POSTULATION OF CURVES CONTAINED IN
A UNION OF HYPERPLANES OF $\mathbb{P}^4$

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Abstract: Let $A \subset \mathbb{P}^4$ be a union of $m$ distinct hyperplanes. In this note for many $m, d$ we prove the existence of reduced, connected and nodal curves $C \subset A$ with $\deg(C) = d$, $p_a(C) = 0$ and maximal rank, i.e. $h^0(A, I_{C,A}(t)) \cdot h^1(A, I_{C,A}(t)) = 0$ for all $t \in \mathbb{N}$.

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1. The Statements

Let $A \subset \mathbb{P}^r$ be a reduced, but reducible hypersurface. For most quadruples $(d, g, r, \deg(A))$ there are reduced and connected and (say) nodal curve with degree $d$ and arithmetic genus $g$ which are contained in no irreducible hypersurface of degree $\deg(A)$ of $\mathbb{P}^r$. Hence allowing reducible hypersurface it is quite easy to construct reducible curve $C \subset A$ with prescribed degree and genera and some other properties (e.g. good postulation) (see [3], [2]). Here “good postulation” means that $C$ has maximal rank in $A$, i.e. for each integer $t > 0$ the restriction map $r_{A,Y,t} : H^0(A, \mathcal{O}_A(t)) \to H^0(C, \mathcal{O}_C(t))$ has maximal rank as a linear map (i.e. it is either injective or surjective). Since $A$ is arithmetically Cohen-
Macaulay, $C$ has maximal rank if and only if $h^0(A, \mathcal{I}_C(t)) \cdot h^1(A, \mathcal{I}_C(t)) = 0$ for all $t \in \mathbb{Z}$. In this paper we take $r = 4$ and as $A$ a reduced union of $m$ hyperplanes.

Let $A \subset \mathbb{P}^4$ be a reduced union of $m \geq 2$ hyperplanes. For each integer $k > m$ we have $h^0(A, \mathcal{O}_A(k)) = \binom{k+4}{4} - \binom{k-m+4}{4}$. This equality explains the integers appearing in the statements of Theorems 1 and 2.

**Theorem 1.** Fix integers $k \geq 2m \geq 4$ and $d \geq 4$ such that $kd + 1 \leq \binom{k+4}{4} - \binom{k-m+4}{4}$. Let $A = H_1 \cup \cdots \cup H_m$ be a union of $m$ hyperplanes such that $H_i \neq H_j$ for all $i \neq j$, the intersection of any 3 of them is a line, the intersection of any 4 of them is a point and no 5 of them has a common point. Then there exists a reduced, connected and nodal curve $C \subset A$ such that $\deg(C) = d$, $p_a(C) = 0$ and $h^1(A, \mathcal{I}_{C,A}(k)) = 0$.

**Theorem 2.** Fix integers $k \geq 2m \geq 4$ and $d \geq 4$ such that $kd + 1 \geq \binom{k+4}{4} - \binom{k-m+4}{4}$. Let $A = H_1 \cup \cdots \cup H_m$ be a union of $m$ hyperplanes such that $H_i \neq H_j$ for all $i \neq j$, the intersection of any 3 of them is a line, the intersection of any 4 of them is a point and no 5 of them has a common point. Then there exists a reduced, connected and nodal curve $C \subset A$ such that $\deg(C) = d$, $p_a(C) = 0$ and $h^0(A, \mathcal{I}_{C,A}(k)) = 0$.

Fix integers $m, k, d$ such that $m \geq k+2 \geq 4$, $d \geq 4$, $kd + 1 \leq \binom{k+4}{4} - \binom{k-m+4}{4}$ and $(k-1)d + 1 > \binom{k+3}{4} - \binom{k-m+3}{4}$ and $A$ as in Theorems 1 and 2. We do not claim the existence of $C \subset A$ as in Theorem 1 with the additional condition $h^0(A, \mathcal{I}_{C,A}(k-1)) = 0$, i.e. we do not claim that $C$ has maximal rank, i.e. the same curve may be used for Theorem 1 and for the case $k' := k-1$ of Theorem 2. In most cases this is a byproduct of our proof, but for numerical reasons we are unable to prove it in all cases. We prove in the case $m = 2$, i.e. we prove the following result.

**Theorem 3.** Fix hyperplanes $H_i \subset \mathbb{P}^4$, $i = 1, 2$ such that $H_1 \neq H_2$. Set $A := H_1 \cup H_2$. For each integer $d > 0$ there is a connected and nodal curve $C \subset A$ such that $\deg(C) = d$, $p_a(C) = 0$ and $C$ has maximal rank.

The curve $C$ in Theorems 1, 2 and 3 satisfies $h^1(C, \mathcal{O}_C(1)) = 0$. It seems obvious how to extend these theorems to curves with arithmetic genus $g$ at least if $d \gg g$. This in left to any interested reader (the only problems should be numerical, the most annoying ones being the ones related to Lemma 1 and Claim 3 of the proof of Theorem 1).
2. The Proofs

For any hyperplane $H \subset \mathbb{P}^4$ and all integers $a \geq 0$ and $b \geq 0$ let $Z(H, a, b)$ denote the set of all smooth curve $E \subset H$ which are the disjoint union of a smooth rational curve of degree $a$ and $b$ lines. $Z(H, a, b)$ is a quasi-projective variety of dimension $4a + 4b$.

Let $H \subset \mathbb{P}^4$ be any hyperplane. Let $Z \subset H$ be any closed subscheme with dim$(Z) \leq 1$. We do not assume that $Z$ is equidimensional, because we need the concepts which are discussing here for the scheme $E_i \cap H_i$ which is the disjoint union of a very nice curve and finitely many points. If either $\dim(Z) = 0$ or each connected component of the one-dimensional part of $Z$ has degree 1, then set $e(Z) = -1$. In all other cases let $e(Z)$ be the first integer $t \geq 0$ such that $h^1(Z, O_Z(t)) = 0$. We have $h^1(Z, O_Z(t)) = 0$ for all $t \geq e(Z)$ and either $\dim(Z) = 0$ or $h^1(Z, O_Z(t)) > 0$ for each $t < e(Z)$. Since each connected component of $C_i$ is a reduced curve with arithmetic genus 0 and one connected component of $C_i$ is not a line, we have $e(C_i) = 0$. Since $E_i \cap H_i$ is the disjoint union of $C_i$ and finitely many points, we have $e(E_i \cap H_i) = 0$. Let $c(Z)$ be the minimal integer $t > e(Z)$ such that $h^0(Z, O_Z(t)) \leq (\frac{t+3}{3})$. Fix an integer $x > e(Z)$ and assume $h^1(H, \mathcal{I}_{Z,H}(x)) = 0$. Since $x > e(Z)$, we have $h^2(H, \mathcal{I}_{Z,H}(x-1)) = h^1(Z, O_Z(x-1)) = 0$. Hence Castelnuovo-Mumford’s lemma gives $h^1(H, \mathcal{I}_{Z,H}(t)) = 0$ for all $t \geq x$. Hence $c(Z)$ is the first integer $t > e(Z)$ such that $h^1(H, \mathcal{I}_{Z,H}(t)) = 0$. Assume (as in our cases with $C_i$ or $E_i \cap H$ as $Z$) that $h^0(Z, O_Z(e(Z)) \geq (\frac{e(Z)+3}{3})$. We say that $Z$ has maximal rank if $h^1(H, \mathcal{I}_{Z,H}(c(Z))) = 0$ and $h^1(H, \mathcal{I}_{Z,H}(c(Z)-1))$. In our set-up $C_i$ has maximal rank, while $E_i \cap H$ has maximal rank if and only if $h^1(H_i, \mathcal{I}_{E_i \cap H_i}(k+i-1)) = 0$.

Lemma 1. Let $H \subset \mathbb{P}^4$ be a hyperplane. Fix a reduced curve $D \subset H$ such that each connected component of $D$ has arithmetic genus 0. Set $d := \deg(D)$ and let $u$ be the number of the connected components of $D$. Assume that $D$ has maximal rank. Fix integers $b \geq 0$ and $f > 0$. Take a finite set $B \subset H \setminus D$ such that $\sharp(B) = b$ and $h^0(H, \mathcal{I}_{D \cup B}(t)) = \max\{0, h^0(H, \mathcal{I}_D(t)) - b\}$ for all $t > 0$. Set $w := c(D \cup B)$ and $a := h^0(H, \mathcal{I}_{D \cup B}(w)) = (\frac{w+3}{3}) - wd - u - b$. If $f > a$, then assume $f + d \leq (\frac{w+2}{2})$. Fix a plane $M \subset H$ such that $B \cap M = \emptyset$ and no irreducible component of $D$ is contained in $M$. Then $D \cup B$ and $D \cup B \cup F$ has maximal rank.

Proof. Obviously $D \cup B$ has maximal rank. The integer $c(D \cup B)$ is the first positive integer $t$ such that $td + u + b \leq (\frac{t+3}{3})$. We order the points $P_1, \ldots, P_f$ of $F$. For each integer $x$ set $S_x := \cup_{i \leq x} P_i$. Notice that $S_0 = \emptyset$ and
Define the integers \( x \). Set \( x \), and assume defined the integers \( S = E \). Ballico 

\[ e(1)) = 0. \]

Since \( e(D \cup B \cup F) = 0 \), we have \( h^0(H, I_{D \cup B \cup F}(w)) = 0 \). Since \( e(D \cup B \cup F) = e(D) \leq 0 \), Castelnuovo-Mumford’s lemma shows that \( D \cup B \cup F \) has maximal rank, it is sufficient to prove \( h^1(H, I_{D \cup B \cup F}(w)) = 0 \).

Let \( y \) be the maximal integer \( \leq m \) such \( h^1(H, I_{D \cup B \cup F}(w)) = 0 \). Assume \( y < f \). Since \( h^1(H, I_{D \cup B \cup F}(w)) = 0 \), we have \( h^0(H, I_{D \cup B \cup F}(w)) = a - y \).

Since \( h^1(H, I_{D \cup B \cup F}(w)) = 0 \) and \( P_{y+1} \) is a general in \( M \), \( M \) is in the base locus of \( |I_{D \cup B \cup F}| \). Since \( M \) contains no irreducible component of \( D \) and \( B \cap M = \emptyset \), we get \( h^0(H, I_{D \cup B}(w-1)) \geq a - y > 0 \), a contradiction. Now assume \( f > a \). Take any \( F' \subset F \) such that \( \sharp(F') = a \). The first part of the proof gives \( h^0(H, I_{D \cup B \cup F'}(w)) = 0 \). Hence \( h^0(H, I_{D \cup B \cup F}(w)) = 0 \). Hence to prove that \( D \cup B \cup F \) has maximal rank it is sufficient to prove that \( h^1(H, I_{D \cup B \cup F}(w+1)) = 0 \). Let \( z \) be the maximal integer \( \leq f \) such that \( h^1(H, I_{D \cup B \cup F}(w+1)) = 0 \). Assume \( z < f \). We have \( h^0(H, I_{D \cup B \cup F}(w+1)) = (w+4)/(w+1)d - u - b - z \).

Since \( P_{z+1} \) is a general point of \( M \), we get that \( M \) is contained in the base locus of \( |I_{D \cup B \cup F}(w+1)| \). Hence \( h^0(H, I_{D \cup B \cup F}(w+1)) = h^0(H, I_{D \cup B}(w)) \), i.e. \( (w+4)/(w+1)d - u - b - z = (w+3)/(w+1) - wd - u - b \). Hence \( m > z = (w+2) - d \), a contradiction.

**Proof of Theorem 1.** For all integers \( k > m \geq 1 \) define the integers \( d_{k,m} \) and \( a_{k,m} \) by the following relations

\[
k d_{k,m} + 1 + a_{k,m} = \left(\frac{k+4}{4}\right) - \left(\frac{k+4-m}{4}\right), \quad 0 \leq a_{k,m} \leq k - 1 \quad (1)
\]

Set \( x_{m,k,m} := d_{k-m+1,1} \) and \( y_{m,k,m} := a_{k-k+1,1} \). Fix an integer \( i \) such that \( 1 \leq i < m \) and assume defined the integers \( x_{i,k,m} \) and \( y_{i,k,m} \) for all \( j \in \{i+1, \ldots, m\} \).

Define the integers \( x_{i,k,m} \) and \( y_{i,k,m} \) by the relations

\[
(k + 1 - i)x_{i,k,m} + 1 + y_{i,k,m} - y_{i+1,k,m} + \sum_{j=i+1}^{m} x_{i,k,m} = \left(\frac{k - i + 1}{3}\right), \quad 0 \leq y_{i,k,m} \leq k - i \quad (2)
\]

Hence \( d_{k,m} = \sum_{i=1}^{m} x_{i,k,m} \)

**Claim 1.** \( x_{i,k,m} > 0 \) and \( x_{i,k,m} \leq (k+4-i)(k+3-i)(k+2-j)/6(k+1-i) \) for all \( i \).

**Proof of Claim 1.** We have \( x_{m,k,m} = \lfloor((k+4-m)/3 - 1)/k \rfloor > 0 \). Hence \( 0 < x_{m,k,m} \leq (k + 4 - m)(k + 3 - m)(k + 2 - m)/6(k + 1 - m) \). Fix an
integer $i \in \{1, \ldots, m\}$ and assume $0 < x_{j,k,m} \leq (k + 4 - j)(k + 3 - j)/6$ for all $j \in \{i, i + 1, \ldots, m\}$. Since $x_{j,k,m} > 0$ for all $j > 0$, we have $x_i \leq (k + 4 - i)(k + 3 - i)(k + 2 - j)/6(k + 1 - i)$. Since $(m - i)(k + 3 - i)(k + 2 - i)/6(k - i) + 1 + (k + 1 - i) + (k - i) \leq \binom{k - i + 1}{3}$, we get $b_{i,k,m} > 0$, concluding the proof of Claim 1.

Claim 2. Assume $k \geq m + 2$. Then $x_{i,k,m} \geq x + 1 - i$.

Proof of Claim 2. Assume $x_{i,k,m} \leq x - i$. First assume $i = m$. By the case $m = 1$ and $k' := k - m + 1$ of (1) we get $(k + 1 - m)^2 + 1 > \binom{k + 4 - m}{3}$, i.e. $6(k + 1 - m)^2 \geq (k + 4 - m)(k + 3 - m)(k + 2 - m)$, contradicting the assumption $k > m$. Now assume $i < m$ and that $i$ is the maximal positive integer for which Claim 2 fails. Claim 1 gives $x_{j,k,m} \leq (k + 4 - j)(k + 3 - j)(k + 2 - j)/6(k + 1 - j)$ for all $j$. We have $y_{i,k,m} \leq k - i$ and $y_{i+1,k,m} \geq 0$. Hence from (2) we get

$$(k + 1 - i)(k + 1 - i) + \sum_{j=i+1}^{m} (k + 4 - j)(k + 3 - j)(k + 2 - j)/6(k + 1 - j) \leq \binom{k + 1 - i}{3}$$

Hence

$$6(k + 1 - i)(k + 1 - i)(k - i) + (m - i)(k + 3 - i)(k + 2 - i)(k + 1 - i) \leq (k + 4 - i)(k + 3 - i)(k + 2 - i)(k - i)$$

with strict inequality if $i \neq m - 1$. First assume $i \neq m - 1$. Since $k \geq m + 2$ and $i < m$, we have $k + 4 - i \geq m + 6 - i$. Hence we get a contradiction. Now assume $i = m - 1$. From (4) we get $6(k + 2 - m)(k + 3 - m)(k + 1 - m) + (k + 4 - m)(k + 3 - m)(k + 2 - m) \leq (k + 5 - m)(k + 4 - m)(k + 3 - m)(k + 1 - m)$, a contradiction. Let $C_m \subset H_m$ be a general union of $y_{m,k,m}$ lines and a smooth rational curve of degree $x_{m,k,m} - y_{m,k,m}$. By [2], $C_m$ has maximal rank in $H_m$. By our definition of the integers $x_{m,k,m}$ and $y_{m,k,m}$ we have $h^i(H_m, \mathcal{I}_{C_m,H_m}(k - m + 1)) = 0$, $j = 1, 2$. Let $N_{C_m,H_m}$ denote the normal bundle of $C_m$ in $H_m$. Since each connected component of $C_m$ is smooth and rational and the tangent bundle of $H_m$ is a quotient of $\mathcal{O}_{H_m}(1)^{\oplus 4}$ by the Euler’s sequence, we have $h^1(C_m, N_{C_m,H_m}(-1)) = 0$. By [4] we get that for general $C_m$ each set $C_m \cap H_i$, $1 \leq i < m$, is formed by $x_{m,k,m}$ general points of $H_i$. Fix an integer $i \in \{1, \ldots, m - 1\}$ and assume defined the curves $C_j \subset H_j$, $i + 1 \leq j \leq m$, with the following properties:

(a) $C_j \subset H_j$ is a disjoint union of $y_{j,k,m}$ lines and a smooth rational curve of degree $x_{y,k,m} - y_{j,k,m}$;
(b) Each curve $E_j := \cup_{h=j}^{m} C_h$ is nodal, with exactly $y_{j,k,m}$ connected components, each of them with arithmetic genus 0;

c) Each $C_j$ is transversal to $H_h$ for all $h \neq j$;

d) no $C_j$ contains a point common to 3 of the planes $H_h$;

e) for each $h, j$ such that $1 \leq h < j \leq m$ the set $E_j \cap H_h$ is general in $H_h$.

Part (b) implies that each irreducible component of $E_j$ is smooth and rational. Since $x_{i,k,m} \geq k + 1 - i$ we may find a disjoint union $C_i \subset H_i$ of $x_{i,k,m} - y_{i,k,m}$ lines and a smooth rational curve, such that no point of $C_i$ is contained in two other hyperplanes $H_h$, $h \neq i$, and $E_i := E_{i-1} \cup C_i$ satisfies properties (b) and (c) above. Since for a general $S \subset H_i \cap H_{i-1}$ with $\sharp(S) = x_{i,k,m}$ there is $E \in Z(H_i, x_{i,k,m} - y_{i,k,m}, y_{i,k,m})$ containing it, we may also assume that $C_i$ is a general element of $Z(H_i, x_{i,k,m} - y_{i,k,m}, y_{i,k,m})$ (seen as an abstract curve in $H_i$, independently of the curve $E_{i-1}$ constructed before). Hence $C_i$ has maximal rank as a curve as a curve of $H_i$. We may also assume that $E_i := E_{i+1} \cup C_i$ satisfies condition (d). Set $E_m := C_m$. Since $E_m = C_m \subset H_m$, we have $C_m = E_m \cap H_m$ (scheme-theoretic intersection). For any $i \in \{1, \ldots, m-1\}$ the set $E_i \cap H_i$ is a disjoint union of $C_i$ and the points of $E_{i+1} \cap H_i$ not contained in $C_i$. Each of these points corresponds to a reduced connected component of the scheme-theoretic intersection $E_i \cap H_i$, because $E_i$ is a nodal curve. Now fix $P \in C_i \cap E_{i-1}$ (if any). We have $P \in H_i \cap H_{i-1}$ and $P \in C_{i-1}$. Since $E_i$ is nodal and the tangent line to $C_{i-1}$ is not contained in $H_i$, $E_i \cap H_i$ and $C_i$ coincide in a neighborhood of $P$. Hence the scheme-theoretic and the set-theoretic intersections of $E_i$ and $H_i$ are the same. We again remark that this construction make sense, because $\sum_{j=i+1}^{m} b_j - y_{i-1,k,m} \geq 0$; indeed, $y_{i-1,k,m} \leq (k-i) \leq b_{i-1,k,m} \leq \sum_{j=i+1}^{m} b_j,k,m$, the second inequality being true by Claim 2. For all $i \in \{1, \ldots, m-1\}$ the set $E_i \cap H_i \setminus C_i$ has exactly $\sum_{j=i+1}^{m} b_j,k,m - y_{i-1,k,m}$ points.

**Claim 3.** For each $i \in \{1, \ldots, m\}$ we have $h^j(H_i, I_{E_i \cap H_i} \cdot H_i, H_i(k+i-1)) = 0$, $j = 0, 1$.

**Proof of Claim 3.** The case $i = m$ is true by our choice of the curve $C_m$. Now assume $i \in \{1, \ldots, m-1\}$. The scheme $E_i \cap H_i$ is the disjoint union of $C_i$ and finitely many points. We have $h^0(C_i, \mathcal{O}_{C_i}(k+i-1)) = (k+i-1)x_{i,k,m} + 1 + y_{i,k,m}$. By (3) we have $h^0(E_i \cap H_i, \mathcal{O}_{E_i \cap H_i}(k+1-i)) = \binom{k+1-i}{3}$. Hence $h^0(H_i, I_{E_i \cap H_i} \cdot H_i, H_i(k+i-1)) = h^0(H_i, I_{E_i \cap H_i} \cdot H_i, H_i(k+i-1)) = 0$. Thus it is sufficient to show that either $h^0(H_i, I_{E_i \cap H_i}(k+i-1)) = 0$ or $h^1(H_i, I_{E_i \cap H_i}(k+i-1)) = 0$. In our set-up we have $H := H_i$, $D := C_i$ (which have maximal rank) and
\[ f := \#(S) = -y_{i+1,k,m} + \sum_{j=i+1}^{m} x_{j,k,m}. \] However, \( S \) is not general in \( H_i \), since we prescribed the values of the cardinalities of the sets \( S_j := S \cap H_j, i + 1 \leq j \leq m \) and \( S \subset H_{j+1} \cup \cdots \cup H_m \). We apply \( m - i + 1 \) times Lemma 1.

**Claim 4.** For each \( i \in \{1, \ldots, m\} \) have \( h^j(H_i \cup \cdots \cup H_m, I_{E_i,H_i \cup \cdots \cup H_m}(k)) = 0, j = 0, 1. \)

**Proof of Claim 4.** Since \( E_m = C_m \), Claim 4 is true for \( i = m \). Now assume \( i < m \) and that Claim 4 is true for the integer \( i + 1 \). Since the set-theoretic and the scheme-theoretic intersection of \( H_{i-1} \) and \( E_i \) are the same, we have an exact sequence of sheaves on \( H_i \cup \cdots \cup H_m \):

\[ 0 \to I_{E_{i-1},H_{i-1} \cup \cdots \cup H_m}(k-i) \to I_{E_i,H_i \cup \cdots \cup H_m}(k) \to I_{E_i \cap H_i,H_i}(k+1-i) \to 0 \quad (5) \]

The inductive assumption gives \( h^j((H_{i+1}, i \cup \cdots \cup H_m, I_{E_{i-1},H_{i-1} \cup \cdots \cup E_m}(k-i)) = 0, j = 0, 1. \) Claim 3 gives \( h^j(H_i, I_{E_i \cap H_i}(k+1-i)) = 0, j = 0, 1. \) Apply (5).

**Claim 5.** We have \( h^1(A, I_{E_1,A}(k)) = 0. \)

**Proof of Claim 5.** This is the case \( i = m \) of Claim 4.

Recall that \( d \leq d_{k,m} = \sum_{i=1}^{m} x_{i,k,m} \). Let \( c \) be the minimal integer \( \leq m \) such that \( d \leq \sum_{i=c}^{m} x_{i,k,m} \). First assume \( c = m \). In this case we may take as \( C \) a general element of \( Z(H_m, d, 0) \). Since \( d \leq x_{k+1-m,k,m} \) and \( C \) has maximal rank \((3)\), we have \( h^1(H_m, I_{C,H_m}(k+1-m)) = 0. \) Hence \( h^1(H_m, I_{C,H_m}(k)) = 0. \) Since the restriction map \( H^0(A, O_A(k)) \to H^0(H_m, O_{H_m}(k)) \) is surjective, we get \( h^1(A, I_{C,A}(k)) = 0. \) Hence we may assume \( 1 \leq c < m \). First assume \( d \geq \sum_{i=c+1}^{m} x_{i,k,m} + y_{c+1,k,m} + 1. \) Take as \( C \) the union of \( E_{c-1} \) with a general \( F \in Z(H_c, d - x_{c+1,k,m}, 0) \) with the only restriction that \( F \cap H_{c-1} \) contains exactly one point of each of the connected components of \( E_{c-1} \). Since \( d \geq x_{c-1,k,m} + y_{c-1,k,m} + 1, F \) may be considered as a general smooth rational curve of degree \( d - x_{c+1,k,m} \) of \( H_c \). Hence it has maximal rank \((3)\). As in the proofs of Claims 3 and 4 we get \( h^1(H_c \cup \cdots \cup C_m, I_{C,H_c \cup \cdots \cup C_m}(k+1-c)) = 0. \) Hence \( h^1(H_c \cup \cdots \cup C_m, I_{C,H_c \cup \cdots \cup C_m}(k)) = 0. \) Hence \( h^1(A, I_{C,A}(k)) = 0. \) Now assume \( d \leq \sum_{i=c+1}^{m} x_{i,k,m} + y_{c+1,k,m}. \) Instead of \( E_{c+1} \) we take the following curve \( F_{c+1} \).

We start with \( E_{c+2} \) (with the convention \( E_{c+2} = \emptyset \) if \( c = m - 1 \). Let \( D_{c+1} \) be a general element of \( Z(H_{c+1}, x_{c+1,k,m}, 0) \) with the only condition that \( D_{c+1} \cap H_{c+2} \) contains exactly one point of each of the \( y_{c+2,k,m} + 1 \) connected components of \( E_{c+2} \).

**Proof of Theorem 2.** Take the proof just given for the integer \( k' := k + 1 \) and make minimal modifications.

**Proof of Theorem 3.** Take \( A = H_1 \cup H_2 \subset \mathbb{P}^4 \) with \( H_1 \) and \( H_2 \) hyperplanes and \( H_1 \neq H_2 \). For all integers \( k \geq 3 \) we have \( h^0(A, O_A(k)) = \binom{k+4}{4} - \binom{k+2}{4} = \)
\[ 2^{(k+3)/3} - \binom{k+2}{2} = (k + 2)(k + 1)(2k + 3)/6. \] We have \( h^0(A, \mathcal{O}_A(2)) = 14 \) and \( h^0(A, \mathcal{O}_A(1)) = 4. \) Set \( d_1 := 4, \alpha_1 := 0, d_2 := 6 \) and \( \alpha_2 := 1. \) For all integer \( k \geq 3 \) define the integers \( d_k \) and \( \alpha_k \) by the relations

\[ kd_k + \alpha_k = \binom{k+4}{4} - \binom{k+2}{4}, \quad 0 \leq \alpha_k \leq k - 1 \]  

(a) We have \( d_k := \lfloor ((k + 2)(k + 1)(2k + 3) - 6)/6k \rfloor = \lfloor (2k^2 + 9k + 13)/6 \rfloor. \) Hence

\[ (2k^2 + 9k + 13)/6 - 1 \leq d_k \leq (2k^2 + 9k + 13)/6 \]  

Claim 1. \( d_k > d_{k-1} \) for all \( k \geq 2. \)

Proof of Claim 1. Since \( d_3 = 10, \) we may assume \( k \geq 4. \) Subtracting the case \( k' := k - 1 \) of (6) from (6) we get

\[ (k - 1)(d_k - d_{k-1}) + d_k + \alpha_k - \alpha_{k-1} = (k + 1)^2 \]  

Assume \( d_k \leq d_{k-1}, \) Since \( \alpha_k \leq k - 1, \) step (a) gives a contradiction.

Fix an integer \( d > 0 \) and let \( k \) be the minimal positive integer such that \( d \leq d_k. \) Claim 1 gives that \( k \) is the only integer such that \( d_{k-1} < d \leq d_k. \) To prove Theorem 3 for the integer \( d \) it is sufficient to prove the existence of a reduced, connected and nodal curve \( C \subset A \) such that \( \deg(C) = d, \) \( p_a(C) = 0, \) \( h^0(A, \mathcal{I}_{C,A}(k - 1)) = 0 \) and \( h^1(A, \mathcal{I}_{C,A}(t)) = 0 \) for all \( t \geq k. \) Since \( h^1(C, \mathcal{O}_C) = 0 \) for any reduced and connected curve with arithmetic genus zero, Castelnuovo-Mumford’s lemma gives that it is sufficient to prove the existence of a reduced, connected and nodal curve \( C \subset A \) such that \( \deg(C) = d, \) \( p_a(C) = 0, \) \( h^0(A, \mathcal{I}_{C,A}(k - 1)) = 0 \) and \( h^1(A, \mathcal{I}_{C,A}(t)) = 0. \) The case \( k = 1, \) i.e. \( d \leq 4, \) is obvious.

(b) Assume \( k = 2, \) i.e. assume \( 5 \leq d \leq 7. \) Take as \( C = A_1 \cup A_2 \) the union of rational normal curve \( A_1 \subset H_1 \) and a general \( A_2 \in Z(H_2, d - 3, 0) \) meeting \( C_1 \) at one point.

(c) From now on we assume \( k \geq 3. \) For all integers \( t > 0, \) define the integers \( u_t \) and \( \gamma_t \) by the relations

\[ tu_t + 1 + \gamma_t = \binom{t+3}{3}, \quad 0 \leq \gamma_t \leq t - 1 \]  

The explicit values of the integers \( u_t \) and \( \gamma_t \) given in [3] show that \( u_t > \gamma_t \) for all \( t \geq 2. \) Let \( C_1 \subset H_1 \) be a general union of a smooth rational curve of
degree \( u_{k-1} - \gamma_{k-1} \) and \( \gamma_{k-1} \) lines. By [2] we have \( h^i(H_1, I_{C_1, H_1}(k - 1)) = 0 \), \( i = 0, 1 \). Let \( M_1 \subset H_1 \) be a general union of a smooth rational curve of degree \( u_{k-2} - \gamma_{k-2} \) and \( \gamma_{k-2} \) lines. By [2] we have \( h^i(H_1, I_{M_1, H_1}(k - 2)) = 0 \), \( i = 0, 1 \).

(d) In this step we assume

\[
(k - 1)(d - u_{k-1}) + 1 + u_{k-1} - \gamma_{k-1} \geq \left( \frac{k + 2}{3} \right)
\]  

(10)

**Claim 2.** Assume (10). Then \( d - u_{k-1} \geq \gamma_{k-1} + 1 \).

*Proof of Claim 2.* Assume \( d - u_{k-1} \leq \gamma_{k-1} \). From (10) we get \( (k - 2)\gamma_{k-1} + 1 + u_{k-1} \geq \left( \frac{k + 2}{3} \right) \). Since \( u_{k-1} \leq \left( \frac{k + 2}{3} \right)/(k - 1) \) and \( \gamma_{k-1} \leq k - 2 \), we get a contradiction.

Let \( C_1 \) be a general element of \( Z(H_1, d - u_{k-1}, 0) \), with the only restriction that \( C_1 \cap H_2 \) contains exactly one point of each connected component of \( C_2 \). Set \( C := C_1 \cup C_2 \). Since \( d \leq d_k \), Lemma 1 gives \( h^1(H_1, I_{C \cap H_1, H_1}(k)) = 0 \).

Since \( h^1(H_2, I_{C_2, H_2}(k - 1)) = 0 \), we get \( h^1(A, I_{C, A}(k)) = 0 \). From (10) and the generality of the set \( C_2 \cap H_1 \) in the plane \( H_1 \cap H_2 \), we get \( h^0(H_1, I_{C \cap H_1, H_1}(k - 1)) = 0 \). Since \( h^0(H_2, I_{C_2, H_2}(k - 1)) = 0 \), we get \( h^0(A, I_{C, A}(k - 1)) = 0 \).

(e) In this step we assume \( d \leq d_k - u_{k-1} + u_{k-2} \) and

\[
(k - 1)(d - u_{k-1}) + u_{k-1} - \gamma_{k-1} \leq \left( \frac{k + 2}{3} \right)
\]  

(11)

Let \( M_1 \subset H_1 \) be a general element of \( Z(H_1, d - u_{k-2}, 0) \), with the only restriction that \( M_1 \cap H_2 \) contains exactly one point of each of the components of \( M_2 \); since \( d > d_{k-1} \) and \( d_{k-1} \geq u_{k-2} + k - 3 \), this is obviously possible. In this case we take \( C := M_1 \cup M_2 \). We first check that \( h^0(A, I_{C, A}(k - 1)) = 0 \). Since \( d > d_{k-1} \), this is just a modification of step (b), taking \( k - 1 \) instead of \( k \). The scheme \( C \cap H_1 \) is a general union of a smooth rational curve of degree \( d - u_{k-2} \) and \( u_{k-2} - \gamma_{k-2} \) general points of \( H_2 \). The assumption “\( d \leq d_k - u_{k-1} + u_{k-2} \)” gives \( h^0(C \cap H_1, O_{C \cap H_1}(k)) \leq \left( \frac{k + 3}{3} \right) \). Hence \( h^1(H_1, I_{C_1 \cap H_1, H_1}(k)) = 0 \) (Lemma 1). Hence \( h^1(A, I_{C, A}(k)) = 0 \).

(g) In this step we assume \( d \geq d_k - u_{k-1} + u_{k-2} \) and prove that (10) is satisfied. Indeed, we have \( (k - 1)(d - u_{k-1}) + 1 + u_{k-1} - \gamma_{k-1} \geq (k - 1)(d_k - u_{k-1}) + 1 + u_{k-1} - \gamma_{k-1} + (k + 2)/3 - \gamma_{k-1} \). \( \square \)
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References


