MORE ON THE DIOPHANTINE EQUATION $2^x + 19^y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $2^x + 19^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(3, 0, 3)$.

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1. Introduction

In 2011, Suvarnamani [11] present that $(3, 0, 3)$ is a unique non-negative integer solution for the Diophantine equation $2^x + p^y = z^2$ where x, y and z are non-negative integers and where p is a positive prime number. In 2013, Chotchaisthit [2] suggested that the Suvarnamani's proof is not completed, and then he gave a completed proof that $(3, 0, 3)$ is a unique non-negative integer solution for the Diophantine equation $2^x + 11^y = z^2$ where x, y and z are non-negative integers. In the same year, we [9] also gave a completed proof that $(3, 0, 3)$ is a unique non-negative integer solution for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. For the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers, Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) . For related papers, we

refer to [3, 5, 6, 7, 8, 10, 12]. In this paper, we prove that $(3, 0, 3)$ is a unique non-negative integer solution for the Diophantine equation $2^x + 19^y = z^2$ where x, y and z are non-negative integers.

2. Preliminaries

In 2004, Mihailescu [4] proved the Catalan's conjecture as follows.

Proposition 2.1. [4] $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.

We refer to the two lemmas as follows.

Lemma 2.2. [1] $(3, 3)$ is a unique solution (x, z) for the Diophantine equation $2^x + 1 = z^2$ where x and z are non-negative integers.

Lemma 2.3. [6] The Diophantine equation $1 + 19^y = z^2$ has no non-negative integer solution where y and z are non-negative integers.

3. Results

Theorem 3.1. $(3, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $2^x + 19^y = z^2$ where x, y and z are non-negative integers.

Proof. Let x, y and z be non-negative integers such that $2^x + 19^y = z^2$. By Lemma 2.3, we have $x \geq 1$. It follows that z is odd. Then $z = 2t + 1$ for some a non-negative integer t . Thus, $2^x + 19^y = 4(t^2 + t) + 1$. This implies that $19^y \equiv 1 \pmod{4}$. It follows that y is even. Then $y = 2k$ for some a non-negative integer k . Now, we will divide the number y into two cases.

Case $y = 0$. By Lemma 2.2, we have $x = 3$ and $z = 3$.

Case $y \geq 2$. Then $k \geq 1$. Then $z^2 - 19^{2k} = 2^x$. Then $(z - 19^k)(z + 19^k) = 2^x$. Thus, $z - 19^k = 2^u$ where u is a non-negative integer. Then $z + 19^k = 2^{x-u}$. Thus, $2(19^k) = 2^{x-u} - 2^u = 2^u(2^{x-2u} - 1)$. Next, we will divide the number u into two subcases.

Subcase $u = 0$. Then $z - 19^k = 1$. It follows that z is even. This is a contradiction.

Subcase $u = 1$. Then $2^{x-2} - 1 = 19^k$. Then $2^{x-2} = 19^k + 1 \geq 19 + 1 = 20$. Then $x - 2 \geq 5$, i.e. $x \geq 7$. Moreover, $2^{x-2} - 19^k = 1$. By Proposition 2.1, we have $k = 1$. Then $2^{x-2} = 20$. This is impossible.

Therefore, $(3, 0, 3)$ is a unique solution (x, y, z) for the equation $2^x + 19^y = z^2$. $\square$

Corollary 3.2. [6] $(1, 0, 3)$ is a unique solution (u, y, z) for the Diophantine equation $8^u + 19^y = z^2$ where u, y and z are non-negative integers.

Proof. Let x, y and z be non-negative integers such that $8^u + 19^y = z^2$. Let $x = 3u$. Then $2^x + 19^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (3, 0, 3)$. Then $3u = x = 3$. Thus, $u = 1$. Therefore, $(1, 0, 3)$ is a unique solution (u, y, z) for the equation $8^u + 19^y = z^2$. $\square$

Corollary 3.3. The Diophantine equation $2^x + 19^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers.

Proof. Suppose that there are non-negative integers x, y and w such that $2^x + 19^y = w^4$. Let $z = w^2$. Then $2^x + 19^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (3, 0, 3)$. Then $w^2 = z = 3$. This is a contradiction. Hence, the equation $2^x + 19^y = w^4$ has no non-negative integer solution. $\square$

4. Open Problem

Recently, we know that what is the set of all solutions (x, y, z) for the Diophantine equation $8^x + 17^y = z^2$ where x, y and z are non-negative integers. Thus, we may pose a question that what is the set of all solutions (x, y, z) for the Diophantine equation $2^x + 17^y = z^2$ where x, y and z are non-negative integers.

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