RANKS OF LINEAR SUBSPACES WITH RESPECT TO AN EMBEDDED VARIETY

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Abstract: Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety and $V \subset \mathbb{P}^r$ a k -dimensional linear subspace. For any $P \in \mathbb{P}^r$ the X -rank of P is the minimal cardinality of a subset of X whose linear span contains P . Here we study several integers related to the X -rank of V , e.g. the minimal X -rank of all elements of a basis of V or the minimal sum of the X -ranks of a basis of V .

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1. Introduction

For all integer $r \geq k \geq 0$ let $G(k, r)$ denote the Grassmannian of all k -dimensional linear subspaces of $\mathbb{P}^r$. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Recall that for each $P \in \mathbb{P}^r$ the X -rank $r_X(P)$ of P is the minimal cardinality of a finite subset $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \cdot \rangle$ denote the linear span ([10]). For each $V \in G(k, r)$ the X -rank $\rho_X(V)$ of V is a minimal cardinality of a finite subset $S \subset X$ such that $V \subseteq \langle S \rangle$ ([3]). The *scheme X -rank* $z_X(V)$ (resp. *smoothable X -scheme rank* $zm_X(V)$, resp. *curvilinear scheme X -rank* $zc_X(V)$) is the minimal degree of a zero-dimensional

(resp. zero-dimensional and smoothable, resp. zero-dimensional and curvilinear) scheme $Z \subset X$ such that $V \subseteq \langle Z \rangle$. We have $z_X(V) \leq zm_X(V) \leq zc_X(V) \leq \rho_X(V)$. In this note we introduce several other notions of X -ranks for the elements of $G(k, r)$. Fix $V \in G(k, r)$. Set $r_X''(V) := \max_{P \in V} r_X(P)$, $r_X'(V) := \min_{P \in V} r_X(P)$, $z_X''(V) := \max_{P \in V} z_X(P)$, $z_X'(V) := \min_{P \in V} z_X(P)$, $zm_X''(V) := \max_{P \in V} zm_X(P)$, $zm_X'(V) := \min_{P \in V} zm_X(P)$. For any $t \in \mathbb{N}$ set $V_X(t) := \{P \in V : r_X(P) = t\}$, $V_{z,X}(t) := \{P \in V : z_X(P) = t\}$ and $V_{zm,X}(t) := \{P \in V : zm_X(P) = t\}$. It is easy to check that each set $V_X(t)$ is a constructible subset. Hence the value $r_X(P)$ is constant in a non-empty open subset Ω of V . Set $r_X(V) := r_X(P)$ for any $P \in \Omega$. Let $e_X(V)$ denote the minimal integer such that $V_X(t)$ contains a subset of V spanning V . Let $w_X(V)$ be the minimal sum $\sum r_X(P)$, where $\sum$ is over any set of points of V spanning V . We may take a corresponding definition $zw_X(P)$ taking in the sum the scheme X -rank $z_X(P)$ (or the smoothable scheme X -rank $zm_X(P)$, or the curvilinear scheme X -rank $zc_X(P)$) of P . Obviously $r_X'(V) \leq e_X(V) \leq r_X(V) \leq r_X''(V) \leq \rho(V) \leq (k+1)e_X(V)$ and $w_X(V) \leq (k+1)e_X(V)$. When $V \cap X = \emptyset$ the integer $r_X'(V)$ has a geometric interpretation in term of a certain rank for the linear projection of X from V into $\mathbb{P}^{r-k-1}$ ([?]), but this interpretation is lost when $V \cap X \neq \emptyset$, because $r_X'(V) = 1$ if $V \cap X \neq \emptyset$. The *border rank* $b_X(V)$ of V is the first integer b such that $V \subseteq \sigma_b(X)$, where $\sigma_b(X)$ (the b -secant variety of X) is the closure in $\mathbb{P}^r$ of the union of all linear spaces $\langle A \rangle$ with $A \subset X$ and $\sharp(A) = b$. Let $\sigma_b^0(X) \subseteq \sigma_b(X)$ be the union of all linear spaces $\langle A \rangle$ with $A \subset X$ and $\sharp(A) = b$. For any integral and non-degenerate variety $X \subset \mathbb{P}^r$ let $\rho(X)$ (resp. $\rho'(X)$, resp. $\rho''(X)$) be the maximal integer t such that $\dim(\langle Z \rangle) = \deg(Z) - 1$ for all zero-dimensional schemes (resp. zero-dimensional and smoothable schemes, resp. finite set) $Z \subset X$ such that $\deg(Z) \leq t$.

Here there is a sample of our results.

Proposition 1. *Let $X \subset \mathbb{P}^r$ be an integral variety. Fix positive integers a, k such that $3a \leq \rho''(X)$. Fix $V \in G(k, r)$ such that $r_X(V) \leq a$. Then there is a set $A \subset X$ such that $\sharp(A) \leq a$ and $V \subseteq \langle S \rangle$.*

Proposition 2. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Fix positive integers a, k such that $k < r$. Set $\alpha := \dim(\sigma_a(X))$ and assume $k + \alpha < r$. Let $\mathcal{S}$ be the set of all $V \in G(k, r)$ such that $e_X(V) = a$.*

(a) *The constructible set $\mathcal{S}$ is an irreducible subset of $G(k, r)$ of dimension $(k+1)\alpha$.*

(b) *Let Δ be the set of all $S \subset \sigma_a^0(X)$ such that $\sharp(S) = k+1$ and $\dim(\langle S \rangle) = k$. The set $\{\langle S \rangle\}_{S \in \Delta}$ covers a non-empty open subset of $\mathcal{S}$ and a general $V \in \mathcal{S}$*

is associated to a unique $S \in \Delta$.

(c) We have $\cup_{V \in \mathcal{S}} V \subseteq \sigma_{(k+1)a}(X)$ and $\cup_{V \in \mathcal{S}} V$ contains a non-empty open subset of $\sigma_{(k+1)a}(X)$

Corollary 1. Fix a general $V \in \mathcal{S}$. Then $w_X(V) = (k+1)a$. If $\rho'(X) \leq a$, then $r'_X(V) = a$ and $zmw_X(V) = (k+1)a$.

Theorem 1. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Fix integers $a \geq 2$ and $k \in \{1, \dots, r\}$. Assume $\dim(\sigma_{a-1}(X)) \leq r - k$. Fix any integer e such that $2a - 1 \leq e \leq (k+1)a$. Then there exists $V \in G(k, n)$ such that $w_X(V) = e$ and $e_X(V) = a$.

We work over an algebraically closed base field $\mathbb{K}$ such that $\text{char}(\mathbb{K}) = 0$.

2. Proofs and Other Results

Remark 1. The integer $r_X(V)$ is the first integer t such that $\sigma_t^0(X)$ contains a non-empty open subset of V . Hence $V \subset \sigma_{r_X(V)}(X)$, i.e. $r_X(V) \geq b_X(V)$.

Remark 2. Fix $X \subset \mathbb{P}^r$ and a positive integer $a \leq \rho'(X)$. Then $\sigma_a(X)$ is the union of all linear spaces $\langle Z \rangle$, where $Z \subset X$ is a smoothable zero-dimensional scheme of degree a ([6], Proposition 11, [8], [7]).

Remark 3. We have $\rho_X(V) = k+1$ if and only if $e_V(X) = 1$.

Remark 4. If $k \geq r - \dim(X)$, then $r'_X(V) = 1$ for all $V \in G(k, r)$.

Proposition 3. Let $X \subset \mathbb{P}^r$ be an integral and m -dimensional variety. Fix positive integers a, k such that $a \leq \rho'(X)$ and $3(k+1) \leq \rho(X)$. Then for any $V \in G(k, r)$ such that $V \subset \sigma_a(X)$ there is a smoothable zero-dimensional scheme $Z \subset X$ such that $\deg(Z) = a$ and $V \subseteq \langle Z \rangle$.

Proof. Since $k > 0$, there are 3 collinear points $O_1, O_2, O_3 \in V$. Since $a \leq \rho'(X)$, there is a zero-dimensional and smoothable scheme $Z_i \subset X$ such that $O_i \in \langle Z_i \rangle$, $i = 1, 2, 3$ (Remark 2). Set $Z := Z_1 \cup Z_2 \cup Z_3$. Since $\deg(Z_1 \cup Z_2 \cup Z_3) \leq \rho(X)$, Grassmann's formula applied to $\langle Z_1 \cup Z_2 \cup Z_3 \rangle$, to $\langle Z_1 \cup Z_2 \rangle$, to $\langle Z_3 \rangle$ and to $\langle (Z_1 \cup Z_2) \cap Z_3 \rangle$ gives $\langle (Z_1 \cup Z_2) \rangle \cap \langle Z_3 \rangle = \langle Z_3 \cap (Z_1 \cup Z_2) \rangle$. Hence $O_1 \in \langle Z_3 \rangle$. Varying O_1 in V , we conclude. $\square$

Proof of Proposition 1. Set $e := r_X(V)$. By assumption there is a non-empty open subset $U \subset V$ such that $U \subset \sigma_e^0(X)$. Take 3 distinct collinear points $O_1, O_2, O_3 \in U$. Fix $S_i \subset X$ such that $\#(S_i) = e$ and $O_i \in \langle S_i \rangle$, $i =$

1, 2, 3. Since $\sharp(S_1 \cup S_2 \cup S_3) \leq 3e \leq \rho''(X)$, Grassmann's formula easily gives $\langle (S_1 \cup S_2) \cap S_3 \rangle = \langle S_1 \cup S_2 \rangle \cap \langle S_3 \rangle$. Hence $O_1 \in \langle S_3 \rangle$. Hence $\langle S_3 \rangle$ contains a general element of V . Hence $V \subseteq \langle S_3 \rangle$. $\square$

Proof of Proposition 1. Let $Y \subseteq \mathbb{P}^r$ be any integral and non-degenerate variety. Set $y := \dim(Y)$ and assume $k + y < r$. Let $A \subset Y$ be a general subset of Y with cardinality y . Since we are in characteristic zero, a general codimension y linear section of Y is formed by $\deg(Y)$ points in linearly general position ([1], p. 109). Since A is general in Y and $k + 1 + y \leq r$, we get that $\langle A \rangle$ has dimension k and that A is the scheme-theoretic intersection of $\langle A \rangle$ and Y . Hence if $B \subset Y$ and $\langle B \rangle = \langle A \rangle$, then $B = A$. Since $\alpha + r < k$, we may apply this observation with $Y = \sigma_a(Y)$. We get that many elements of $\mathcal{S}$ are spanned by a unique subset of $\sigma_a(X)$ and that this subset has cardinality $k + 1$.

Fix $V \in \mathcal{S}$. Since $e_X(V) = a$, there is $S \subset \cup_{1 \leq e \leq a} \sigma_e^0(X)$ such that $\sharp(S) = k + 1$ and $\langle S \rangle = V$. Since $\dim(V) = \sharp(S) - 1$, S is linearly independent. Each finite set $A \subset \sigma_e(X)$, $e \leq a$, is a flat limit of a family of subsets of $\sigma_e^0(X)$. Hence A is a flat limit of a family of subsets of $\sigma_e^0(X)$. Since S is linearly independent and linear independence is an open condition, S is a flat limit of a family of elements of Δ . Conversely, every $A \in \Delta$ gives an element $E := \langle A \rangle \in G(r, k)$ such that $e_X(E) \leq a$. Since $\Delta \cap \sigma_e^0(X) = \emptyset$ for all $e < a$, to prove that $e_X(E) = a$ it is sufficient to prove that $E \cap \sigma_a(X) = A$. We proved it above taking $Y := \sigma_a(X)$. $\square$

Proof of Corollary 1. For the first part copy the proof of Proposition 1. For the second part use Remark 2. $\square$

Obviously in the set-up of Propositions 3 and 1 we also get $k \leq a + 1$. When X is a smooth linearly normal curve and $k = a + 1$ C. Voisin proved a result stronger than Proposition 3. See [4] and [12] for statements and proofs related to Proposition 1.

Proposition 4. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate m -dimensional variety. For each integer $x > 0$ set $\alpha(X, x) := \dim(\sigma_x(X))$. Fix an integer $k \in \{0, \dots, r - 1\}$ and a general $V \in G(k, r)$. Let c be the first integer such that $\sigma_c(X) = \mathbb{P}^r$ and b the first integer such that $\alpha(X, b) \geq r - k$. We have $r'_X(V) = e_X(V) = b$ and $r_X(V) = c$.*

Proof. Since V is general, we have $V \cap \sigma_{b-1}(X) = \emptyset$ (and hence $r'_X(V) \geq b$) and $V \cap \sigma_b^0(X) \neq \emptyset$ (and hence $r'_X(V) \leq b$). Hence $r'_X(V) = b$ and $V \cap \sigma_b(X) \subset \sigma_b^0(X)$. Take a flag $V = V_k \subset V_{k+1} \subset \dots \subset V_{r-1} \subset V_r = \mathbb{P}^r$. Since $\sigma_b(X)$ is integral and non-degenerate, the scheme $V \cap \sigma_b(X)$ spans V . Since $\sigma_b(X)$ is an integral variety and V is general, the scheme $V \cap \sigma_b(X)$ is reduced. Fix any hyperplane $H \subset \mathbb{P}^r$ and any integral and non-degenerate variety. Since Y is

integral, we have $h^1(\mathcal{I}_Y) = 0$. Since Y is non-degenerate, we have $h^0(\mathcal{I}_Y(1)) = 0$. The exact sequence

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_Y(1) \rightarrow \mathcal{I}_{Y \cap H}(1) \rightarrow 0$$

gives $h^0(H, \mathcal{I}_{Y \cap H}(1)) = 0$, i.e. the scheme $Y \cap H$ spans H . If $Y \cap H$ is reduced then the set-theoretic intersection $Y \cap H$ spans H . If $Y \cap H$ is integral, then we may continue. After $r - \alpha(X, b)$ steps we get that the set $\sigma_b(X) \cap V$ spans V . The set $\sigma_b(X) \setminus \sigma_b^0(X)$ is a finite union of locally closed sets of dimension $< \alpha$. Since V is general, we get $V \cap \sigma_{b-1}(X) = \emptyset$ and $V \cap (\sigma_b(X) \setminus \sigma_b^0(X)) = \emptyset$. Hence $e_X(V) = b$. Since V is general and $r_X(P) = c$ for a general $P \in \mathbb{P}^r$, we have $r_X(V) = c$. $\square$

Proof of Theorem 1. Since $2a - 1 \leq e \leq (k + 1)a$, there are integers $f_j \in \{1, \dots, a\}$, $1 \leq j \leq k$, such that $\sum_{j=1}^k f_j = e - a$. If $f_j = 0$, then set $S_j := \emptyset$. If $f_j > 0$, then let S_j be a general subset of $\sigma_j(X)$ such that $\sharp(S_j) = f_j$. Fix a general $O \in \sigma_a(X)$. Set $S := \{O\} \cup \bigcup_{j=1}^k S_j$ and $V := \langle V \rangle$. Since X is non-degenerate and $k \leq r$, the generality of S with the only condition of the generality of S with prescribed $S \cap \sigma_i(X)$, $1 \leq i \leq a$, we have $\dim(V) = k$. Each $\sigma_i(X)$ is irreducible. Since $\sigma_{a-1}(X) \neq \mathbb{P}^r$, we have $\sigma_i(X) \subsetneq \sigma_{i+1}(X)$ for all $i < a$ ([2], Observation 1.2). Since $\dim(\sigma_{a-1}(X)) \leq r - k$ and each S_j is general, as in the first part of the proof of Proposition 1 we get $S \cap \sigma_i(X) = S_i$ for all $i < a$. Hence $e_X(V) = a$ and $w_X(V) = e$. $\square$

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