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ROTATION-INVARIANT L^p-FUNCTIONS

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Abstract: This paper show that the space of rotation-invariant L^p-functions with respect to a Gaussian measure can established as an even L^p-space on $\mathbb{R}$ with respect to some non-Gaussian measure. The space of holomorphic rotation-invariant L^p-functions with respect to a complex Gaussian measure can established as a holomorphic even L^p-space on $\mathbb{C}$ with respect to some non-complex Gaussian measure. We give a condition for a rotation-invariant function which the image of the Segal-Bargmann transform to be in the space of holomorphic rotation-invariant L^q-functions with respect to a complex Gaussian measure.

AMS Subject Classification: Segal-Bargmann transform, Segal-Bargmann space, rotation-invariance

1. Introduction

The Segal-Bargmann transform is an integral transform B which maps $L^2(\mathbb{R}^d, \rho)$, the set of all functions on $\mathbb{R}^d$ that are square integrable with respect to the real

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Gaussian measure $\rho(x)dx = (2\pi)^{-d/2}e^{-x^2/2}dx$, onto $\mathcal{H}L^2(\mathbb{C}^d, \mu)$, the set of all holomorphic functions on $\mathbb{C}^d$ that are square integrable with respect to the complex Gaussian measure $\mu(z)dz = (\pi)^{-d}e^{-|z|^2}dz$. The transform B is given by this formula

$$(Bf)(z) = \int_{\mathbb{R}^d} f(x)\frac{e^{-(z-x)^2/2}}{(2\pi)^{d/2}}dx$$

for all $f \in L^2(\mathbb{R}^d, \rho)$ and $z \in \mathbb{C}^d$. Here we use notation $x^2 = x_1^2 + x_2^2 + \cdots + x_d^2$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The space $\mathcal{H}L^2(\mathbb{C}^d, \mu)$ is also called the Segal-Bargmann space. See [1], [2], [3], [4], [5], [6], [8] for details about the importance of this space.

In [9], [10] and [11], Chaiworn and Lewkeeratiyutkul consider the subspace of the Segal-Bargmann which is invariant under the action of the complex special orthogoal group. They construct the non-Gaussian measure γ on $\mathbb{R}$ which the space $L^2(\mathbb{R}^d, \rho)^{SO(d)}$ is unitarily equivalent to the space $L^2(\mathbb{R}, \gamma)^e$. In the same way they also construct the non-Gaussian measure λ on $\mathbb{C}$ which the Segal-Bargmann space is unitarily equivalent to the Hilbert space $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$.

In this paper we will denote $L^p(\mathbb{R}^d, \rho)^{SO(d)}$ by the set of all $SO(d)$-invariant L^p functions with respect to a Gaussian measure and $\mathcal{H}L^p(\mathbb{C}^d, \mu)^{SO(d,\mathbb{C})}$ by the space of all $SO(d, \mathbb{C})$-invariant holomorphic L^p-functions on $\mathbb{C}^d$ with respect to a complex Gaussian measure. The $SO(d)$-invariance of real valued f is determined by it value on $(x, 0, \ldots, 0) \cong \mathbb{R}$ and it is an even function. Thus the space $L^p(\mathbb{R}^d, \rho)^{SO(d)}$ and $L^p(\mathbb{R}, \gamma)^e$ are isometrically isomorphic. Similarly a complex $SO(d, \mathbb{C})$-invariant function F is determined by its values on $(z, 0, \ldots, 0) \cong \mathbb{C}$ and it is a complex even function. Then we have, the space $\mathcal{H}L^p(\mathbb{C}^d, \mu)^{SO(d,\mathbb{C})}$ and $\mathcal{H}L^p(\mathbb{C}, \lambda)^e$ are isometrically isomorphic. We consider the Segal-Bargmann transform applied to the function on $L^p(\mathbb{R}^d, \rho)^{SO(d)}$ with $1 < p < \infty$. In [7], Hall obtain that if $F = B(f)$ for some $f \in L^p(\mathbb{R}^d, \rho)$, then

$$|F(x + iy) \leq Ce^{y^2/2}e^{x^2/(p-1)}.$$

In this paper we will use this bound and Theorem 7 of [7] to give a sufficient condition for $f \in L^p(\mathbb{R}^d, \rho)^{SO(d)}$ that $B(f)$ to be in the space $\mathcal{H}L^q(\mathbb{C}^d, \mu)^{SO(d,\mathbb{C})}$, when $\frac{1}{q} + \frac{1}{p} = 1$, $0 < q \leq 2$ and $p \geq \frac{q}{2} + 1$.

2. Rotation-Invariant Function Spaces

Denote by $SO(d)$ the set of $d \times d$ real orthogonal matrices with determinant one and by $SO(d, \mathbb{C})$ the set of $d \times d$ complex orthogonal matrices with determinant one.

Definition 1. Let F be a function on $\mathbb{F}^d$ where $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$ and let G be a group of $d \times d$ matrices. We say that F is **G-invariant** if

$$F(Ax) = F(x) \quad \text{for all } A \in G \text{ and all } x \in \mathbb{F}^d.$$

Note that if F is an $SO(d)$-invariant holomorphic function on $\mathbb{C}^d$, then by analytic continuation it is $SO(d, \mathbb{C})$-invariant.

Denote by $\mathcal{B}_d$ the Borel σ-algebra in $\mathbb{R}^d$ and by $\mathcal{B}$ the Borel σ-algebra in $\mathbb{R}$. Define the maps $\Psi_i \colon (\mathbb{R}^d, \mathcal{B}_d, \rho) \to (\mathbb{R}, \mathcal{B})$, $i = 1, 2$ by

$$\Psi_1(x) = |x| \quad \text{and} \quad \Psi_2(x) = -|x|$$

for all $x \in \mathbb{R}^d$. For each $E \in \mathcal{B}$ let

$$\gamma_i(E) = \rho(\Psi_i^{-1}(E)),$$

and let $\gamma = (\gamma_1 + \gamma_2)/2$. It is easy to check that γ is a Borel measure on $\mathbb{R}$ and for any measurable function g and any $E \in \mathcal{B}$

$$\int_E g \, d\gamma \; = \; \frac{1}{2} \int_{\Psi_1^{-1}(E)} g \circ \Psi_1 \, d\rho + \frac{1}{2} \int_{\Psi_2^{-1}(E)} g \circ \Psi_2 \, d\rho.$$

Denote by $\mathcal{F}(\mathbb{R}^d)^{SO(d)}$ the set of all $SO(d)$-invariant functions on $\mathbb{R}^d$ and $\mathcal{F}(\mathbb{R})^e$ the set of all even functions on $\mathbb{R}$. We write

$$L^p(\mathbb{R}^d, \rho)^{SO(d)} = \mathcal{F}(\mathbb{R}^d)^{SO(d)} \cap L^p(\mathbb{R}^d, \rho)$$
$$L^p(\mathbb{R}, \gamma)^e = \mathcal{F}(\mathbb{R})^e \cap L^p(\mathbb{R}, \gamma).$$

Form [10] we know that the measure γ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ with density given by

$$\Delta(s) = \frac{\sigma(S^{d-1})}{(2\pi)^{d/2}} \, |s|^{d-1} e^{-s^2/2}. \tag{1}$$

where S^{d-1} is a unit sphere on $\mathbb{R}^d$ and σ is the surface measure on S^{d-1}.

Theorem 2. *[10] For any $d \geq 2$, the map $\Phi \colon \mathcal{F}(\mathbb{R}^d)^{SO(d)} \to \mathcal{F}(\mathbb{R})^e$ defined by*

$$\Phi(G)(s) = G(s, 0, \ldots, 0)$$

for all $G \in \mathcal{F}(\mathbb{R}^d)^{SO(d)}$ and all $s \in \mathbb{R}$, is a linear isomorphism whose inverse is given by

$$\Psi(g)(x) = g\left(\sqrt{(x, x)}\right) = g(|x|)$$

for all $g \in \mathcal{F}(\mathbb{R})^e$ and all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

Proof. See Theorem 1 in [10]. □

Theorem 3. *The spaces $L^p(\mathbb{R}^d,\rho)^{SO(d)}$ and $L^p(\mathbb{R},\gamma)^e$ are isometrically isomorphic.*

Proof. From Theorem 2 we have that the function

$$\Psi\colon \mathcal{F}(\mathbb{R})^e \to \mathcal{F}(\mathbb{R}^d)^{SO(d)}$$

is a linear isomorphism. We consider the restriction of Ψ to the space $L^p(\mathbb{R},\gamma)^e$. Let $g\in\mathcal{F}(\mathbb{R})^e$ and $G\in\mathcal{F}(\mathbb{R}^d)^{SO(d)}$ be such that $G=\Psi(g)$. Thus

$$\begin{aligned}\int_{\mathbb{R}} |g|^p d\gamma &= \frac{1}{2}\int_{\Psi_1^{-1}(\mathbb{R})} |g\circ\Psi_1(x)|^p\,\rho(x)\,dx + \frac{1}{2}\int_{\Psi_2^{-1}(\mathbb{R})} |g\circ\Psi_2(x)|^p\,\rho(x)\,dx\\ &= \int_{\mathbb{R}^d} |g(|x|)|^p\,\rho(x)\,dx\\ &= \int_{\mathbb{R}^d} |\Psi(g)\,(x)|^p\,\rho(x)\,dx\\ &= \int_{\mathbb{R}^d} |G(x)|^p\,\rho(x)\,dx.\end{aligned}$$

So $\|g\|_{L^p(\mathbb{R},\gamma)} = \|G\|_{L^p(\mathbb{R}^d,\rho)}$. Hence, $G\in L^p(\mathbb{R}^d,\rho)^{SO(d)}$ if and only if $g\in L^p(\mathbb{R},\gamma)^e$. This shows that Ψ is an isometry map from $L^p(\mathbb{R},\gamma)^e$ onto $L^p(\mathbb{R}^d,\rho)^{SO(d)}$. □

In the same way denote by $\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ the set of all $SO(d,\mathbb{C})$-invariant holomorphic functions on $\mathbb{C}^d$, and $\mathcal{H}(\mathbb{C})^e$ the set of all holomorphic even functions on $\mathbb{C}$. Therefore we have the following theorem for complex case.

Theorem 4. *For any $d\geq 2$, the map $\phi\colon \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}\to\mathcal{H}(\mathbb{C})^e$ defined by*

$$\phi(f)(\xi) = f(\xi,0,\ldots,0)$$

for any $f\in\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ and any $\xi\in\mathbb{C}$, is a linear isomorphism whose inverse is given by

$$\psi(g)(z) = g\left(\sqrt{(z,z)}\right)$$

for any $g\in\mathcal{H}(\mathbb{C})^e$ and any $z\in\mathbb{C}^d$ where $(z,z)=z_1^2+z_2^2+\cdots+z_d^2$.

Note that since g is even, the value of $\phi(g)(z)$ is independent of the choice of square root of (z,z).

In the same way denote by $\mathcal{B}(\mathbb{C}^d)$ the Borel σ-algebra in $\mathbb{C}^d$ and by $\mathcal{B}(\mathbb{C})$ the Borel σ-algebra in $\mathbb{C}$ and define $\Phi_i\colon (\mathbb{C}^d,\mathcal{B}(\mathbb{C}^d),\mu)\to(\mathbb{C},\mathcal{B}(\mathbb{C}))$, $i=1,2$ to be the branch of $\sqrt{(z,z)}$ with a smaller and larger argument respectively and for each $E \in \mathcal{B}(\mathbb{C})$ define

$$\lambda_i(E) = \mu(\Phi_i^{-1}(E)),$$

and let $\lambda = (\lambda_1 + \lambda_2)/2$.

Define

$$\mathcal{H}L^p(\mathbb{C}^d, \mu)^{SO(d,\mathbb{C})} = \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \cap L^p(\mathbb{C}^d, \mu)$$

and

$$\mathcal{H}L^p(\mathbb{C}, \lambda)^e = \mathcal{H}(\mathbb{C})^e \cap L^p(\mathbb{C}, \lambda).$$

We now have the following theorem whose proof is similar to that of Theorem 3.

Theorem 5. *The spaces $\mathcal{H}L^p(\mathbb{C}^d, \mu)^{SO(d,\mathbb{C})}$ and $\mathcal{H}L^p(\mathbb{C}, \lambda)^e$ are isometrically isomorphic.*

Theorem 6. *[9] The measure λ is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$ with density given by*

$$\Lambda(w) = \frac{|w|^{2d-2}}{(\pi)^d} \int_{S_1} e^{-|wz|^2} d\alpha(z). \tag{2}$$

Proof. See Proposition 6 in [9]. □

In [9], we established that the density Λ is equivalent to the function $\beta(w) = |w|^{d-1} \dfrac{e^{-|w|^2}}{\pi}$ for all $w \in \mathbb{C}$ bounded away from zero.

Lemma 7. *Let p be a number with $1 < p < \infty$. The norms $\|\cdot\|_{L^p(\mathbb{C},\beta)}$ and $\|\cdot\|_{L^p(\mathbb{C},\lambda)}$ are equivalent, i.e., there are constants $k, K > 0$, depending on d, such that*

$$k\|f\|_{L^p(\mathbb{C},\beta)} \leq \|f\|_{L^p(\mathbb{C},\lambda)} \leq K\|f\|_{L^p(\mathbb{C},\beta)}, \tag{3}$$

for all $f \in \mathcal{H}L^p(\mathbb{C}, \lambda)$.

Proof. First, we will show that there is a constant $D > 0$, depending on d, such that

$$\|f\|^p_{L^p(\mathbb{C},\beta)} \leq D\,\|f\|^p_{L^p(\mathbb{C}-\mathbb{D},\lambda)}$$

for any $f \in \mathcal{H}L^p(\mathbb{C}, \lambda)$, where $\mathbb{D} = \{w \in \mathbb{C} : |w| \leq 1\}$.

Let $w \in \mathbb{D}$. Denote by $A(w)$ the annulus $\{z \in \mathbb{C} \mid 2 \leq |z-w| \leq 3\}$. For any $v \in A(w)$, we use the polar coordinates with the origin at w so that $v-w = re^{i\theta}$. If $f \in \mathcal{H}L^p(\mathbb{C},\lambda)$, then we expand f as a power series around $v = w$:

$$f(v) = f(w) + \sum_{n=1}^{\infty} a_n (v-w)^n.$$

Hence,

$$\int_{A(w)} f(v)\, dv = f(w)(9\pi - 4\pi) + \sum_{n=1}^{\infty} \int_2^3 \int_0^{2\pi} a_n r^n e^{in\theta}\, d\theta dr = 5\pi f(w).$$

Therefore

$$\begin{aligned} f(w) &= \frac{1}{5\pi} \int_{A(w)} f(v)\, dv \\ &= \frac{1}{5\pi} \int_{\mathbb{C}-\mathbb{D}} \chi_{A(w)}(v) \frac{1}{\Lambda(v)} f(v) \Lambda(v)\, dv. \end{aligned}$$

By Hölder's inequality we have that

$$|f(w)| \leq \frac{1}{5\pi} \left\| \chi_{A(w)} \frac{1}{\Lambda} \right\|_{L^q(\mathbb{C}-\mathbb{D},\lambda)} \|f\|_{L^p(\mathbb{C}-\mathbb{D},\lambda)}$$

provided $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Since Λ is strictly positive and continuous on $A(w)$, $\frac{1}{\Lambda}$ is bounded on $A(w)$. Thus $\left\| \chi_{A(w)} \frac{1}{\Lambda} \right\|_{L^q(\mathbb{C}-\mathbb{D},\lambda)}$ is finite. However, for each $w \in \mathbb{D}$,

$$\frac{1}{5\pi} \left\| \chi_{A(w)} \frac{1}{\Lambda} \right\|_{L^q(\mathbb{C}-\mathbb{D},\lambda)} \leq \frac{1}{5\pi} \left\| \chi_{A^*} \frac{1}{\Lambda} \right\|_{L^q(\mathbb{C}-\mathbb{D},\lambda)} := c < \infty$$

where $A^* = \{z \in \mathbb{C} : 1 < |z| < 4\}$, which contains each $A(w)$, $w \in \mathbb{D}$. Hence for any $w \in \mathbb{D}$

$$|f(w)| \leq c \|f\|_{L^p(\mathbb{C}-\mathbb{D},\lambda)}.$$

Since the density Λ is equivalent to the function $\beta(w) = |w|^{d-1} \dfrac{e^{-|w|^2}}{\pi}$ for all $w \in \mathbb{C}$ bounded away from zero, we have that

$$\begin{aligned} \int_{\mathbb{C}} |f(w)|^p \beta(w)\, dw &= \int_{\mathbb{D}} |f(w)|^p \beta(w) dw + \int_{\mathbb{C}-\mathbb{D}} |f(w)|^p \beta(w)\, dw \\ &\leq c^2 \|f\|^p_{L^p(\mathbb{C}-\mathbb{D},\lambda)} \int_{\mathbb{D}} \beta(w) dw + \frac{1}{m} \|f\|^p_{L^p(\mathbb{C}-\mathbb{D},\lambda)} \end{aligned}$$

$$\leq D\|f\|^p_{L^p(\mathbb{C}-\mathbb{D},\lambda)}$$

for some constant $D > 0$ depending on d. This give the first inequality in (3). The second inequality in (3) can be proved in the same way. □

3. The Segal-Bargmann Transform of Rotation-Invariant L^p Functions

For any $f \in L^p(\mathbb{R}^d, \rho)$ with $1 < p < \infty$. Define the Segal-Bargmann transform $B(f)$ of f by

$$(Bf)(z) = \int_{\mathbb{R}^d} f(x)\frac{e^{-(z-x)^2/2}}{(2\pi)^{d/2}}\,dx, z \in \mathbb{C}^d.$$

By Morera's Theorem, we have that $B(f) \in \mathcal{H}(\mathbb{C}^d)$. If $f \in L^p(\mathbb{R}^d, \rho)$ and $B(f) = 0$, then $f = 0$ i.e., B is $1-1$.

Theorem 8. *A function $f \in L^p(\mathbb{R}^d, \rho)$ is $SO(d)$-invariant if and only if $B(f) \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$.*

Proof. Let $f \in L^p(\mathbb{R}^d, \rho_t)$ and $F = B(f)$.
($\Rightarrow$) Assume that f is $SO(d)$-invariant. Define a bilinear form $(\cdot,\cdot)$ on $\mathbb{F}^d$ by

$$(x, y) = x_1y_1 + x_2y_2 + \cdots + x_dy_d$$

for all $x, y \in \mathbb{F}^d$. Then the elements of $SO(d)$ and $SO(d,\mathbb{C})$ preserve the bilinear form on $\mathbb{R}^d$ and $\mathbb{C}^d$ respectively. If $A \in SO(d)$ and $z \in \mathbb{C}^d$, then

$$\begin{aligned} F(Az) &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)e^{-(Az-x)^2/2}\,dx \\ &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)e^{-(z-A^{-1}x)^2/2}\,dx \\ &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(Ax)e^{-(z-x)^2/2}\,d(Ax) \\ &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)e^{-(z-x)^2/2}\,dx \\ &= F(z). \end{aligned}$$

We use the fact that $\det(A) = 1$ in the change of variables above. Hence, F is $SO(d)$-invariant. By analytic continuation, F is $SO(d, \mathbb{C})$-invariant.
($\Leftarrow$) Assume that F is $SO(d, \mathbb{C})$-invariant. Let $A \in SO(d)$ and

$$g(x) = f(Ax) \quad \text{for all } x \in \mathbb{R}^d.$$

Then $g \in L^p(\mathbb{R}^d, \rho)$ and for any $z \in \mathbb{C}^d$

$$\begin{aligned} B(g)(z) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g(x) e^{-(z-x)^2/2}\, dx \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(Ax) e^{-(z-x)^2/2}\, dx \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-(z-A^{-1}x)^2/2}\, d(A^{-1}x) \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-(Az-x)^2/2}\, dx \\ &= F(Az) = F(z) = B(f)(z). \end{aligned}$$

Since B is 1-1, we must have $g = f$. Hence f is $SO(d)$-invariant.

□

Define the operator $\mathcal{A} \colon \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \to \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ by

$$(\mathcal{A}F)(z) = (z, z)F(z)$$

for all $z \in \mathbb{C}^d$ where $(z, z) = z_1^2 + \cdots + z_d^2$.

Lemma 9. *For any $F \in \mathcal{H}(C^d)^{SO(d,\mathbb{C})}$, $f = \phi(F)$ and any $n \in \mathbb{N}$,*

$$\|\mathcal{A}^n(F)\|_{L^p(\mathbb{C}^d,\mu)} = \|w^{2n} f\|_{L^p(\mathbb{C},\lambda)}.$$

Proof. Let $F \in \mathcal{H}(C^d)^{SO(d,\mathbb{C})}$ and $f = \phi(F)$. Then for all $w \in \mathbb{C}$, $f(w) = F(w, 0, \ldots, 0)$ and for all $n \in \mathbb{N}$

$$\begin{aligned} \|\mathcal{A}^n(F)\|^p_{L^p(\mathbb{C}^d,\mu)} &= \int_{\mathbb{C}^d} |(\mathcal{A}^n F)(z)|^p \mu(z) dz \\ &= \int_{\mathbb{C}} |\phi(\mathcal{A}^n F)(w)|^p \lambda(w) dw \\ &= \int_{\mathbb{C}} |(\mathcal{A}^n F)(w, 0, \ldots, 0)|^p \lambda(w) dw \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{C}} |w^{2n} F(w, 0, \dots, 0)|^p \lambda(w) dw \\
&= \int_{\mathbb{C}} |w^{2n} f(w)|^p \lambda(w) dw \\
&= \|w^{2n}(f)\|^p_{L^p(\mathbb{C},\lambda)},
\end{aligned}
$$

so the lemma is proved. □

Proposition 10. *Let p be a number with $1 < p < \infty$. For any $f \in \mathcal{H}(\mathbb{C})$, if $w^2 f \in L^p(\mathbb{C}, \lambda)$, then $f \in L^p(\mathbb{C}, \lambda)$. In particular, for any $n \in \mathbb{N}$ if $w^{2n} f \in \mathcal{H}L^p(\mathbb{C}, \lambda)^e$ then $f \in \mathcal{H}L^p(\mathbb{C}, \lambda)^e$.*

Proof. Let $\mathbb{D} = \{w \in \mathbb{C} \colon |w| \leq 1\}$. Then $\|f\|_{L^p(\mathbb{C}-\mathbb{D},\lambda)} \leq \|w^2 f\|_{L^p(\mathbb{C}-\mathbb{D},\lambda)} < \infty$. Next, we will show that there is a constant $C > 0$ such that

$$\|f\|_{L^p(\mathbb{C},\lambda)} \leq C\|f\|_{L^p(\mathbb{C}-\mathbb{D},\lambda)}.$$

Let $w \in \mathbb{D}$. Denote by $\mathcal{C}(w)$ the annulus $\{v \in \mathbb{C} \colon 2 \leq |v - w| \leq 3\}$. For any $v \in \mathcal{C}(w)$, we use the polar coordinates with the origin at w so that $v - w = re^{i\theta}$. Since $f \in \mathcal{H}(\mathbb{C})$, we expand f as a power series around $v = w$:

$$f(v) = f(w) + \sum_{n=1}^{\infty} a_n (v - w)^n.$$

Hence,

$$
\begin{aligned}
\int_{\mathcal{C}(w)} f(v) dv &= 5\pi f(w) + \sum_{n=1}^{\infty} \int_2^3 \int_0^{2\pi} a_n r^{n+1} e^{i\theta} d\theta dr \\
&= 5\pi f(w).
\end{aligned}
$$

Thus

$$
\begin{aligned}
f(w) &= \frac{1}{5\pi} \int_{\mathcal{C}(w)} f(v) dv \\
&= \frac{1}{5\pi} \int_{\mathbb{C}-\mathbb{D}} \chi_{\mathcal{C}(w)}(v) \frac{1}{\Lambda(v)} f(v) \Lambda(v) dv.
\end{aligned}
$$

By Hölder's inequality we have that

$$|f(w)| \leq \frac{1}{5\pi} \|\chi_{\mathcal{C}(w)} \frac{1}{\Lambda}\|_{L^q(\mathbb{C}-\mathbb{D},\lambda)} \|f\|_{L^p(\mathbb{C}-\mathbb{D},\lambda)}$$

where $\frac{1}{q}+\frac{1}{p}=1$. Since Λ is strictly positive and continuous on $\mathcal{C}(w)$, $\frac{1}{\Lambda}$ is bounded on $\mathcal{C}(w)$. Thus $\|\chi_{\mathcal{C}(w)}\frac{1}{\Lambda}\|_{L^q(\mathbb{C}-\mathbb{D},\lambda)}$ is finite and

$$\frac{1}{5\pi}\|\chi_{\mathcal{C}(w)}\frac{1}{\Lambda}\|_{L^q(\mathbb{C}-\mathbb{D},\lambda)} \leq \frac{1}{5\pi}\|\chi_{\mathcal{C}^*}\frac{1}{\Lambda}\|_{L^q(\mathbb{C}-\mathbb{D},\lambda)} < \infty$$

where $\mathcal{C}^* = \{v \in \mathbb{C}\colon 1 < |z| < 4\}$ which $\mathcal{C}(w) \subset \mathcal{C}^*$ for all $w \in \mathbb{D}$. It follows that there exists a constant c such that

$$|f(w)| \leq c\|f\|_{L^p(\mathbb{C}-\mathbb{D},\lambda)}.$$

Hence

$$\begin{aligned}\int_{\mathbb{C}} |f(w)|^p d\lambda(w) &= \int_{\mathbb{D}} |f(w)|^p d\lambda(w) + \int_{\mathbb{C}-\mathbb{D}} |f(w)|^p d\lambda(w)\\ &\leq c^p\|f\|^p_{L^p(\mathbb{C}-\mathbb{D},\lambda)}\lambda(\mathbb{D}) + \|f\|^p_{L^p(\mathbb{C}-\mathbb{D},\lambda)}\\ &\leq C\|f\|^p_{L^p(\mathbb{C}-\mathbb{D},\lambda)},\end{aligned}$$

so the proposition is proved. □

Corollary 11. *Let p be a number with $1 < p < \infty$ and $F \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$. For any $n \in \mathbb{N}$ if $\mathcal{A}^n(F) \in \mathcal{H}L^p(\mathbb{C}^d,\mu)^{SO(d,\mathbb{C})}$ then $F \in \mathcal{H}L^p(\mathbb{C}^d,\mu)^{SO(d,\mathbb{C})}$.*

Lemma 12. *Let p be a positive number with $1 < p < \infty$ and $F \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$. If there is $f \in L^p(\mathbb{R}^d,\rho)^{SO(d)}$ such that $B(f) = F$ and for all multi-indices α,β, $x^\alpha\left(\frac{\partial}{\partial x}\right)^\beta f(x) \in L^p(\mathbb{R}^d,\rho)$ then for any n there is a constant C_n such that*

$$|F(z)| \leq C_n e^{y^2/2}e^{x^2/2(p-1)}\frac{1}{\Big(1+|(z,z)|^n\Big)\Big(1+|(z,z)|^{d-1}\Big)}, \qquad x+iy = (z,z).$$

Proof. From the proof of Theorem 7 of [7] tell us that for all positive integer n_i, $z_i^{n_i}F(z)$ is the Segal-Bargmann transform of some function in $L^p(\mathbb{R}^d,\rho)$. Thus $(z,z)^{n+d-1}F(z) = (z_1^2+z_2^2+\cdots+z_d^2)^{n+d-1}F(z)$ is also the Segal-Bargman transform of a function in $L^p(\mathbb{R}^d,\rho)^{SO(d)}$. By Theorem 2 of [7] there exists a constant C_n such that

$$|(z,z)|^{n+d-1}F(z)| \leq C_n e^{y^2/2}e^{x^2/2(p-1)} \qquad x+iy = z.$$

But the bilinear form $(\cdot,\cdot)$ and F are $SO(d,\mathbb{C})$-invariant and

$$|(z,z)| = \inf\{|Az|^2\colon A \in SO(d,\mathbb{C})\}.$$

Therefore for all $z \in \mathbb{C}^d$,

$$|F(z)| \leq C_n e^{y^2/2} e^{x^2/2(p-1)} \frac{1}{\Big(1+|(z,z)|^n\Big)\Big(1+|(z,z)|^{d-1}\Big)}$$

where $x+iy=(z,z)$. □

Theorem 13. *Let p be a positive number with $1<p<\infty$ and $F \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$. If there is $f \in L^p(\mathbb{R}^d,\rho)^{SO(d)}$ such that $B(f)=F$ and for all multi-indices α,β, $\quad x^\alpha \Big(\frac{\partial}{\partial x}\Big)^\beta f(x) \in L^p(\mathbb{R}^d,\rho)$ then $\mathcal{A}^n(F) \in \mathcal{H}L^q(\mathbb{C}^d,\mu)^{SO(d,\mathbb{C})}$ for all $n \in \mathbb{N}$ when $\frac{1}{p}+\frac{1}{q}=1$, $0<q\leq 2$ and $p \geq \frac{q}{2}+1$.*

Proof. For any $n \in \mathbb{N}$, by Lemma 12 there is a constant C_n such that

$$|F(z)| \leq \frac{C_n e^{y^2/2} e^{x^2/2(p-1)}}{\Big(1+|(z,z)|^n\Big)\Big(1+|(z,z)|^{(d-1)}\Big)}, \qquad x+iy=(z,z).$$

Following Lemma 7 we have that the L^q-norm with respect to λ is equivalent to the L^q-norm with respect to the measure β. Then

$$\begin{aligned}
\|\mathcal{A}^n(F)\|^q_{L^q(\mathbb{C}^d,\mu)} &= \int_{\mathbb{C}^d} |(z,z)^n F(z)|^q \mu(z)dz \\
&\leq \int_{\mathbb{C}^d} |(z,z)^n|^q \frac{C e^{qy^2/2} e^{qx^2/2(p-1)}}{\Big(1+|(z,z)|^n\Big)^q\Big(1+|(z,z)|^{(d-1)}\Big)^q} \mu(z)dz \\
&= \int_{\mathbb{C}} |w|^{2nq} \frac{C e^{qy^2/2} e^{qx^2/2(p-1)}}{(1+|w|^{2(d-1)})^q(1+|w|^{2n})^q} \lambda(w)dw \\
&\leq \int_{\mathbb{C}} K|w|^{2nq} \frac{C e^{qy^2/2} e^{qx^2/2(p-1)}}{(1+|w|^{2(d-1)})^q(1+|w|^{2n})^q} \frac{e^{-|w|^2}|w|^{d-1}}{\pi} dw
\end{aligned}$$

Since $0<q\leq 2$ and $p \geq \frac{q}{2}+1$, $\frac{q}{2}-1 \leq 0$ and $\frac{q}{2(p-1)} \leq 0$. Thus

$$\|\mathcal{A}^n(F)\|^q_{L^q(\mathbb{C}^d,\mu)} \leq \int_{\mathbb{C}} K|w|^{2nq} \frac{C e^{qy^2/2} e^{qx^2/2(p-1)}}{(1+|w|^{2(d-1)})^q(1+|w|^{2n})^q} \frac{e^{-|w|^2}|w|^{d-1}}{\pi} dw < \infty.$$

Hence for all $n \in \mathbb{N}$, $\mathcal{A}^n(F) \in \mathcal{H}L^q(\mathbb{C}^d,\mu)^{SO(d,\mathbb{C})}$. □

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