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PAIRS OF INTERVAL NEGATIONS AND INTERVAL IMPLICATIONS

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Abstract: In this paper, we investigate the properties of pairs of interval negations and interval implications on a bounded lattice. In particular, we study the relations between pairs of negations and implications and pairs of interval negations and interval implications.

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1. Introduction

Bedregal and Takahashi [4] introduced interval fuzzy connectives as an extension for fuzzy connectives. This concept provides tools for approximate reasoning and decision making with a frame work to deal with uncertainty and incompleteness of information [1-3]. Georgescu and Popescue [5-7] introduced pseudo t-norms and generalized residuated lattices in a sense as non-commutative property. Generally, it induces two implications and two negations.

In this paper, we investigate the properties of pairs of interval negations and interval implications on a bounded lattice. For examples, we consider a pair of two implications defined by $a \Rightarrow b = \bigvee\{c \mid a \odot c \leq b\}$ and $a \rightarrow b = \bigvee\{c \mid$

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$c \odot a \leq b\}$. Moreover, we consider a pair of two negations defined by $a \Rightarrow 0$ and $a \rightarrow 0$. In particular, we study the relations between pairs of negations and implications and pairs of interval negations and interval implications.

2. Preliminaries

In this paper, we assume that $(L, \vee, \wedge, \perp, \top)$ is a bounded lattice with a bottom element $\perp$ and a top element $\top$. Moreover, we define the following definitions in a sense as non-commutative [5-7] and interval property [1-4].

Definition 1. A pair (n_1, n_2) with maps $n_i : L \rightarrow L$ is called a *pair of negations* if it satisfies the following conditions:
(N1) $n_i(\top) = \perp, n_i(\perp) = \top$ for all $i \in \{1, 2\}$.
(N2) $n_i(x) \geq n_i(y)$ for $x \leq y$ and $i \in \{1, 2\}$.
(N3) $n_1(n_2(x)) = n_2(n_1(x)) = x$ for all $x \in X$.

Definition 2. A pair (I_1, I_2) with maps $I_1, I_2 : L \times L \rightarrow L$ is called a *pair of implications* if it satisfies the following conditions:
(I1) $I_i(\top, \top) = I_i(\perp, \top) = I_i(\perp, \perp) = \top, I_i(\top, \perp) = \perp$ for all $i \in \{1, 2\}$.
(I2) If $x \leq y$, then $I_i(x, z) \geq I_i(y, z)$ for all $i \in \{1, 2\}$.
(I3) $I_i(\top, x) = x$ for all $x \in L$ and $i \in \{1, 2\}$.
(I4) $I_1(x, I_2(y, z)) = I_2(y, I_1(x, z))$ for all $x, y, z \in X$.
(I5) $I_1(I_2(x, \perp), \perp) = I_2(I_1(x, \perp), \perp) = x$.

Remark 3. Let $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, \top, \perp)$ be a complete generalized residuated lattice with the law of double negation defined as $a = n_1(n_2(a)) = n_2(n_1(a))$ where $n_1(x) = x \Rightarrow \perp$ and $n_2(x) = x \rightarrow \perp$(ref. [6,7]).
(1) A pair (n_1, n_2) is a pair of negations.
(2) A pair $(\Rightarrow, \rightarrow)$ is a pair of implications because $a \rightarrow (b \Rightarrow c) = b \Rightarrow (a \rightarrow c)$.

Let $L^{[2]} = \{[x_1, x_2] \mid x_1 \leq x_2, x_1, x_2 \in L\}$ where $[x_1, x_2] = \{x \in L \mid x_1 \leq x \leq x_2\}$. We define

$$[x_1, x_2] \leq [y_1, y_2], \text{ iff } x_1 \leq y_1,\ x_2 \leq y_2$$

$$[x_1, x_2] \subset [y_1, y_2], \text{ iff } y_1 \leq x_1 \leq x_2 \leq y_2$$

$$l([x_1, x_2]) = x_1, \quad r([x_1, x_2]) = x_2.$$

Definition 4. A pair $(\mathcal{N}_1, \mathcal{N}_2)$ with maps $\mathcal{N}_i : L^{[2]} \rightarrow L^{[2]}$ is called a *pair of interval negations* if it satisfies the following conditions:

(IN1) $\mathcal{N}_i([\top,\top]) = [\bot,\bot]$, $\mathcal{N}_i([\bot,\bot]) = [\top,\top]$ for all $i \in \{1,2\}$.
(IN2) If $[x_1,x_2] \le [y_1,y_2]$, then $\mathcal{N}_i([y_1,y_2]) \le \mathcal{N}_i([x_1,x_2])$ for all $i \in \{1,2\}$.
(IN3) If $[x_1,x_2] \subset [y_1,y_2]$, then $\mathcal{N}_i([x_1,x_2]) \subset \mathcal{N}_i([y_1,y_2])$ for all $i \in \{1,2\}$.
(IN4) $\mathcal{N}_1(\mathcal{N}_2([x_1,x_2])) = \mathcal{N}_2(\mathcal{N}_1([x_1,x_2])) = [x_1,x_2]$ for all $[x_1,x_2] \in L^{[2]}$.

Definition 5. A pair $(\mathcal{I}_1,\mathcal{I}_2)$ with maps $\mathcal{I}_1,\mathcal{I}_2 : L^{[2]} \times L^{[2]} \to L^{[2]}$ is called a *pair of interval implications* if it satisfies the following conditions:

(II1) $\mathcal{I}_i([\top,\top],[\top,\top]) = \mathcal{I}_i([\bot,\bot],[\top,\top]) = \mathcal{I}_i([\bot,\bot],[\bot,\bot]) = [\top,\top]$, $\mathcal{I}_i([\top,\top],[\bot,\bot]) = [\bot,\bot]$ for all $i \in \{1,2\}$.

(II2) If $[x_1,x_2] \le [y_1,y_2]$, then $\mathcal{I}_i([x_1,x_2],[z_1,z_2]) \ge \mathcal{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

(II3) If $[x_1,x_2] \subset [y_1,y_2]$, then $\mathcal{I}_i([x_1,x_2],[z_1,z_2]) \subset \mathcal{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

(II4) $\mathcal{I}_i([\top,\top],[x_1,x_2]) = [x_1,x_2]$ for all $i \in \{1,2\}$.

(II5) $\mathcal{I}_1([x_1,x_2],\mathcal{I}_2([y_1,y_2],[z_1,z_2])) = \mathcal{I}_2([y_1,y_2],\mathcal{I}_1([x_1,x_2],[z_1,z_2]))$ for all $[x_1,x_2],[y_1,y_2],[z_1,z_2] \in L^{[2]}$.

(II6) $\mathcal{I}_1(\mathcal{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{I}_2(\mathcal{I}_1([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = [x_1,x_2]$.

3. Pairs of Interval Negations and Interval Implications

Theorem 6. *Let $\mathcal{N}_i : L^{[2]} \to L^{[2]}$ be a pair of interval negations. Then we have the following properties.*

(1) *Define maps $\underline{\mathcal{N}_i},\overline{\mathcal{N}_i} : L \to L$ as*

$$\underline{\mathcal{N}_i}(x) = l(\mathcal{N}_i([x,x])),\ \ \overline{\mathcal{N}_i}(x) = r(\mathcal{N}_i([x,x])).$$

Then $\mathcal{N}_i([x_1,x_2]) = [\underline{\mathcal{N}_i}(x_2),\overline{\mathcal{N}_i}(x_1)]$.

(2) *$(\underline{\mathcal{N}_1},\underline{\mathcal{N}_2})$ is a pair of negations such that*

$$\underline{\mathcal{N}_1} = \overline{\mathcal{N}_1},\ \ \underline{\mathcal{N}_2} = \overline{\mathcal{N}_2}.$$

(3) *We define maps $\mathcal{I}_i : L^{[2]} \times L^{[2]} \to L^{[2]}$ as*

$$\mathcal{I}_1([x_1,x_2],[y_1,y_2]) = \mathcal{N}_1([x_1,x_2]) \vee [y_1,y_2],$$

$$\mathcal{I}_2([x_1,x_2],[y_1,y_2]) = \mathcal{N}_2([x_1,x_2]) \vee [y_1,y_2].$$

Then $(\mathcal{I}_1,\mathcal{I}_2)$ is a pair of interval implications.

Proof. (1) Since $[x_1,x_2] \le [x_2,x_2]$, then $\mathcal{N}_i([x_2,x_2]) \le \mathcal{N}_i([x_1,x_2])$. So, $l(\mathcal{N}_i([x_2,x_2])) \le l(\mathcal{N}_i([x_1,x_2]))$. Since $[x_2,x_2] \subset [x_1,x_2]$, then $\mathcal{N}_i([x_2,x_2]) \subset \mathcal{N}_i([x_1,x_2])$. So, $l(\mathcal{N}_i([x_1,x_2])) \le l(\mathcal{N}_i([x_2,x_2]))$. Hence

$$l(\mathcal{N}_i([x_1,x_2])) = l(\mathcal{N}_i([x_2,x_2])) = \underline{\mathcal{N}_i}(x_2).$$

Since $[x_1,x_2] \ge [x_1,x_1]$, then $\mathcal{N}_i([x_1,x_1]) \ge \mathcal{N}_i([x_1,x_2])$. So, $r(\mathcal{N}_i([x_1,x_1])) \ge r(\mathcal{N}_i([x_1,x_2]))$. Since $[x_1,x_1] \subset [x_1,x_2]$, then $\mathcal{N}_i([x_1,x_1]) \subset \mathcal{N}_i([x_1,x_2])$. So, $r(\mathcal{N}_i([x_1,x_1])) \le r(\mathcal{N}_i([x_1,x_2]))$. Hence

$$r(\mathcal{N}_i([x_1,x_2])) = r(\mathcal{N}_i([x_1,x_1])) = \overline{\mathcal{N}_i}(x_1).$$

Moreover, we have:

$$\begin{aligned}\mathcal{N}_i([x_1,x_2]) &= [l(\mathcal{N}_i([x_1,x_2])), r(\mathcal{N}_i([x_1,x_2]))]\\ &= [l(\mathcal{N}_i([x_2,x_2])), r(\mathcal{N}_i([x_1,x_1]))]\\ &= [\underline{\mathcal{N}_i}(x_2), \overline{\mathcal{N}_i}(x_1)].\end{aligned}$$

(2) (N1) $\underline{\mathcal{N}_i}(\bot) = l(\mathcal{N}_i([\bot,\bot])) = l([\top,\top]) = \top$ and $\underline{\mathcal{N}_i}(\top) = l(\mathcal{N}_i([\top,\top])) = l([\bot,\bot]) = \bot$.

(N2) If $x \le y$, then $\mathcal{N}_i([x,x]) \ge \mathcal{N}_i([y,y])$ for all $i \in \{1,2\}$. Thus $\underline{\mathcal{N}_i}(x) \ge \underline{\mathcal{N}_i}(y)$.

(N3) Suppose $\mathcal{N}_1([x,x]) = [y,z]$ such that $y < z$. Then $\mathcal{N}_2(\mathcal{N}_1([x,x])) = \mathcal{N}_2([y,z]) = [x,x]$. Since $[z,z] \subset [y,z]$, then $\mathcal{N}_2([y,z]) = \mathcal{N}_2([z,z]) = [x,x]$. Hence $\mathcal{N}_2$ is not injective. It is a contradiction. Hence $l(\mathcal{N}_1([x,x])) = r(\mathcal{N}_1([x,x])$. Similarly, $l(\mathcal{N}_2([x,x])) = r(\mathcal{N}_2([x,x])$.

$$\begin{aligned}\underline{\mathcal{N}_1}(\underline{\mathcal{N}_2}(x)) &= \underline{\mathcal{N}_1}(l\mathcal{N}_2([x,x]))\\ &= l(\mathcal{N}_1([l(\mathcal{N}_2([x,x])), l(\mathcal{N}_2([x,x]))]))\\ &= l(\mathcal{N}_1([l(\mathcal{N}_2([x,x])), r(\mathcal{N}_2([x,x]))]))\\ &= l(\mathcal{N}_1(\mathcal{N}_2([x,x]))) = x.\end{aligned}$$

$$\begin{aligned}\mathcal{N}_1(\mathcal{N}_2([x,x])) &= \mathcal{N}_1([\underline{\mathcal{N}_2}(x), \overline{\mathcal{N}_2}(x)])\\ &= [\underline{\mathcal{N}_1}(\overline{\mathcal{N}_2}(x)), \overline{\mathcal{N}_1}(\underline{\mathcal{N}_2}(x))])\\ &= [x,x]\end{aligned}$$

So, $\underline{\mathcal{N}_1}(\overline{\mathcal{N}_2}(x)) = x$. Thus $\overline{\mathcal{N}_2}(x) = \underline{\mathcal{N}_2}(\underline{\mathcal{N}_1}(\overline{\mathcal{N}_2}(x))) = \underline{\mathcal{N}_2}(x)$. Hence $\overline{\mathcal{N}_2}(x) = \underline{\mathcal{N}_2}(x)$. Similarly, $\underline{\mathcal{N}_2}(\underline{\mathcal{N}_1}(x)) = x$ and $\overline{\mathcal{N}_1}(x) = \underline{\mathcal{N}_1}(x)$. Thus $(\underline{\mathcal{N}_1}, \underline{\mathcal{N}_2})$ is a pair of negations.

(3) (II1)

$$\mathcal{I}_i([\top,\top],[\bot,\bot]) = \mathcal{N}_i([\top,\top]) \vee [\bot,\bot)] = [\bot,\bot],$$
$$\mathcal{I}_i([\bot,\bot],[\top,\top]) = \mathcal{I}_i([\bot,\bot],[\bot,\bot]) = [\top,\top] = \mathcal{I}_i([\top,\top],[\top,\top]).$$

(II2) If $[x_1,x_2]\le[y_1,y_2]$, then $\mathcal{N}_i([x_1,x_2])\ge\mathcal{N}_i([y_1,y_2])$.

$$\mathcal{I}_1([x_1,x_2],[z_1,z_2])=\mathcal{N}_i([x_1,x_2])\vee[z_1,z_2]\ge\mathcal{N}_i([y_1,y_2])\vee[z_1,z_2]$$
$$=\mathcal{I}_1([y_1,y_2],[z_1,z_2]).$$

(II3) If $[x_1,x_2]\subset[y_1,y_2]$, then $\mathcal{N}_i([x_1,x_2])\subset\mathcal{N}_i([y_1,y_2])$.

$$\mathcal{I}_1([x_1,x_2],[z_1,z_2])=\mathcal{N}_i([x_1,x_2])\vee[z_1,z_2]\subset\mathcal{N}_i([y_1,y_2])\vee[z_1,z_2]$$
$$=\mathcal{I}_1([y_1,y_2],[z_1,z_2]).$$

(II4)

$$\mathcal{I}_i([\top,\top],[z_1,z_2])=\mathcal{N}_i([\top,\top])\vee[z_1,z_2]=[z_1,z_2].$$

(II5)

$$\mathcal{I}_1([x_1,x_2],\mathcal{I}_2([y_1,y_2],[z_1,z_2]))\qquad=\mathcal{N}_1([x_1,x_2]\vee\mathcal{N}_2([y_1,y_2])\vee[z_1,z_2]$$
$$=\mathcal{I}_2([y_1,y_2],\mathcal{I}_1([x_1,x_2],[z_1,z_2])).$$

(II6)

$$\mathcal{I}_1(\mathcal{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot])$$
$$=\mathcal{N}_1(\mathcal{N}_2([x_1,x_2]))=[x_1,x_2].$$

□

Example 7. Put $L=\{(x,y)\in R^2 \mid (\frac{1}{2},1)\le(x,y)\le(1,0)\}$ with a bottom element $(\frac{1}{2},1)$ and a top element $(1,0)$ where

$$(x_1,y_1)\le(x_2,y_2)\Leftrightarrow x_1<x_2 \text{ or } x_1=x_2, y_1\le y_2.$$

(1) Define $(\mathcal{N}_1,\mathcal{N}_2)$ is defined $\mathcal{N}_i:L^{[2]}\to L^{[2]}$ as

$$\begin{aligned}\mathcal{N}_1([(x_1,y_1),(x_2,y_2)]) &= [(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})],\\ \mathcal{N}_2([(x_1,y_1),(x_2,y_2)]) &= [(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})].\end{aligned}$$

(IN1) $\mathcal{N}_1([(1,0),(1,0)])=[(\frac{1}{2},1),(\frac{1}{2},1)]$ and $\mathcal{N}_1([(\frac{1}{2},1),(\frac{1}{2},1)])=[(1,0),(1,0)]$.

(IN2) If $[(x_1,y_1),(x_2,y_2)]\le[(z_1,w_1),(z_2,w_2)]$, then $(\frac{1}{2x_i},\frac{1-y_i}{x_i})\ge(\frac{1}{2z_i},\frac{1-w_i}{z_i})$ and $(\frac{1}{2x_i},1-\frac{y_i}{2x_i})\ge(\frac{1}{2z_i},1-\frac{w_i}{2z_i})$ for $i=1,2$. Hence $\mathcal{N}_i([(z_1,w_1),(z_2,w_2)])\le\mathcal{N}_i([(x_1,y_1),(x_2,y_2)])$.

(IN3) If $[(x_1,y_1),(x_2,y_2)]\subset[(z_1,w_1),(z_2,w_2)]$, then $(z_1,w_1)\le(x_1,y_1)\le(x_2,y_2)\le(z_2,w_2)$. So,

$$(\frac{1}{2z_1},\frac{1-w_1}{z_1})\ge(\frac{1}{2x_1},\frac{1-y_1}{x_1})\ge(\frac{1}{2x_2},\frac{1-y_2}{x_2})\ge(\frac{1}{2z_2},\frac{1-w_2}{z_2}),$$

$$(\frac{1}{2z_1},1-\frac{w_1}{2z_1}) \geq (\frac{1}{2x_1},1-\frac{y_1}{2x_1}) \geq (\frac{1}{2x_2},1-\frac{y_2}{2x_2}) \geq (\frac{1}{2z_2},1-\frac{w_2}{2z_2}).$$

Hence $\mathcal{N}_i([(x_1,y_1),(x_2,y_2)]) \subset \mathcal{N}_i([(z_1,w_1),(z_2,w_2)])$.

(IN4) $\mathcal{N}_1(\mathcal{N}_2([(x_1,y_1),(x_2,y_2)])) = [(x_1,y_1),(x_2,y_2)]$ and $\mathcal{N}_2(\mathcal{N}_1([(x_1,y_1),(x_2,y_2)])) = [(x_1,y_1),(x_2,y_2)]$.

Then $(\mathcal{N}_1,\mathcal{N}_2)$ is a pair of interval negations.

By Theorem 6(2), we obtain a pair of negations $(\underline{\mathcal{N}_1},\underline{\mathcal{N}_2})$ as

$$\begin{aligned}
\underline{\underline{\mathcal{N}_1}}(x,y) &= l(\mathcal{N}_1([(x,y),(x,y)])) = l([(\tfrac{1}{2x},\tfrac{1-y}{x}),(\tfrac{1}{2x},\tfrac{1-y}{x})]),\\
\overline{\overline{\mathcal{N}_1}}(x,y) &= r(\mathcal{N}_1([(x,y),(x,y)])) = r([(\tfrac{1}{2x},\tfrac{1-y}{x}),(\tfrac{1}{2x},\tfrac{1-y}{x})]),\\
\underline{\mathcal{N}_1}(x,y) &= \overline{\mathcal{N}_1}(x,y) = (\tfrac{1}{2x},\tfrac{1-y}{x})\\
\underline{\underline{\mathcal{N}_2}}(x,y) &= l(\mathcal{N}_1([(x,y),(x,y)])) = l([(\tfrac{1}{2x},1-\tfrac{y}{2x}),(\tfrac{1}{2x},1-\tfrac{y}{2x})]),\\
\overline{\overline{\mathcal{N}_2}}(x,y) &= r(\mathcal{N}_1([(x,y),(x,y)])) = r([(\tfrac{1}{2x},1-\tfrac{y}{2x}),(\tfrac{1}{2x},1-\tfrac{y}{2x})]),\\
\underline{\mathcal{N}_2}(x,y) &= \overline{\mathcal{N}_2}(x,y) = (\tfrac{1}{2x},1-\tfrac{y}{2x}).
\end{aligned}$$

By Theorem 6(1),

$$\begin{aligned}
\mathcal{N}_1([(x_1,y_1),(x_2,y_2)]) &= [\underline{\mathcal{N}_1}(x_2,y_2),\overline{\mathcal{N}_1}(x_1,y_1)],\\
\mathcal{N}_2([(x_1,y_1),(x_2,y_2)]) &= [\underline{\mathcal{N}_2}(x_2,y_2),\overline{\mathcal{N}_2}(x_1,y_1)].
\end{aligned}$$

From Theorem 6(3), we can obtain a pair of interval implications $(\mathcal{I}_1,\mathcal{I}_2)$ defined as $\mathcal{I}_i : L^{[2]} \times L^{[2]} \to L^{[2]}$

$$\begin{aligned}
&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&= \mathcal{N}_1([(x_1,y_1),(x_2,y_2)]) \vee [(z_1,w_1),(z_2,w_2)]\\
&= [(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}) \vee (z_1,w_1),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1}) \vee (z_2,w_2)],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&= \mathcal{N}_2([(x_1,y_1),(x_2,y_2)]) \vee [(z_1,w_1),(z_2,w_2)]\\
&= [(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}) \vee (z_1,w_1),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1}) \vee (z_2,w_2)].
\end{aligned}$$

(2) Define $\mathcal{N}_i : L^{[2]} \to L^{[2]}$ as

$$\begin{aligned}
\mathcal{N}_1([(x_1,y_1),(x_2,y_2)]) &= [(\tfrac{1}{2x_2},\tfrac{1-y_2^2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1^2}{x_1})],\\
\mathcal{N}_2([(x_1,y_1),(x_2,y_2)]) &= [(\tfrac{1}{2x_2},\sqrt{1-\tfrac{y_2}{2x_2}}),(\tfrac{1}{2x_1},\sqrt{1-\tfrac{y_1}{2x_1}})].
\end{aligned}$$

By a similar method as (1), $(\mathcal{N}_1,\mathcal{N}_2)$ is a pair of interval negations.

By Theorem 6(2), we obtain a pair of negations $(\underline{\mathcal{N}_1}, \underline{\mathcal{N}_2})$ as

$$\begin{aligned}
\underline{\mathcal{N}_1}(x,y) &= l(\mathcal{N}_1([(x,y),(x,y)])) = l([(\tfrac{1}{2x}, \tfrac{1-y^2}{x}), (\tfrac{1}{2x}, \tfrac{1-y^2}{x})]),\\
\overline{\mathcal{N}_1}(x,y) &= r(\mathcal{N}_1([(x,y),(x,y)])) = r([(\tfrac{1}{2x}, \tfrac{1-y^2}{x}), (\tfrac{1}{2x}, \tfrac{1-y^2}{x})]),\\
\underline{\mathcal{N}_1}(x,y) &= \overline{\mathcal{N}_1}(x,y) = (\tfrac{1}{2x}, \tfrac{1-y^2}{x})\\
\underline{\mathcal{N}_2}(x,y) &= l(\mathcal{N}_1([(x,y),(x,y)])) = l([(\tfrac{1}{2x}, \sqrt{1-\tfrac{y}{2x}}), (\tfrac{1}{2x}, \sqrt{1-\tfrac{y}{2x}})]),\\
\overline{\mathcal{N}_2}(x,y) &= r(\mathcal{N}_1([(x,y),(x,y)])) = r([(\tfrac{1}{2x}, \sqrt{1-\tfrac{y}{2x}}), (\tfrac{1}{2x}, \sqrt{1-\tfrac{y}{2x}})]),\\
\underline{\mathcal{N}_2}(x,y) &= \overline{\mathcal{N}_2}(x,y) = (\tfrac{1}{2x}, \sqrt{1-\tfrac{y}{2x}}).
\end{aligned}$$

By Theorem 6(1),

$$\begin{aligned}
\mathcal{N}_1([(x_1,y_1),(x_2,y_2)]) &= [\underline{\mathcal{N}_1}(x_2,y_2), \overline{\mathcal{N}_1}(x_1,y_1)],\\
\mathcal{N}_2([(x_1,y_1),(x_2,y_2)]) &= [\underline{\mathcal{N}_2}(x_2,y_2), \overline{\mathcal{N}_2}(x_1,y_1)].
\end{aligned}$$

From Theorem 6(3), we can obtain a pair of interval implications $(\mathcal{I}_1, \mathcal{I}_2)$ defined as $\mathcal{I}_i : L^{[2]} \times L^{[2]} \to L^{[2]}$

$$\begin{aligned}
&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&= \mathcal{N}_1([(x_1,y_1),(x_2,y_2)]) \vee [(z_1,w_1),(z_2,w_2)]\\
&= [(\tfrac{1}{2x_2}, \tfrac{1-y_2^2}{x_2}) \vee (z_1,w_1), (\tfrac{1}{2x_1}, \tfrac{1-y_1^2}{x_1}) \vee (z_2,w_2)],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&= \mathcal{N}_2([(x_1,y_1),(x_2,y_2)]) \vee [(z_1,w_1),(z_2,w_2)]\\
&= [(\tfrac{1}{2x_2}, \sqrt{1-\tfrac{y_2}{2x_2}}) \vee (z_1,w_1), (\tfrac{1}{2x_1}, \sqrt{1-\tfrac{y_1}{2x_1}}) \vee (z_2,w_2)].
\end{aligned}$$

Theorem 8. *Let $(\mathcal{I}_1, \mathcal{I}_2)$ be a pair of interval implications on $L^{[2]}$. We define*

$$\underline{\mathcal{I}_i}(x,y) = l(\mathcal{I}_i([x,x],[y,y])), \ \ \overline{\mathcal{I}_i}(x,y) = r(\mathcal{I}_i([x,x],[y,y])).$$

Then we have the following properties:

(1) *If $[y_1,y_2] \le [z_1,z_2]$, then*

$$\mathcal{I}_1([x_1,x_2],[y_1,y_2]) \le \mathcal{I}_1([x_1,x_2],[z_1,z_2]).$$

(2) *If $[y_1,y_2] \subset [z_1,z_2]$, then*

$$\mathcal{I}_1([x_1,x_2],[y_1,y_2]) \subset \mathcal{I}_1([x_1,x_2],[z_1,z_2]).$$

(3) $\mathcal{I}_i([x_1,x_2],[y_1,y_2]) = [\underline{\mathcal{I}_i}(x_2,y_1), \overline{\mathcal{I}_i}(x_1,y_2)]$.

(4) *If, for each* $x, y \in L$*, there exists* $z \in L$ *such that* $\mathcal{I}_i([x,x],[y,y]) = [z,z]$, $i = 1,2$*, then* $(\underline{\mathcal{I}_1}, \underline{\mathcal{I}_2})$ *is a pair of implications such that*

$$\underline{\mathcal{I}_1} = \overline{\mathcal{I}_1},\ \underline{\mathcal{I}_2} = \overline{\mathcal{I}_2}.$$

(5) *Define maps* $\mathcal{N}_i : L^{[2]} \to L^{[2]}$ *as*

$$\mathcal{N}_1([x_1,x_2]) = \mathcal{I}_1([x_1,x_2],[\perp,\perp]),$$

$$\mathcal{N}_2([x_1,x_2]) = \mathcal{I}_2([x_1,x_2],[\perp,\perp]).$$

Then $(\mathcal{N}_1, \mathcal{N}_2)$ *is a pair of interval negations.*

(6)

$$\mathcal{I}_1(\mathcal{N}_2([y_1,y_2]),\mathcal{N}_2([x_1,x_2])) = \mathcal{I}_2([x_1,x_2],[y_1,y_2]),$$
$$\mathcal{I}_2(\mathcal{N}_1([y_1,y_2]),\mathcal{N}_1([x_1,x_2])) = \mathcal{I}_1([x_1,x_2],[y_1,y_2]).$$

Proof. (1) If $[y_1,y_2] \le [z_1,z_2]$, then

$$\mathcal{I}_1([x_1,x_2],[y_1,y_2]) \le \mathcal{I}_1([x_1,x_2],[z_1,z_2]).$$

Since $\mathcal{I}_1([y_1,y_1],[\perp,\perp]) \ge \mathcal{I}_1([z_1,z_1],[\perp,\perp])$ from (II2), we have

$$\begin{aligned}
\mathcal{I}_1([x_1,x_2],[y_1,y_2]) &= \mathcal{I}_1([x_1,x_2],\mathcal{I}_2(\mathcal{I}_1([y_1,y_1],[\perp,\perp]),[\perp,\perp])) \\
&= \mathcal{I}_2(\mathcal{I}_1([y_1,y_1],[\perp,\perp]),\mathcal{I}_1([x_1,x_2],[\perp,\perp])) \\
&\le \mathcal{I}_2(\mathcal{I}_1([z_1,z_1],[\perp,\perp]),\mathcal{I}_1([x_1,x_2],[\perp,\perp])) \\
&= \mathcal{I}_1([x_1,x_2],\mathcal{I}_2(\mathcal{I}_1([z_1,z_2],[\perp,\perp]),[\perp,\perp])) \\
&= \mathcal{I}_1([x_1,x_2],[y_1,y_2]).
\end{aligned}$$

(2) If $[y_1,y_2] \subset [z_1,z_2]$, then $\mathcal{I}_i([y_1,y_2],[\perp,\perp]) \subset \mathcal{I}_i([z_1,z_2],[\perp,\perp])$ from (II3), we have

$$\begin{aligned}
\mathcal{I}_1([x_1,x_2],[y_1,y_2]) &= \mathcal{I}_1([x_1,x_2],\mathcal{I}_2(\mathcal{I}_1([y_1,y_1],[\perp,\perp]),[\perp,\perp])) \\
&= \mathcal{I}_2(\mathcal{I}_1([y_1,y_1],[\perp,\perp]),\mathcal{I}_1([x_1,x_2],[\perp,\perp])) \\
&\subset \mathcal{I}_2(\mathcal{I}_1([z_1,z_1],[\perp,\perp]),\mathcal{I}_1([x_1,x_2],[\perp,\perp])) \\
&= \mathcal{I}_1([x_1,x_2],\mathcal{I}_2(\mathcal{I}_1([z_1,z_2],[\perp,\perp]),[\perp,\perp])) \\
&= \mathcal{I}_1([x_1,x_2],[z_1,z_2]).
\end{aligned}$$

Thus,

$$\mathcal{I}_1([x_1,x_2],[y_1,y_2]) \subset \mathcal{I}_1([x_1,x_2],[z_1,z_2]).$$

(3) Since $[x_2,x_2] \subset [x_1,x_2]$ and $[y_1,y_1] \subset [y_1,y_2]$, then $\mathcal{I}_i([x_2,x_2],[y_1,y_1]) \subset \mathcal{I}_i([x_1,x_2],[y_1,y_2])$. Hence $l(\mathcal{I}_i([x_1,x_2],[y_1,y_2])) \le l(\mathcal{I}_i([x_2,x_2],[y_1,y_1]))$. On the other hand, since $[x_1,x_2] \le [x_2,x_2]$ and $[y_1,y_1] \le [y_1,y_2]$, then

$$\mathcal{I}_i([x_2,x_2],[y_1,y_1]) \le \mathcal{I}_i([x_1,x_2],[y_1,y_2]).$$

Hence $l(\mathcal{I}_i([x_1,x_2],[y_1,y_2])) \geq l(\mathcal{I}_i([x_2,x_2],[y_1,y_1]))$. Thus

$$l(\mathcal{I}_i([x_1,x_2],[y_1,y_2])) = l(\mathcal{I}_i([x_2,x_2],[y_1,y_1])).$$

Since $[x_1,x_1] \subset [x_1,x_2]$ and $[y_2,y_2] \subset [y_1,y_2]$, then $\mathcal{I}_i([x_1,x_1],[y_2,y_2]) \subset \mathcal{I}_i([x_1,x_2],[y_1,y_2])$. Hence $r(\mathcal{I}_i([x_1,x_2],[y_1,y_2])) \geq r(\mathcal{I}_i([x_1,x_1],[y_2,y_2]))$. On the other hand, since

$$[x_1,x_1] \leq [x_1,x_2] \text{ and } [y_1,y_2] \leq [y_2,y_2],$$

then $\mathcal{I}_i([x_1,x_2],[y_1,y_2]) \leq \mathcal{I}_i([x_1,x_1],[y_2,y_2])$. Hence $r(\mathcal{I}_i([x_1,x_2],[y_1,y_2])) \leq r(\mathcal{I}_i([x_1,x_1],[y_2,y_2]))$. Thus

$$r(\mathcal{I}_i([x_1,x_2],[y_1,y_2])) = r(\mathcal{I}_i([x_1,x_1],[y_2,y_2])).$$

$$\begin{aligned} \mathcal{I}_i([x_1,x_2],[y_1,y_2]) &= [l(\mathcal{I}_i([x_1,x_2],[y_1,y_2])), r(\mathcal{I}_i([x_1,x_2],[y_1,y_2]))] \\ &= [l(\mathcal{I}_i([x_2,x_2],[y_1,y_1])), r(\mathcal{I}_i([x_1,x_1],[y_2,y_2]))] \\ &= [\underline{\mathcal{I}_i}(x_2,y_1), \overline{\mathcal{I}_i}(x_1,y_2)]. \end{aligned}$$

(4) For each $x,y \in L$, there exists $z \in L$ such that $\mathcal{I}_i([x,x],[y,y]) = [z,z]$, $i = 1,2$, then $l(\mathcal{I}_2([y,y],[z,z])) = r(\mathcal{I}_2([y,y],[z,z]))$ and $l(\mathcal{I}_1([x,x],[z,z])) = r(\mathcal{I}_1([x,x],[z,z]))$. Thus

$$\underline{\mathcal{I}_1}(x,y) = l(\mathcal{I}_1([x,x],[y,y])) = r(\mathcal{I}_1([x,x],[y,y])) = \overline{\mathcal{I}_1}(x,y),$$

$$\underline{\mathcal{I}_2}(x,y) = l(\mathcal{I}_2([x,x],[y,y])) = r(\mathcal{I}_2([x,x],[y,y])) = \overline{\mathcal{I}_2}(x,y).$$

(I1)

$$\begin{aligned} \underline{\mathcal{I}_i}(\top,\bot) &= l(\mathcal{I}_i([\top,\top],[\bot,\bot])) = l([\bot,\bot]) = \bot, \\ \underline{\mathcal{I}_i}(\bot,\top) &= l(\mathcal{I}_i([\bot,\bot],[\top,\top])) = l([\top,\top]) = \top, \\ \underline{\mathcal{I}_i}(\bot,\bot) &= l(\mathcal{I}_i([\bot,\bot],[\bot,\bot]) = l([\top,\top]) = \top, \\ \underline{\mathcal{I}_i}(\top,\top) &= l(\mathcal{I}_i([\top,\top],[\top,\top])) = l([\top,\top]) = \top. \end{aligned}$$

(I2) If $x \leq y$, then

$$\underline{\mathcal{I}_i}(x,z) = l(\mathcal{I}_i([x,x],[z,z])) \geq l(\mathcal{I}_i([y,y],[z,z])) = \underline{\mathcal{I}_i}(y,z).$$

(I3)

$$\underline{\mathcal{I}_i}(\top,x) = l(\mathcal{I}_i([\top,\top],[x,x]) = l([x,x]) = x.$$

(I4) For each $x, y \in L$, there exists $z \in L$ such that $\mathcal{I}_i([x,x],[y,y]) = [z,z]$, $i = 1,2$, then $l(\mathcal{I}_i([y,y],[z,z])) = r(\mathcal{I}_i([y,y],[z,z]))$. Thus

$$\begin{aligned}
&\underline{\mathcal{I}_1}(x, \underline{\mathcal{I}_2}(y,z)) = \underline{\mathcal{I}_1}(x, l(\mathcal{I}_2([y,y],[z,z])))\\
&= l(\mathcal{I}_1([x,x],[l(\mathcal{I}_2([y,y],[z,z])), l(\mathcal{I}_2([y,y],[z,z]))]))\\
&= l(\mathcal{I}_1([x,x],[l(\mathcal{I}_2([y,y],[z,z])), r(\mathcal{I}_2([y,y],[z,z]))]))\\
&= l(\mathcal{I}_1([x,x],\mathcal{I}_2([y,y],[z,z]))\\
&= l(\mathcal{I}_2([y,y],\mathcal{I}_1([x,x],[z,z]))\\
&= l(\mathcal{I}_2([y,y],[l(\mathcal{I}_1([x,x],[z,z])), r(\mathcal{I}_1([x,x],[z,z]))]))\\
&= l(\mathcal{I}_2([y,y],[l(\mathcal{I}_1([x,x],[z,z])), l(\mathcal{I}_1([x,x],[z,z]))]))\\
&= \underline{\mathcal{I}_2}(y, \underline{\mathcal{I}_1}(x,z)).
\end{aligned}$$

(I5)

$$\begin{aligned}
\underline{\mathcal{I}_1}(\underline{\mathcal{I}_2}(x,\bot),\bot) &= l(\mathcal{I}_1([l(\mathcal{I}_2([x,x],[\bot,\bot])), l(\mathcal{I}_2([x,x],[\bot,\bot]))],[\bot,\bot]))\\
&= l(\mathcal{I}_1([l(\mathcal{I}_2([x,x],[\bot,\bot])), r(\mathcal{I}_2([x,x],[\bot,\bot]))],[\bot,\bot]))\\
&= l(\mathcal{I}_1(\mathcal{I}_2([x,x],[\bot,\bot]),[\bot,\bot])) = l([x,x]) = x.
\end{aligned}$$

(5) (IN1)

$$\mathcal{N}_i([\bot,\bot]) = \mathcal{I}_i([\bot,\bot],[\bot,\bot]) = [\top,\top],$$
$$\mathcal{N}_i([\top,\top]) = \mathcal{I}_i([\top,\top],[\bot,\bot]) = [\bot,\bot].$$

(IN2) If $[x_1,x_2] \le [y_1,y_2]$, then

$$\mathcal{N}_1([x_1,x_2]) = \mathcal{I}_1([x_1,x_2],[\bot,\bot]) \ge \mathcal{I}_1([y_1,y_2],[\bot,\bot]) = \mathcal{N}_i([x_1,x_2])$$

for all $i \in \{1,2\}$.

(IN3) If $[x_1,x_2] \subset [y_1,y_2]$, then

$$\begin{aligned}
\mathcal{N}_i([x_1,x_2]) &= \mathcal{I}_1([x_1,x_2],[\bot,\bot])\\
&\subset \mathcal{I}_1([y_1,y_2],[\bot,\bot]) = \mathcal{N}_i([y_1,y_2]).
\end{aligned}$$

(IN4) $\mathcal{N}_1(\mathcal{N}_2([x_1,x_2])) = \mathcal{N}_2(\mathcal{N}_1([x_1,x_2])) = [x_1,x_2]$ for all $[x_1,x_2] \in L^{[2]}$.

$$\begin{aligned}
\mathcal{N}_1(\mathcal{N}_2([x_1,x_2])) &= \mathcal{I}_1(\mathcal{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot])\\
&= [x_1,x_2].
\end{aligned}$$

(6)

$$\begin{aligned}
&\mathcal{I}_1(\mathcal{N}_2([y_1,y_2]),\mathcal{N}_2([x_1,x_2]))\\
&= \mathcal{I}_1(\mathcal{N}_2([y_1,y_2]),\mathcal{I}_2([x_1,x_2],[\bot,\bot]))\\
&= \mathcal{I}_2([x_1,x_2],\mathcal{I}_1(\mathcal{N}_2([y_1,y_2]),[\bot,\bot]))\\
&= \mathcal{I}_2([x_1,x_2],\mathcal{I}_1((\mathcal{I}_2([y_1,y_2]),[\bot,\bot]),[\bot,\bot]))\\
&= \mathcal{I}_2([x_1,x_2],[y_1,y_2]).
\end{aligned}$$

□

Example 9. Put $L = \{(x,y) \in R^2 \mid (\frac{1}{2},1) \leq (x,y) \leq (1,0)\}$ with a bottom element $(\frac{1}{2},1)$ and a top element $(1,0)$ where

$$(x_1,y_1) \leq (x_2,y_2) \Leftrightarrow x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2.$$

Let $L^{[2]} = \{[(x_1,y_1),(x_2,y_2)] \mid (x_1,y_1),(x_2,y_2) \in L\}$.

(1) Define $\mathcal{I}_1, \mathcal{I}_2 : L^{[2]} \times L^{[2]} \to L^{[2]}$ as follows:

$$\begin{aligned}
&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\
&= [(\tfrac{z_1}{x_2}, \tfrac{w_1-y_2}{x_2}) \wedge (1,0), (\tfrac{z_2}{x_1}, \tfrac{w_2-y_1}{x_1}) \wedge (1,0)] \\
&\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\
&= [(\tfrac{z_1}{x_2}, w_1 - \tfrac{z_1 y_2}{x_2}) \wedge (1,0), (\tfrac{z_2}{x_1}, w_2 - \tfrac{z_2 y_1}{x_1}) \wedge (1,0)].
\end{aligned}$$

(II1)

$$\begin{aligned}
&\mathcal{I}_i([(1,0),(1,0)],[(\tfrac{1}{2},1),(\tfrac{1}{2},1)]) = [(\tfrac{1}{2},1),(\tfrac{1}{2},1)], \\
&\mathcal{I}_i([(\tfrac{1}{2},1),(\tfrac{1}{2},1)],[(\tfrac{1}{2},1),(\tfrac{1}{2},1)]) = [(1,0),(1,0)], \\
&\mathcal{I}_i([(\tfrac{1}{2},1),(\tfrac{1}{2},1)],[(1,0),(1,0)]) = [(1,0),(1,0)], \\
&\mathcal{I}_i([(1,0),(1,0)],[(1,0),(1,0)]) = [(1,0),(1,0)].
\end{aligned}$$

(II2) If $[(x_1,y_1),(x_2,y_2)] \leq [(a_1,b_1),(a_2,b_2)]$, then $(x_i,y_i) \leq (a_i,b_i)$ iff $x_i < a_i$ or $x_i = a_i, y_i \leq b_i$. It follows

$$\begin{aligned}
&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\
&= [(\tfrac{z_1}{x_2}, \tfrac{w_1-y_2}{x_2}) \wedge (1,0), (\tfrac{z_2}{x_1}, \tfrac{w_2-y_1}{x_1}) \wedge (1,0)] \\
&\geq [(\tfrac{z_1}{a_2}, \tfrac{w_1-b_2}{a_2}) \wedge (1,0), (\tfrac{z_2}{a_1}, \tfrac{w_2-b_1}{a_1}) \wedge (1,0)] \\
&= \mathcal{I}_1([(a_1,b_1),(a_2,b_2)],[(z_1,w_1),(z_2,w_2)]), \\
&\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\
&= [(\tfrac{z_1}{x_2}, w_1 - \tfrac{z_1 y_2}{x_2}) \wedge (1,0), (\tfrac{z_2}{x_1}, w_2 - \tfrac{z_2 y_1}{x_1}) \wedge (1,0)] \\
&\geq [(\tfrac{z_1}{a_2}, w_1 - \tfrac{z_1 b_2}{a_2}) \wedge (1,0), (\tfrac{z_2}{a_1}, w_2 - \tfrac{z_2 b_1}{a_1}) \wedge (1,0)] \\
&= \mathcal{I}_2([(a_1,b_1),(a_2,b_2)],[(z_1,w_1),(z_2,w_2)]).
\end{aligned}$$

(II3) If $[(x_1,y_1),(x_2,y_2)] \subset [(a_1,b_1),(a_2,b_2)]$, then $(a_1,b_1) \leq (x_1,y_1) \leq (x_2,y_2) \leq (a_2,b_2)$. Thus

$$(\frac{z_1}{a_2}, \frac{w_1-b_2}{a_2}) \leq (\frac{z_1}{x_2}, \frac{w_1-y_2}{x_2}) \leq (\frac{z_1}{x_1}, \frac{w_2-y_1}{x_1}) \leq (\frac{z_2}{a_2}, \frac{w_2-b_1}{a_1})$$

$$(\frac{z_1}{a_2}, w_1 - \frac{z_1 b_2}{a_2}) \leq (\frac{z_1}{x_2}, w_1 - \frac{z_1 y_2}{x_2}) \leq (\frac{z_2}{x_1}, w_2 - \frac{z_2 y_1}{x_1}) \leq (\frac{z_2}{a_2}, w_2 - \frac{z_2 b_1}{a_1}).$$

It follows

$$\begin{aligned}
&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=[(\tfrac{z_1}{x_2},\tfrac{w_1-y_2}{x_2})\wedge(1,0),(\tfrac{z_2}{x_1},\tfrac{w_2-y_1}{x_1})\wedge(1,0)]\\
&\subset[(\tfrac{z_1}{a_2},\tfrac{w_1-b_2}{a_2})\wedge(1,0),(\tfrac{z_2}{a_1},\tfrac{w_2-b_1}{a_1})\wedge(1,0)]\\
&=\mathcal{I}_1([(a_1,b_1),(a_2,b_2)],[(z_1,w_1),(z_2,w_2)]),\\
&\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=[(\tfrac{z_1}{x_2},w_1-\tfrac{z_1y_2}{x_2})\wedge(1,0),(\tfrac{z_2}{x_1},w_2-\tfrac{z_2y_1}{x_1})\wedge(1,0)]\\
&\subset[(\tfrac{z_1}{a_2},w_1-\tfrac{z_1b_2}{a_2})\wedge(1,0),(\tfrac{z_2}{a_1},w_2-\tfrac{z_2b_1}{a_1})\wedge(1,0)]\\
&=\mathcal{I}_2([(a_1,b_1),(a_2,b_2)],[(z_1,w_1),(z_2,w_2)]).
\end{aligned}$$

(II4)

$$\begin{aligned}
&\mathcal{I}_1([(1,0),(1,0)],[(z_1,w_1),(z_2,w_2)])=[(z_1,w_1),(z_2,w_2)]\\
&\mathcal{I}_2([(1,0),(1,0)],[(z_1,w_1),(z_2,w_2)])=[(z_1,w_1),(z_2,w_2)].
\end{aligned}$$

(II5)

$$\begin{aligned}
&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],\mathcal{I}_2([(z_1,w_1),(z_2,w_2)],[(p_1,r_1),(p_2,r_2)]))\\
&=\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(\tfrac{p_1}{z_2},r_1-\tfrac{p_1w_2}{z_2})\wedge(1,0),(\tfrac{p_2}{z_1},r_2-\tfrac{p_2w_1}{z_1})\wedge(1,0)])\\
&=[(\tfrac{p_1}{x_2z_2},\tfrac{z_2r_1-p_1w_2-z_2y_2}{x_2z_2})\wedge(1,0),(\tfrac{p_2}{x_1z_1},\tfrac{z_1r_2-p_2w_1-z_1y_1}{x_1z_1})\wedge(1,0)].
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}_2([(z_1,w_1),(z_2,w_2)],\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(p_1,r_1),(p_2,r_2)]))\\
&=\mathcal{I}_2([(z_1,w_1),(z_2,w_2)],[(\tfrac{p_1}{x_2},\tfrac{r_1-y_2}{x_2})\wedge(1,0),(\tfrac{p_2}{x_1},\tfrac{r_2-y_1}{x_1})\wedge(1,0)])\\
&=[(\tfrac{p_1}{x_2z_2},\tfrac{z_2r_1-p_1w_2-z_2y_2}{x_2z_2})\wedge(1,0),(\tfrac{p_2}{x_1z_1},\tfrac{z_1r_2-p_2w_1-z_1y_1}{x_1z_1})\wedge(1,0)].
\end{aligned}$$

(II6)

$$\begin{aligned}
\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(\tfrac12,1),(\tfrac12,1)]) &=[(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})]\\
\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(\tfrac12,1),(\tfrac12,1)]) &=[(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}_2(\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(\tfrac12,1),(\tfrac12,1)]),[(\tfrac12,1),(\tfrac12,1)])=[(x_1,y_1),(x_2,y_2)]\\
&\mathcal{I}_1(\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(\tfrac12,1),(\tfrac12,1)]),[(\tfrac12,1),(\tfrac12,1)])=[(x_1,y_1),(x_2,y_2)].
\end{aligned}$$

Hence $(\mathcal{I}_1,\mathcal{I}_2)$ is a pair of interval implications.

Since, for each $x,y\in L$, there exists $z\in L$ such that $\mathcal{I}_i([x,x],[y,y])=[z,z]$, $i=1,2$, by Theorem 9 (4), $(\underline{\mathcal{I}_1},\underline{\mathcal{I}_2})$ is a pair of interval implications such that

$$\begin{aligned}
\underline{\mathcal{I}_1}((x,y),(z,w)) &=l(\mathcal{I}_1([(x,y),(x,y)],[(z,w),(z,w)]))\\
&=l([(\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0),(\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0)])\\
&=r([(\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0),(\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0)])\\
&=(\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0)=\overline{\mathcal{I}_1}((x,y),(z,w)),\\
\underline{\mathcal{I}_2}((x,y),(z,w)) &=l([(\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0),(\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0)])\\
&=r([(\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0),(\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0)])\\
&=(\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0)=\overline{\mathcal{I}_2}((x,y),(z,w)).
\end{aligned}$$

Moreover, by Theorem 9 (3), we have

$$\begin{aligned}&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\&=[\underline{\mathcal{I}_1}((x_2,y_2),(z_1,w_1),\overline{\mathcal{I}_1}((x_1,y_1),(z_2,w_2))]\\&\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\&=[\underline{\mathcal{I}_2}((x_2,y_2),(z_1,w_1),\overline{\mathcal{I}_2}((x_1,y_1),(z_2,w_2))].\end{aligned}$$

We obtain an pair of interval negations pair $(\mathcal{N}_1,\mathcal{N}_2)$ such that $\mathcal{N}_1,\mathcal{N}_2 : L^{[2]}\to L^{[2]}$ as follows:

$$\begin{aligned}\mathcal{N}_1([(x_1,y_1),(x_2,y_2)])&=\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(\tfrac{1}{2},1),(\tfrac{1}{2},1)])\\&=[(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})]\\\mathcal{N}_2([(x_1,y_1),(x_2,y_2)])&=\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(\tfrac{1}{2},1),(\tfrac{1}{2},1)])\\&=[(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})].\end{aligned}$$

(2) Define $\mathcal{I}_1,\mathcal{I}_2 : L^{[2]}\times L^{[2]}\to L^{[2]}$ as follows:

$$\begin{aligned}&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\&=[(\tfrac{z_1}{x_2},w_1-2z_1+\tfrac{2z_1-2z_1y_2}{x_2})\wedge(1,0),(\tfrac{z_2}{z_1},w_2-2z_2+\tfrac{2z_2-2z_2y_1}{x_1})\wedge(1,0)]\\&\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\&=[(\tfrac{z_1}{x_2},1-\tfrac{y_2+2-2w_1}{2x_1})\wedge(1,0),(\tfrac{z_2}{x_1},1-\tfrac{y_1+2-2w_2}{2x_1})\wedge(1,0)].\end{aligned}$$

$$\begin{aligned}&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],\mathcal{I}_2([(z_1,w_1),(z_2,w_2)],[(p_1,r_1),(p_2,r_2)]))\\&=\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(\tfrac{p_1}{z_2},1-\tfrac{w_2+2-2r_1}{2z_2})\wedge(1,0),(\tfrac{p_2}{z_1},1-\tfrac{w_1+2-2r_2}{2z_1})\wedge(1,0)])\\&=[(\tfrac{p_1}{z_2w_2},\tfrac{2z_2x_2-z_2y_2-2z_2+2z_2r_1-4p_1x_1+4p_1-4p_1w_2}{2z_2w_2})\wedge(1,0),\\&\quad(\tfrac{p_2}{z_1w_1},\tfrac{2z_1x_1-z_1y_1-2z_1+2z_1r_2-4p_2x_2+4p_2-4p_2w_1}{2z_1w_1})\wedge(1,0)].\end{aligned}$$

$$\begin{aligned}&\mathcal{I}_2([(z_1,w_1),(z_2,w_2)],\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(p_1,r_1),(p_2,r_2)]))\\&=\mathcal{I}_2([(z_1,w_1),(z_2,w_2)],[(\tfrac{p_1}{x_2},r_1-2p_1+\tfrac{2p_1-2p_1y_2}{x_2})\wedge(1,0),\\&\quad(\tfrac{p_2}{x_1},r_2-2p_2+\tfrac{2p_2-2p_2y_1}{x_1})\wedge(1,0)])\\&=[(\tfrac{p_1}{z_2w_2},\tfrac{2z_2x_2-z_2y_2-2z_2+2z_2r_1-4p_1x_1+4p_1-4p_1w_2}{2z_2w_2})\wedge(1,0),\\&\quad(\tfrac{p_2}{z_1w_1},\tfrac{2z_1x_1-z_1y_1-2z_1+2z_1r_2-4p_2x_2+4p_2-4p_2w_1}{2z_1w_1})\wedge(1,0)].\end{aligned}$$

By a similar method as in (1), $(\mathcal{I}_1,\mathcal{I}_2)$ is a pair of interval implications. Since, for each $x,y\in L$, there exists $z\in L$ such that $\mathcal{I}_i([x,x],[y,y])=[z,z]$,

$i = 1, 2$, by Theorem 9(4), $(\underline{\mathcal{I}_1}, \underline{\mathcal{I}_2})$ is a pair of interval implications such that

$$\begin{array}{l}\underline{\mathcal{I}_1}((x,y),(z,w)) = l(\mathcal{I}_1([(x,y),(x,y)],[(z,w),(z,w)]))\\ = l([(\frac{z}{x}, w-2z+\frac{2z-2zy}{x}) \wedge (1,0), (\frac{z}{x}, w-2z+\frac{2z-2zy}{x}) \wedge (1,0)])\\ = r([(\frac{z}{x}, w-2z+\frac{2z-2zy}{x}) \wedge (1,0), (\frac{z}{x}, w-2z+\frac{2z-2zy}{x}) \wedge (1,0)])\\ = (\frac{z}{x}, w-2z+\frac{2z-2zy}{x}) \wedge (1,0) \wedge (1,0) = \overline{\mathcal{I}_1}((x,y),(z,w)),\\ \underline{\mathcal{I}_2}((x,y),(z,w)) = l([(\frac{z}{x}, 1-\frac{y+2-2w}{2x}) \wedge (1,0), (\frac{z}{x}, 1-\frac{y+2-2w}{2x}) \wedge (1,0)])\\ = r([(\frac{z}{x}, 1-\frac{y+2-2w}{2x}) \wedge (1,0), (\frac{z}{x}, 1-\frac{y+2-2w}{2x}) \wedge (1,0)])\\ = (\frac{z}{x}, 1-\frac{y+2-2w}{2x}) \wedge (1,0) = \overline{\mathcal{I}_2}((x,y),(z,w)).\end{array}$$

We obtain an pair of interval negations pair $(\mathcal{N}_1, \mathcal{N}_2)$ such that $\mathcal{N}_1, \mathcal{N}_2 : L^{[2]} \to L^{[2]}$ as follows:

$$\begin{array}{ll}\mathcal{N}_1([(x_1,y_1),(x_2,y_2)]) & = \mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(\frac{1}{2},1),(\frac{1}{2},1)])\\ & = [(\frac{1}{2x_2}, \frac{1-y_2}{x_2}), (\frac{1}{2x_1}, \frac{1-y_1}{x_1})]\\ \mathcal{N}_2([(x_1,y_1),(x_2,y_2)]) & = \mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(\frac{1}{2},1),(\frac{1}{2},1)])\\ & = [(\frac{1}{2x_2}, 1-\frac{y_2}{2x_2}), (\frac{1}{2x_1}, 1-\frac{y_1}{2x_1})].\end{array}$$

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