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M_2-EDGE COLORING AND MAXIMUM MATCHING OF GRAPHS

Kristína Budajová[1], Július Czap[2 §]

[1]Department of Aerodynamics and Simulations
Faculty of Aeronautics
Technical University of Košice
Rampová 7, SK-041 21 Košice, SLOVAKIA

[2]Department of Applied Mathematics and Business Informatics
Faculty of Economics
Technical University of Košice
B. Němcovej 32, SK-040 01 Košice, SLOVAKIA

Abstract: An edge coloring φ of a graph G is called M_2-edge coloring if $|\varphi(v)| \leq 2$ for every vertex v of G, where $\varphi(v)$ is the set of colors of edges incident with v. Let $\alpha(G)$ denote the size of a maximum matching in G. Every graph G with maximum degree at least 2 has an M_2-edge coloring with at least $\alpha(G) + 1$ colors. We prove that this bound is tight even for connected planar graphs. We show that for any $n \in \mathbb{N}$ and $\delta \in \{1, 2, 3, 4, 5\}$ there is a connected planar graph G on at least n vertices with minimum degree δ such that the maximum number of colors used in an M_2-edge coloring of G is equal to $\alpha(G) + 1$.

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1. Introduction

We use the standard terminology according to Bondy and Murty [1], except

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§Correspondence author

for few notations defined throughout. However, we recall some frequently used terms.

Let $G = (V, E)$ be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. An edge coloring φ of a graph G is an assignment of colors to the edges of G. The coloring φ is proper if no two adjacent edges have the same color. Unless otherwise stated, edge colorings of graphs in this paper are not necessarily proper.

Given a simple graph G and one of its edges $e = uv$, the contraction of e consists of replacing u and v by a new vertex adjacent to all the former neighbors of u and v, and removing the loop corresponding to the edge e. In addition, if two vertices are joined with more than one edge, then we remove all of them but one.

Given two vertices $u, v \in V(G)$, let $E(uv)$ be the set of edges joining u and v in G. The multiplicity $\mu(uv)$ of an edge uv is the size of $E(uv)$. Set $\mu(v) = \max\{\mu(uv) : u \in V(G)\}$, which is called the multiplicity of a vertex v.

Let the degree of a vertex v in a graph G be denoted by $\deg_G(v)$, or by $\deg(v)$ if G is known from the context.

Let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. An f-coloring of G is an edge coloring of G such that each vertex v is incident with at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is denoted by $\chi'_f(G)$. If $f(v) = 1$ for all $v \in V(G)$, then the f-coloring problem is reduced to the proper edge coloring problem. Hakimi and Kariv [3] obtained the following result:

$$\max_{v \in V(G)} \{\lceil \deg(v)/f(v) \rceil\} \leq \chi'_f(G) \leq \max_{v \in V(G)} \{\lceil (\deg(v) + \mu(v))/f(v) \rceil\}$$

for any multigraph G. When G does not contain multiple edges we have $\mu(v) \leq 1$ for each $v \in V(G)$. Therefore for simple graphs we have the following bounds:

$$\max_{v \in V(G)} \{\lceil \deg(v)/f(v) \rceil\} \leq \chi'_f(G) \leq \max_{v \in V(G)} \{\lceil \deg(v)/f(v) \rceil\} + 1.$$

If the requirement "the number of edges of the same color incident with a vertex v is at most $f(v)$" is replaced by "the number of colors of edges incident with a vertex v is at most $f(v)$", then we obtain a definition of a generalized M_i-edge coloring.

An edge coloring of a graph G is M_i-edge coloring if at most i colors appear at any vertex of G. The problem is to determine the maximum number of colors $K_i(G)$ used in an M_i-edge coloring of G. Clearly, $K_i(G) \leq |E(G)|$ for every i. Moreover, this bound is tight for graphs with maximum degree at most i. M_2-edge colorings of graphs with maximum degree 3 were studied in [2].

2. Results

A splitting of a vertex v of degree 2 is defined in the following way. Assume that v is incident with vertices v_1 and v_2. We replace the vertex v by a pair of vertices v_x, v_y and the edges v_1v, vv_2 by edges v_1v_x, v_yv_2.

Theorem 1. *[2] Let G be a graph on n vertices with maximum degree at most 3. Let m denote the number of vertices of degree 2 in G. Let G' be a graph obtained from G by splitting all vertices of degree 2. Let t denote the maximum number of disjoint cycles in G'. Then $K_2(G) = \frac{n+m}{2} + t$.*

A matching in a graph is a set of pairwise nonadjacent edges. A maximum matching is a matching that contains the largest possible number of edges. The number of edges in a maximum matching of a graph G is denoted by $\alpha(G)$.

A vertex cover of a graph $G = (V, E)$ is a subset K of V such that every edge of G is incident with a vertex in K. A minimum vertex cover is a vertex cover that contains the smallest possible number of vertices. The number of vertices in a minimum vertex cover of G is denoted by $\beta(G)$.

Kőnig in 1931 describes an equivalence between the maximum matching problem and the minimum vertex cover problem in bipartite graphs.

Theorem 2. *[4] Let G be a bipartite graph. Then the size of a maximum matching in G is equal to the size of a minimum vertex cover of G, i.e.*

$$\alpha(G) = \beta(G).$$

Theorem 3. *Let G be a graph with maximum degree at least two. Then $K_2(G) \geq \alpha(G) + 1$. Moreover, this bound is tight even for connected planar graphs.*

Proof. Color the edges in a maximum matching with different colors and the other edges of G with the same color. In this way we obtain an M_2-edge coloring of G which uses $\alpha(G) + 1$ colors.

For the sharpness of the lower bound see Theorems 4 and 5. □

In the rest of the paper we show that for any $n \in \mathbb{N}$ and $\delta \in \{1, 2, 3, 4, 5\}$ there is a connected planar graph G on at least n vertices with minimum degree δ such that the maximum number of colors used in an M_2-edge coloring of G is equal to $\alpha(G) + 1$.

Theorem 4. *For any $\delta \in \{1, 2\}$ and $n \in \mathbb{N}$ there is a connected planar graph G on at least n vertices with minimum degree δ such that $K_2(G) = \alpha(G) + 1$.*

Proof. Let n be a fixed positive integer.

Let G be a graph with vertex set $V(G) = \{v, v_0, v_1, \ldots, v_{2n}\}$ and edge set $E(G) = \{vv_i : i = 0, \ldots, n\} \cup \{v_i v_{i+n} : i = 1, \ldots, n\}$. Clearly, the minimum degree of G is 1. Any matching in G has at most $n + 1$ edges, since this graph has $2n + 2$ vertices. The edges in $\{v_i v_{i+n} : i = 1, \ldots, n\} \cup \{vv_0\}$ are pairwise nonadjacent, hence $\alpha(G) = n + 1$. On the other hand $K_2(G) = n + 2$, because on the edges incident with the vertex v at most 2 colors appear and on the other n edges at most n different colors appear.

Let H be a graph with vertex set $V(H) = \{v, v_0, v_1, \ldots, v_{2n}\}$ and edge set $E(H) = \{vv_i : i = 0, \ldots, 2n\} \cup \{v_i v_{i+1} : i = 1, 3, \ldots, 2n - 1\} \cup \{v_{2n} v_0\}$. The minimum degree of H is 2. The edges in $\{v_i v_{i+1} : i = 1, 3, \ldots, 2n-1\} \cup \{vv_0\}$ are pairwise nonadjacent, therefore $\alpha(H) = n + 1$. Finally, any M_2-edge coloring of H uses at most $n + 2$ colors, because on the edges incident with the vertex v at most 2 colors appear and on the other $n + 1$ edges at most n different colors appear (if all of these $n + 1$ edges have different colors and on the edges incident with v two other colors appear, then the three edges incident with v_{2n} are colored distinctly). □

Theorem 5. *For any $\delta \in \{3, 4, 5\}$ and $n \in \mathbb{N}$ there is a simple planar triangulation G on at least n vertices with minimum degree δ such that $K_2(G) = \alpha(G) + 1$.*

Proof. Let T be a simple plane triangulation on $n \geq 4$ vertices. Let G_i be a graph obtained from T by inserting a configuration H_i, shown in Figure 1, into each of its faces, for $i = 3, 4, 5$.

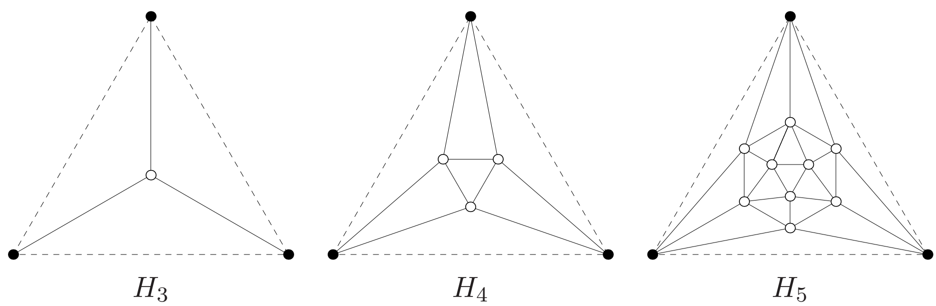

Figure 1: A construction of triangulations with minimum degree 3, 4 and 5.

We say that a vertex v of G_i is black if $v \in V(T)$, otherwise it is white. Clearly, every G_i contains n black vertices. An edge uv of G_i is monochromatic if the vertices u and v have the same color, otherwise it is bichromatic.

Observe that the minimum degree of G_i is i, for $i = 3, 4, 5$. In the following we show that $K_2(G_i) = \alpha(G_i) + 1$.

First we prove that $\alpha(G_3) = n$. Let G'_3 be a graph obtained from G_3 by removing all monochromatic edges. Clearly, the set of all black vertices B is a vertex cover of G'_3, moreover each edge is covered by one vertex. Hence, $|E(G'_3)| = \sum_{v \in B} \deg_{G'_3}(v)$. Assume that K is a vertex cover of G'_3 such that $|K| < |B|$. Observe that every black vertex in G'_3 has degree at least 3 and the white vertices are of degree 3. Since B contains more vertices than K it holds $\sum_{v \in B} \deg_{G'_3}(v) > \sum_{v \in K} \deg_{G'_3}(v)$. As $|E(G'_3)| > \sum_{v \in K} \deg_{G'_3}(v)$ the set K cannot be a vertex cover of G'_3. Consequently, B is a minimum vertex cover of the bipartite graph G'_3. Theorem 2 implies that G'_3 has a matching of size $|B|$. Since G_3 is a supergraph of G'_3, it also has a matching of size at least $|B| = n$.

On the other hand any matching M in G_3 has at most n edges, because every edge in M has an endvertex of color black. Therefore $\alpha(G_3) = n$.

Next we prove that $\alpha(G_4) = 3n - 4$. The white vertices of G_4 induce vertex disjoint cycles of length 3. The graph G_4 contains $2n - 4$ such cycles, because every plane triangulation on n vertices has $2n-4$ faces. Therefore, any matching of G_4 has at most $2n - 4$ edges with both endvertices of color white. The number of edges incident with a black vertex is bounded from above by the number of black vertices in any matching of G_4. These observations imply that $\alpha(G_4) \leq 3n - 4$.

Now we contract all edges of G_4 with both endvertices of color white. In this way we obtain the graph G_3. As it was proved, G_3 has a matching of size n such that each edge in this one is bichromatic. Therefore, G_4 also contains such a matching M. Let $V(M)$ denote the set of endvertices of the edges in M. The graph $G_4 - V(M)$ contains $2n - 4$ pairwise nonadjacent edges with both endvertices of color white, since from each cycle of length 3 induced by 3 white vertices at most one vertex belongs to $V(M)$. Consequently, M can be extended to a matching of size $3n - 4$ in G_4.

Finally we prove that $\alpha(G_5) = 9n - 16$. The white vertices of G_5 induce $2n - 4$ vertex disjoint subgraphs shown in Figure 2.

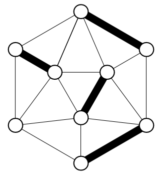

Figure 2: A configuration induced by the white vertices in G_5.

Since these subgraphs have 9 vertices any matching in G_5 has at most $4(2n-4)$ edges with both endvertices of color white. The number of edges incident with a black vertex is at most n in any matching of G_5. Hence, $\alpha(G_5) \leq 9n-16$.

If we contract all edges of G_5 with both endvertices of color white, then we obtain the graph G_3. Hence, similarly as above, let M be a matching of size n in G_5 consisting of bichromatic edges. From each configuration induced by the white vertices in G_5 at most one vertex is in $V(M)$. Therefore, we can choose 4 edges from each maximal subgraph of $G_5 \setminus V(M)$ such that no two of them are adjacent (see Figure 2). The graph $G_5 \setminus V(M)$ has $2n-4$ such subgraphs, hence the matching M can be extended to the matching of size $n+4(2n-4) = 9n-16$.

In the second part of the proof we show that $K_2(G_3) = n+1$, $K_2(G_4) = 3n-3$ and $K_2(G_5) = 9n-15$. By Theorem 3 it is sufficient to show that $K_2(G_3) \leq n+1$, $K_2(G_4) \leq 3n-3$ and $K_2(G_5) \leq 9n-15$.

Let φ_i be an M_2-edge coloring of G_i which uses $K_2(G_i)$ colors, for $i = 3, 4, 5$. For a set of vertices $W \subseteq V(G_i)$, let $\varphi_i(W)$ denote the set of colors which appear on the edges incident with vertices in W under the coloring φ_i.

The graph T is connected, therefore we can number the black vertices $v_1, v_2, \ldots, v_n$ of G_i such that for every $j \geq 2$ the vertex v_j is adjacent to a vertex v_k for some $k < j$.

Now we show, by induction, that $|\varphi_i(v_1, \ldots, v_m)| \leq m+1$ for any $m \in \{1, \ldots, n\}$. Clearly, $\varphi_i(v_1)$ has at most two elements. Assume that $|\varphi_i(v_1, \ldots, v_{m-1})| \leq m$. The vertex v_m is adjacent to a vertex v_k for some $k < m$. There is at most one color which appears on an edge incident with v_m and does not appear in $\varphi_i(v_1, \ldots, v_{m-1})$, since at most two colors appear on the edges incident with v_m and the color of the edge $v_m v_k$ is already in $\varphi_i(v_1, \ldots, v_{m-1})$. Hence, $|\varphi_i(v_1, \ldots, v_m)| \leq m+1$.

If $i = 3$, then every edge of $G_i = G_3$ is incident with a black vertex. Therefore, $K_2(G_3) = |\varphi_3(v_1, \ldots, v_n)| \leq n+1$.

If $i = 4$, then the edges which are incident with no black vertex induce $2n-4$ vertex disjoint cycles of length 3. Let C be one of them. If on the edges of C there are at least two colors which do not belong to $\varphi_4(v_1, \ldots, v_n)$, then at least one vertex of C is incident with 3 colors, which is not possible. Therefore, on these cycles of length 3 at most $2n-4$ colors appear which are not in $\varphi_4(v_1, \ldots, v_n)$. Consequently, φ_4 uses at most $|\varphi_4(v_1, \ldots, v_n)| + 2n-4 \leq 3n-3$ colors.

If $i = 5$, then the edges which are not incident with any black vertex induce $2n-4$ disjoint copies of the graph shown in Figure 2. Let C be one of them. We prove that at most 4 colors different from $\varphi_5(v_1, \ldots, v_n)$ appear on the edges of C under the coloring φ_5. Assume, to the contrary, that there are at least 5 such colors, say $c_1, \ldots, c_5$. Since C has 9 vertices it contains two adjacent edges which have different colors, say c_1 and c_2. Then these colors appear on the edges incident with a vertex of degree 5 in C, say v, since every vertex of degree 3 and 4 in C is incident with an edge of color from $\varphi_5(v_1, \ldots, v_n)$ in G_5. Clearly, every edge incident with v has color c_1 or c_2. Let v_1, v_2, v_3 be the vertices of C of degree 3 or 4 which are adjacent to v. Observe that the colors c_3, c_4, c_5 do not appear on the edges of C which are incident with v_1, v_2, v_3, since each such vertex is incident with an edge of color from $\{c_1, c_2\}$ and with an other edge of color from $\varphi_5(v_1, \ldots, v_n)$. Therefore, the colors c_3, c_4, c_5 appear on the remaining 7 edges of C. Since these edges are incident with only 5 vertices, C contains two adjacent edges from these 7 ones which have different colors, say c_3 and c_4. By the same argument as above, these two edges must be incident with a vertex of degree 5 in C, say w. The vertices v and w are adjacent in C, therefore the color of the edge vw appears in $\varphi_5(v) = \{c_1, c_2\}$ and also in $\varphi_5(w) = \{c_3, c_4\}$, a contradiction. Consequently, φ_5 uses at most $|\varphi_5(v_1, \ldots, v_n)| + 4(2n-4) \leq 9n - 15$ colors. $\square$

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