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ON THE DIOPHANTINE EQUATION $p^x + (p+1)^y = z^2$ WHERE p IS A MERSENNE PRIME

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Abstract: In this paper we show that $(p, x, y, z) = (7, 0, 1, 3)$ and $(p, x, y, z) = (3, 2, 2, 5)$ are the only solutions to the Diophantine equation $p^x + (p+1)^y = z^2$, where x, y, z are non-negative integers and p is a Mersenne prime.

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1. Introduction

In 2012, Sroysang [3] showed that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative solution. Later in 2013, Sroysang [4] also showed that the Diophantine equation $7^x + 8^y = z^2$ has only one solution, namely $(x, y, z) = (0, 1, 3)$, and he posed an open problem regarding the set of all solutions (x, y, z) for the Diophantine equation $p^x + (p+1)^y = z^2$, where x, y and z are non-negative integers. Inspired by this open problem, we therefore aim to study the Diophantine equation $p^x + (p+1)^y = z^2$, where x, y, z are non-negative integers and p is a Mersenne prime.

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2. Preliminaries

A Mersenne number is a number of the from $2^n - 1$, where n is a positive integer. If Mersenne number is prime, then it is called a Mersenne prime. It is well known that if $2^n - 1$ is prime, then n is also prime.

Mersenne numbers have played an important role in number theory since 1644. In particular, there has been long interest in Mersenne prime, which are closely related to finding perfect numbers and large primes. As of April 2013, only 48 Mersenne primes were discovered so far. Among them, the largest known prime (discovered on January 25^{th} , 2013) is $2^{57885161} - 1$, which is 17425170 digits long ; see http://www.mersenne.org/various/57885161.htm and http://primes.utm.edu/mersenne/ for more details.

3. Main Results

In this study, we will use Catalan's conjecture (see [2]), which states that the only solution in integers $a > 1, b > 1, x > 1, y > 1$ to the equation $a^x - b^y = 1$ is $(a, b, x, y) = (3, 2, 2, 3)$.

Now we will consider the Diophantine equation $p^x + (p + 1) = z^2$, where p is a Mersenne prime.

Theorem 1. *The Diophantine equation* $p^x + (p + 1)^y = z^2$, *where* p *is a Mersenne prime, has only two solutions in non-negative integer, namely,* $(p, x, y, z) = (7, 0, 2, 3)$ *and* $(p, x, y, z) = (3, 2, 2, 5)$.

Proof. Let p be a Mersenne prime. Then $p = 2^q - 1$ for some prime q. It is easy to check that z must be odd and $p \equiv 3 (\mathrm{mod}\ 4)$. Thus $z^2 \equiv 1 (\mathrm{mod}\ 4)$ and $p + 1 \equiv 0 (\mathrm{mod}\ 4)$.

Now we consider the Diophantine equation $p^x + (p+1)^y = z^2$ in the following cases:

Case 1 : Suppose $x = 0$. It can be checked easily that, if the Diophantine equation $1 + (p + 1)^y = z^2$ has a solution, then

$$2^{qy} = (p + 1)^y = z^2 - 1 = (z + 1)(z - 1).$$

Hence there exist non-negative integers α, β such that $2^\alpha = z + 1, 2^\beta = z - 1$, where $\alpha > \beta$ and $\alpha + \beta = qy$. Moreover, one can see that

$$2^\beta (2^{\alpha - \beta} - 1) = 2^\alpha - 2^\beta = (z + 1) - (z - 1) = 2.$$

This implies that $\beta = 1$ and $2^{\alpha-1} - 1 = 1$. Then $\alpha = 2$. Since q is prime, we obtain $q = 3, y = 1, p = 7$ and $z = 3$. Therefore $(p, x, y, z) = (7, 0, 1, 3)$ is the unique solution to the Diophantine equation $p^x + (p+1)^y = z^2$ in this case.

Case 2: Suppose $x \geq 1$. Now we have $(p+1)^y \equiv 0 (\text{mod } 4)$ and $z^2 \equiv 1 (\text{mod } 4)$, which implies that $p^x \equiv 1 (\text{mod } 4)$. Since $p \equiv 3 (\text{mod } 4), x$ must be even, i.e., $x = 2k$ for some integer $k \geq 1$. Hence we have

$$2^{qy} = (p+1)^y = z^2 - p^{2k} = (z + p^k)(z - p^k).$$

Thus there exist non-negative integers α, β such that $2^\alpha = z + p^k$ and $2^\beta = z - p^k$, where $\alpha > \beta$ and $\alpha + \beta = qy$. Moreover, one can see that

$$2^\beta(2^{\alpha-\beta} - 1) = 2^\alpha - 2^\beta = (z + p^k) - (z - p^k) = 2p^k.$$

This implies that $\beta = 1, \alpha > \beta = 1$ and

$$2^{\alpha-1} - 1 = p^k \tag{$*$}$$

Hence $z = p^k + 2$.

By Catalan's conjecture, $2^{\alpha-1} - p^k = 1$ has no solution only when $\alpha - 1 > 1$ and $k > 1$. Thus it suffices to consider only the case when $\alpha - 1 \leq 1$ or $k \leq 1$. We know that $\alpha > 1, \alpha + 1 = qy, q$ is prime, and $k \geq 1$. This implies that $q = 3$ and $y = 1$; or $k = 1$.

Case $k = 1$. From $(*)$, we have

$$2^{qy-2} - 1 = 2^{\alpha-1} - 1 = p = 2^q - 1.$$

It follows that $qy - 2 = q$, or equivalently, $q(y-1) = 2$. Since q is prime, we have $q = 2$ and $y = 2$. This implies that $p = 3, x = 2$ and $z = 5$. Therefore $(p, x, y, z) = (3, 2, 2, 5)$ is the unique solution to the Diophantine equation $p^x + (p+1)^y = z^2$ in this case.

Case $q = 3$ and $y = 1$. From $(*)$, we have

$$p^k = 2^{\alpha-1} - 1 = 2^{qy-2} - 1 = 1.$$

This implies that $k = 0$, which contradicts the fact that $k \geq 1$. Hence the Diophantine equation $p^x + (p+1)^y = z^2$ has no solution in this case.

It is easy to verity that $(p, x, y, z) = (7, 0, 1, 3)$ and $(p, x, y, z) = (3, 2, 2, 5)$ satisfy the equation $p^x + (p+1)^y = z^2$. This completes the proof. $\square$

4. Conclusion

Let p be a Mersenne prime. We have shown that the Diophantine equation $p^x + (p+1)^y = z^2$ has no solution except for $p = 3, 7$. Note that the results in [3] and [4] are special cases of our theorem since 7 and 31 are Mersenne primes. Nevertheless, the Diophantine equation $p^x + (p+1)^y = z^2$ where p is not a Mersenne prime remains an open problem.

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