} \\ + \frac{99073874491392x^7}{1148232742907190}.$$

respectively.

Problem 3.

$$Ly(x) = y''(x) - 2(1 + 2x^2)y(x) = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

with analytical solution $y(x) = \exp(x^2)$.

For degree 5, 6 and 7 approximate solution, we have

$$y_5(x) = 1 + \frac{19855060080x^2}{21441580663} + \frac{15314739712x^3}{21441580663} - \frac{22091646208x^4}{21441580663} \\ + \frac{23992840192x^5}{21441580663}.$$

$$y_6(x) = 1 + \frac{14339349817120x^2}{14137226504329} - \frac{3116501231104x^3}{14137226504329} + \frac{54597872159488x^4}{42411679512987} \\ - \frac{46782632538112x^5}{42411679512987} + \frac{31339103866880x^6}{42411679512987}.$$

$$y_7(x) = 1 + \frac{28873617687738500x^2}{28953089655559100} + \frac{1809682067903230x^3}{28953089655559100} \\ + \frac{4638442954437370x^4}{28953089655559100} + \frac{22820567800864700x^5}{28953089655559100} \\ - \frac{20305651810889700x^6}{28953089655559100} + \frac{11916952869830600x^7}{28953089655559100},$$

respectively.

For experiment 4 and 5, we consider non-overdetermined type.

Problem 4.

$$Ly(x) = 2(1+x)y'(x) + y(x) = 0, \quad y(0) = 1, \quad 0 \leq x \leq 1,$$

with analytical solution $y(x) = (1+x)^{-\frac{1}{2}}$, $P_{0,0} = 1$, $P_{1,0} = 2$, $P_{1,1} = 2$, $m = 1$, $f_0 = 0$.

For degree 5, 6 and 7 approximate solution, we have

$$y_5(x) = 1 - \frac{169546x}{339323} + \frac{124272x^2}{339323} - \frac{88160x^3}{339323} + \frac{44800x^4}{339323} - \frac{10752x^5}{339323},$$

$$y_6(x) = 1 - \frac{59392456x}{118799927} + \frac{44274072x^2}{118799927} - \frac{34792960x^3}{118799927} + \frac{23717120x^4}{118799927} - \frac{645120x^5}{6988231} + \frac{2365440x^6}{118799927},$$

$$y_7(x) = 1 - \frac{9239102x}{18478633} + \frac{6918816x^2}{18478633} - \frac{5653568x^3}{18478633} + \frac{4442368x^4}{18478633} - \frac{14224896x^5}{92393165} + \frac{5992448x^6}{92393165} - \frac{1171456x^7}{92393165}.$$

Problem 5.

$$y''(x) - (1-x)y'(x) + y(x) = 0, \quad y(0) = 1, \quad y'(0) = 1, \quad 0 \leq x \leq 1,$$

with analytical solution $\exp(x - \frac{x^2}{2})$.

For degree 5, 6 and 7 approximate solution, we have

$$y_5(x) = 1 + x + \frac{2168x^2}{3272391} - \frac{3315848x^3}{9817173} - \frac{815488x^4}{9817173} + \frac{675584x^5}{9817173},$$

$$y_6(x) = 1 + x - \frac{230528x^2}{515794089} - \frac{504319192x^3}{1547382267} - \frac{173802752x^4}{1547382267} + \frac{149575936x^5}{1547382267} - \frac{51200x^6}{5546173},$$

$$y_7(x) = 1 + x - \frac{57032x^2}{42919912639} - \frac{42921645520x^3}{128759737917} - \frac{10673371904x^4}{128759737917} + \frac{6122414848x^5}{128759737917} + \frac{1202865152x^6}{42919912639} - \frac{455455744x^7}{42919912639}.$$

Below are the maximum errors obtain for each experiment and degree.

Experiment	ε_5	ε_6	ε_7
Problem 1	3.57×10^{-5}	1.97×10^{-5}	1.10×10^{-6}
Problem 2	2.07×10^{-4}	2.96×10^{-5}	4.50×10^{-6}
Problem 3	1.06×10^{-2}	1.23×10^{-3}	1.40×10^{-4}
Problem 4	3.08×10^{-5}	4.24×10^{-6}	6.64×10^{-7}
Problem 5	9.42×10^{-5}	2.62×10^{-5}	1.81×10^{-7}

Table 1: Errors of Experiments 1-5

5. Conclusion

The generalized form of the recursive formulation of the tau method for initial value problems in ordinary differential equations for both overdetermined and non-overdetermined types has been presented. Evidently, the overdetermined type is more cumbersome to derive than for the case of non-overdetermined type.

This study will facilitate an automation of this variant of the tau method towards a software development of the technique.

The derived scheme has been implemented on some selected test problems. Numerical evidences therefrom confirm its accuracy and effectiveness.

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