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RADON-NIKODYM THEOREM WITH DANIELL SCHEME

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Abstract: We give a new formulation of the Radon-Nikodym density based on Daniell scheme. Our main result is applicable to arbitrary measure spaces, in particular, not necessarily localizable. Furthermore, we discuss the relationships between our results and localizable measure spaces, and give an example of the measure space which fails to be localizable but has a certain Radon-Nidkoym density for absolutely continuous measures.

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1. Introduction

The Radon-Nikodym theorem, which dates back to the papers [7, 12], is one of the most important theorems in the classical measure theory, and its theory has been already studied and extended to the framework of Daniell scheme to some extent; see [1, 6, 13, 14, 15] among others. In this paper, we give a more comprehensive discussion on the Radon-Nikodym theorem based on a type of Daniell scheme slightly different from the above literatures. In [1, 6, 14, 15], they

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studied different types of Daniell schemes assuming the underlying space Ω is measurable. In most textbook of analysis, it was essential that we assume that Ω is σ-finite. Further, in [9, 13, 15], the authors considered non-σ-finite cases and found necessary and sufficient conditions under which the Radon-Nikodym theorem holds, where they assumed that the whole space Ω is *measurable*. On the other hand, our scheme covers the situation where Ω is not necessarily measurable. It will propose a new method to study measure spaces such that they are σ-rings and that they are not always σ-finite, as a consequence, we shall newly formulate the Radon-Nikodym density, called a *folder*, which allows us to obtain the Radon-Nikodym theorem on non-σ-finite and/or σ-ring measure spaces, in other words, we do not have to work on the σ-finite measure spaces by the framework of the Daniell integral.

The paper is organized as follows: In the remainder of this section, we discuss briefly the Daniell extension procedure. Section 2 contains the definition of the folder and its basic properties. In Section 3, we describe the Daniell integral induced by the density folder. In Section 4, we give the proof of our Radon-Nikodym theorem. In Section 5, we consider some applications.

Now, we will outline the Daniell scheme. In what follows, $\overline{\mathbb{R}}$ denotes the set of all extended real numbers $\mathbb{R} \cup \{\pm\infty\}$. Its conventions follow as usual, $0 \cdot \infty = 0$, for instance.

A vector space $\mathcal{H}$ consisting of all $\mathbb{R}$-valued functions on a set $\Omega(\neq \emptyset)$ is said to be an *elementary function space* if $\mathcal{H}$ is closed under taking absolute value. The functions in $\mathcal{H}$ are called elementary. The set $\mathcal{H}$ is also called a *vector lattice* or a *Riesz space*, if it is a partially ordered vector space closed under taking pointwise maxima, minima of functions h, k, denoted by $h \vee k$, $h \wedge k$, respectively.

A $\mathbb{R}$-valued linear functional $\int$ on $\mathcal{H}$ satisfying

(1) non-negativity: $\mathcal{H} \ni h \geq 0 \Rightarrow \int h \geq 0$,

(2) continuity: $h_n \searrow 0 \Rightarrow \int h_n \to 0$

is said to be an *elementary integral* or a *Daniell integral* [3, 6, 11, 14]. The triplet $(\Omega, \mathcal{H}, \int)$ is called a *Daniell system*.

We denote by $\mathcal{H}^+$ the class of all pointwise limit functions f which can be expressed as the limit of a sequence of the monotone increasing elementary functions [6, 11, 14]. Here, we understand that any function in $\mathcal{H}^+$ assumes its value in $\overline{\mathbb{R}}$. We define the integral of $f \in \mathcal{H}^+$ by $\int f = \lim \int h_n$, where $\{h_n\}_{n=1}^{\infty}$ is a sequence of the monotone increasing elementary functions. The integral on $\mathcal{H}^+$, for which we still write $\int$, is an $\overline{\mathbb{R}}$-valued functional.

Remark. Obviously, $\mathcal{H} \subset \mathcal{H}^+$ and $\int$ on $\mathcal{H}^+$ extends the elementary integral.

A function $f \in \mathcal{H}^+$ is said to be *integrable* if $\int f < \infty$ and we denote the set of all such f by $\mathcal{H}^+_{\text{int}}$. A subset $Z \subset \Omega$ is said to be a *null set*, if it is realized as a subset of $\{f = +\infty\}$ for some $f \in \mathcal{H}^+_{\text{int}}$ (see [6, 14]). A subset of a null set, and a countable union of null sets are still null sets. When a given property holds on Ω except on a null set, we say that the property holds *almost everywhere* on Ω, or "a.e." for short. For example, we can verify $f \in \mathcal{H}^+_{\text{int}}$ takes in $\mathbb{R}$ almost everywhere.

An $\overline{\mathbb{R}}$-valued φ, defined a.e. on Ω, is said to be *measurable* if it is an a.e. limit of a sequence of elementary functions [6]. The set of all measurable functions is denoted by $\mathcal{M}$. (Here, $f \in \mathcal{M}$ takes values in $\overline{\mathbb{R}}$, and $\mathcal{H}^+ \subset \mathcal{M}$.) A subset $D \subset \Omega$ is said to be a *measurable*, or more precisely *Daniell measurable* if $I(D) \in \mathcal{M}$ and we denote the set of all such D by $\mathcal{D}$. The set of all measurable sets forms a σ-ring (in general it is not necessarily Ω is in $\mathcal{D}$). We note that this definition is essentially different from any other definition in [6, 14] and so on.

A function $\varphi \in \mathcal{M}$ is said to be in $\mathcal{L}^+$ if it can be represented as $\varphi = f - g$ a.e. for some $f \in \mathcal{H}^+$ and $g \in \mathcal{H}^+_{\text{int}}$, and we define $\int \varphi := \int f - \int g \in (-\infty, \infty]$.

Remark. Obviously, $\mathcal{H}^+ \subset \mathcal{L}^+ \subset \mathcal{M}$. The integral $\int$ on $\mathcal{L}^+$ is an extension of $\int$ on $\mathcal{H}^+$. The space $\mathcal{L}^+$ is not a vector space and the extended integral $\int$ on $\mathcal{L}^+$ is not linear. But as far as we ignore the difference on a null set, $\mathcal{L}^+$ is closed under addition, multiplication by non-negative constants, $\vee, \wedge$ and taking limits for increasing sequence of $\mathcal{L}^+$. The extended integral $\int$ is closed under addition, and it has non-negative homogeneity, and continuity of increasing sequence of $\mathcal{L}^+$.

We use frequently the fact that non-negative measurable function $\varphi \in \mathcal{M}$ is in $\mathcal{L}^+$ [6, p.115]. Further, for any non-negative $\varphi \in \mathcal{L}^+$, there exist $f \in \mathcal{H}^+$ and $g \in \mathcal{H}^+_{\text{int}}$ such that $\varphi = f - g$ a.e. Since we can choose $g_n \in \mathcal{H}$ with $g_n \nearrow g$, it follows $\varphi_n := f - g_n \in \mathcal{H}^+$ and $\varphi_n \searrow \varphi$ a.e.

If the integral of $\varphi \in \mathcal{L}^+$ is finite, φ is said to be an *integrable* function [6, 14], and the set of all such functions is denoted by $\mathcal{L}$. We deduce $\mathcal{H} \subset \mathcal{H}^+_{\text{int}} \subset \mathcal{L} \subset \mathcal{L}^+$. As far as we ignore the difference on a null set, $\mathcal{L}$ has a linear structure and the integral $\int$ on $\mathcal{L}$ is a real-valued linear functional. Further, the Monotone Convergence Theorem and the Dominated Convergence Theorem remain valid for $\mathcal{L}$. We will use the fact that any $\varphi \in \mathcal{L}$ is finite almost everywhere.

In addition, throughout of this paper we always suppose that

$$h \in \mathcal{H} \Rightarrow h \wedge 1 \in \mathcal{H},$$

the so-called *Stone condition*([14]). This condition guarantees the measurability of the product of measurable functions.

Remark. The above procedure is called a *Daniell scheme*. Several types of the Daniell scheme are described in [1, 5, 6, 11, 14], with different contents and constructions, and are not equivalent to one another. The scheme we adopted is almost the same as adopted in [6], however, the Stone condition in [6] includes the assumption of σ-finiteness of whole space Ω.

2. Folders

In this section, we introduce the notion of folders so that we can discribe the density of the Radon-Nikodym theorem.

Definition 2.1. (1) A subset $E \subset \Omega$ is said to be an *elementary measurable set* if $I(E) \in \mathcal{H}^+$ and all elementary measurable sets is denoted by $\mathcal{E}$. (2) A subset E is said to be an *elementary integrable set* if there exists $\varphi \in \mathcal{H}$ such that $E = \{x \in \Omega \ : \varphi(x) > 1\}$, and all elementary integrable sets is denoted by $\mathcal{E}_0$.

Remark. Since $\mathcal{H}$ and $\mathcal{H}^+$ are closed under $\vee$, $\wedge$, we deduce $\mathcal{E}_0$ and $\mathcal{E}$ are closed under $\cup$, $\cap$. Further, all elementary measurable(integrable) sets are measurable(integrable) with respect to all integral on $\mathcal{H}$.

Given a sequence of measurable subsets $\{E_n\}_{n=1}^\infty$, we write $E_n \nearrow \bigcup_{n=1}^\infty E_n$ to express that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$.

Proposition 2.1. *Let $\{E_n\}_{n=1}^\infty$ be a sequence of measurable subsets.*

(1) *If $\mathcal{E} \ni E_n \nearrow E$ then $E \in \mathcal{E}$.*

(2) *The following assertions are equivalent:*

(a) $E \in \mathcal{E}$,

(b) *there exists $\varphi \in \mathcal{H}^+$ such that $E = \{\varphi > 1\}$,*

(c) *there exists $\varphi \in \mathcal{H}^+$ such that $E = \{\varphi > 0\}$.*

(3) $\mathcal{E}_0 \subset \mathcal{E}$.

(4) *For any $E \in \mathcal{E}$, there exists $E_n \in \mathcal{E}_0$ $(n = 1, 2, \cdots)$ such that $E_n \nearrow E$.*

Proof. (1) It follows easily from $\mathcal{H}^+ \ni I(E_n) \nearrow I(E) \in \mathcal{H}^+$.

(2) We may assume $\varphi \in \mathcal{H}^+$ to be non-negative in (b) and (c), since $\varphi^+ = \varphi \vee 0$ is in $\mathcal{H}^+$ for all $\varphi \in \mathcal{H}^+$. (a) $\Rightarrow$ (b): If $E \in \mathcal{E}$, then $I(E) \in \mathcal{H}^+$, and hence it suffices to set $\varphi = 2I(E)$. (b) $\Rightarrow$ (c): For $\varphi \in \mathcal{H}^+$ of (b), there exists $h_n \in \mathcal{H}; h_n \nearrow \varphi$. Then $h_n - h_n \wedge 1$ is in $\mathcal{H}$ and converges to $\varphi - \varphi \wedge 1$. Since $h_n - h_n \wedge 1 = (h_n - 1) \vee 0$, we deduce that $h_n - h_n \wedge 1$ converges increasingly to $(\varphi - 1) \wedge 0 = \varphi - \varphi \wedge 1$, and this implies $\varphi - \varphi \wedge 1 \in \mathcal{H}^+$. Then we obtain $E = \{\varphi - \varphi \wedge 1 > 0\}$. (c) $\Rightarrow$ (a): Since $(n\varphi) \wedge 1 \in \mathcal{H}^+$, it follows that $(n\varphi) \wedge 1 \nearrow I(E)$ and that $I(E) \in \mathcal{H}^+$.

(3) It follows from (2). Indeed, remark that $\mathcal{H}^+ \supset \mathcal{H}$.

(4) For any $E \in \mathcal{E}$, there exists $\varphi \in \mathcal{H}^+$ such that $E = \{\varphi > 1\}$ by (2). Choose $\varphi_n \in \mathcal{H}$ so that $\varphi_n \nearrow \varphi$ and put $E_n := \{\varphi_n > 1\}$, then $E_n \in \mathcal{E}_0$ and $E_n \nearrow E$. $\square$

Proposition 2.2. (1) *If $\varphi \in \mathcal{M}$, then $\{\varphi \neq 0\} \in \mathcal{D}$.*

(2) *For any $D \in \mathcal{D}$, there exists $E \in \mathcal{E}$ such that $D \subset E$.*

Remark. Recall that $\mathcal{M}$ is the set of all $\overline{\mathbb{R}}$-valued functions φ, defined a.e. on Ω such that φ is an a.e. limit of a sequence of elementary functions [6]. Since $\varphi \in \mathcal{M}$ is defined almost everywhere, we have

$$\{\varphi \neq 0\} = \{|\varphi| > 0\} \cup \{x \in \Omega\,; \varphi(x) \text{ is undefined}\},$$

so that $\{\varphi \neq 0\} = \{|\varphi| > 0\}$ is not the case.

Proof of Proposition 2.2. (1) If $\varphi \in \mathcal{M}$, then $\infty I(\varphi \neq 0) = \infty|\varphi|$ a.e. and $\infty|\varphi| = \lim_{n\to\infty} nI(\varphi \neq 0) \in \mathcal{L}^+$. Noting $I(\varphi \neq 0) = (\infty I(\varphi \neq 0)) \wedge 1$, we deduce $(\infty I(\varphi \neq 0)) \wedge 1 \in \mathcal{L}^+$ by the Stone condition. This implies $I(\varphi \neq 0) \in \mathcal{L}^+$, and hence $\{\varphi \neq 0\}$ is a measurable set.

(2) Since D is measurable, $I(D)$ is in $\mathcal{L}^+$, and hence there exists $f_n \in \mathcal{H}^+$ such that $0 \leq f_n \searrow I(D)$ holds except for some null set Z. There exists $0 \leq f_0 \in \mathcal{H}^+_{\text{int}}$ such that $Z \subset \{f_0 = +\infty\}$, and hence $I(D) \leq f_1 + f_0$ holds everywhere, because if $x \in Z$ then $I(x \in D) \leq f_1(x) + f_0(x) = f_1(x) + \infty$. Since $f_1 + f_0 \in \mathcal{H}^+$, $E := \{f_1 + f_0 > 0\}$ is a desired elementary measurable set. $\square$

Definition 2.2. (1) Let $(f_E)_{E\in\mathcal{E}}$ be a family of functions defined a.e. We call it a *folder*, if

$$f_F I(E) = f_{E\cap F} \text{ a.e.} \tag{2.1}$$

for any $E, F \in \mathcal{E}$, and write $\langle f\rangle := (f_E)_{E\in\mathcal{E}}$. Each f_E is called a *file*.
(2) If $(f_E)_{E\in\mathcal{E}_0}$ satisfies the condition (2.1), for any $E, F \in \mathcal{E}_0$, then we also denote this system by $\langle f\rangle$ and call it a *prefolder*. Each f_E is called file, too.
(3) Let $\langle f\rangle, \langle g\rangle$ be folders. Then, we say that $\langle f\rangle = (\text{or } \leq)\langle g\rangle$ a.e. if $f_E = (\text{or } \leq) g_E$ a.e. for all $E \in \mathcal{E}$. Similarly, for prefolders $\langle f\rangle$ and $\langle g\rangle$, we define $\langle f\rangle = (\text{or } \leq)\langle g\rangle$ a.e. analogously.

Let $(f_E)_{E\in\mathcal{E}}$ be a folder. Obviously, for $E \in \mathcal{E}$, by letting $E = F$ in (2.1), $f_E I(E) = f_E$ a.e. and for $E, F \in \mathcal{E}$,

$$f_E I(F) = f_F I(E) = f_{E\cap F} \tag{2.2}$$

a.e. holds. In addition, for a folder $(f_E)_{E\in\mathcal{E}}$, the restriction $\langle f\rangle|_{\mathcal{E}_0} = (f_E)_{E\in\mathcal{E}_0}$ is a prefolder.

Example 1. A mapping from $E \in \mathcal{E}$ to the indicator function $I(E) \in \mathcal{M}$ is a folder. We denote this folder by $\langle I\rangle$ and call it the *indicator folder.*

Given any prefolder $\langle h\rangle$, it can be extended uniquelly to the folder $\langle f\rangle$ as the following proposition shows:

Proposition 2.3. *For any prefolder $\langle h\rangle$, there exists a folder $\langle f\rangle$ such that:*
(1) $\langle f\rangle|_{\mathcal{E}_0} = \langle h\rangle$ a.e.
(2) and if there exists a folder $\langle g\rangle$ such that $\langle h\rangle = \langle g\rangle|_{\mathcal{E}_0}$ a.e., then $\langle f\rangle = \langle g\rangle$ a.e.

Proof. (1) For any $E \in \mathcal{E}$, there exists $E_n \in \mathcal{E}_0$ such that $E_n \nearrow E$ by Proposition 2.1 (4). Let $E_0 := \emptyset$, and we set a function defined a.e. as:

$$f_E := \sum_{n=1}^{\infty} h_{E_n} I(E_n \setminus E_{n-1}). \tag{2.3}$$

For any sequence $\mathcal{E}_0 \ni F_m \nearrow E$, we have

$$\begin{aligned} f_E I(F_m) &= \sum_n h_{E_n} I(F_m) I(E_n \setminus E_{n-1}) \\ &= \sum_n h_{F_m} I(E_n) I(E_n \setminus E_{n-1}) \\ &= h_{F_m} I(E) = h_{F_m} I(F_m) I(E) = h_{F_m} I(F_m) = h_{F_m}, \end{aligned}$$

where the above equalities hold a.e., and we obtain $f_E = \lim_m h_{F_m}$ a.e. This implies $f_E = \lim_n h_{E_n}$ a.e. holds independently of the choice of a sequence E_n.

Now, we shall prove that $(f_E)_{E\in\mathcal{E}}$ forms a folder. Let $E, F \in \mathcal{E}$, and let $E_n, F_m \in \mathcal{E}_0$ be approximating sequences of E, F, respectively. Then $E_n \cap F_m \nearrow E \cap F$ as $m, n \to \infty$. Since $h_{F_m} I(E_n) = h_{E_n\cap F_m}$ a.e., we have $f_F I(E) = f_{E\cap F}$ a.e. as $m, n \to \infty$, this implies $E \mapsto h_E$ is a folder, which we denote it by $\langle f\rangle$. Obviously $\langle f\rangle|_{\mathcal{E}_0} = \langle h\rangle$ a.e. follows from (2.3).

(2) For any $E \in \mathcal{E}$, choose $E_n \in \mathcal{E}_0$ so that $E_n \nearrow E$. Then $h_{E_n} = g_{E_n}$ a.e., and hence $f_E = g_E$ a.e. as $n \to \infty$. $\square$

Remark. Let φ be a function defined a.e. and $\langle h\rangle$ be a folder. Then $\mathcal{E} \ni E \mapsto \varphi h_E$ is also a folder. We denote this folder by $\varphi\langle h\rangle$. In particular, if we put $\varphi = I(F)$ $(F \in \mathcal{E})$, then we have $I(F)\langle h\rangle = h_F\langle I\rangle$ a.e.

Definition 2.3. We say $\langle f\rangle$ is a *complete* folder if there exists $E_0 \in \mathcal{E}$ such that $f_F = f_{E_0\cap F}$ a.e. holds for any $F \in \mathcal{E}$. The file f_{E_0} is called a complete file of the folder $\langle f\rangle$.

Remark. (1) We say that $\mathcal{H}$ is σ-finite if $1 \in \mathcal{H}^+$ (cf. [14]). This condition is equivalent to $\Omega \in \mathcal{E}$ so that if $\mathcal{H}$ is σ-finite then all folders are complete because we can choose the complete file as h_Ω whenever we are given a folder $\langle h \rangle$.

(2) By definition, $f_{E_0 \cap F} = f_{E_0} I(F)$ a.e. holds, and this implies the complete folder satisfies $\langle f \rangle = f_{E_0} \langle I \rangle = I(E_0)\langle f \rangle$ a.e. The set $E_0 \in \mathcal{E}$, corresponding to the complete file f_{E_0}, is not unique but the complete file is unique as a function as follows:

Proposition 2.4. *Let $\langle f \rangle, \langle g \rangle$ be folders. Suppose that $\langle f \rangle = \langle g \rangle$ a.e., and that $\langle f \rangle$ is complete. Then*

(1) *$\langle g \rangle$ is also complete.*

(2) *Let f_{E_0}, g_{E_1} be complete files of $\langle f \rangle$, $\langle g \rangle$, respectively. Then it follows $f_{E_0} = g_{E_1}$ a.e.*

Proof. (1) We can choose f_{E_0} as a complete file of $\langle g \rangle$.

(2) Since $f_{E_0} I(F) = g_{E_1} I(F)$ a.e. for any $F \in \mathcal{E}$, taking $F = E_0 \cup E_1$, we obtain $f_{E_0} = g_{E_1}$ a.e. □

Proposition 2.4 says a complete folder $\langle f \rangle$ can be naturally identified with its complete file f_{E_0}. Hereafter, unless stated otherwise, $\langle f \rangle$ is abbreviated to f_{E_0}. For example, $\langle f \rangle \in \mathcal{L}$ means $f_{E_0} \in \mathcal{L}$. In general, the indicator folder $\langle I \rangle$ is not necessarily complete.

We say a folder(or prefolder) $\langle h \rangle$ is *measurable* if all its files are measurable. Note that if $\langle f \rangle$ and $\langle g \rangle$ are folders (or prefolders) with $\langle f \rangle = \langle g \rangle$ a.e. and $\langle f \rangle$ is measurable, then $\langle g \rangle$ is also measurable.

Proposition 2.5. (1) *A folder $\langle h \rangle$ is measurable if and only if the prefolder $\langle h \rangle|_{\mathcal{E}_0}$ is measurable.*

(2) *Let $\langle h \rangle$ be a measurable folder and $\varphi \in \mathcal{M}$. Then $\varphi\langle h \rangle$ is measurable and complete.*

Proof. (1) If $\langle h \rangle$ is measurable, then $\langle h \rangle|_{\mathcal{E}_0}$ is obviously measurable. Conversely, suppose that $\langle h \rangle|_{\mathcal{E}_0}$ is measurable. Since $\mathcal{M}$ is closed under taking the a.e. limit, the measurability of $\langle h \rangle$ follows from the proof of Proposition 2.3.

(2) By the Stone condition, $\mathcal{M}$ is closed under multiplication, and this implies each file φh_E of $\varphi\langle h \rangle$ is measurable. We choose $E_0 \in \mathcal{E}; \{\varphi \neq 0\} \subset E_0$ by Proposition 2.2, then, $\varphi h_F = \varphi I(E_0) h_F = \varphi h_{E_0 \cap F}$ a.e. for any $F \in \mathcal{E}$. □

3. Density

Now we describe how to define the linear functional when we are given a folder $\langle f\rangle$.

Definition 3.1. We say the measurable folder $\langle h\rangle$ is a *density folder*, if for every $f \in \mathcal{H}$, $f\langle h\rangle$ is integrable.

Given a density folder $\langle h\rangle$ and $f \in \mathcal{H}$, the folder $f\langle h\rangle$ is complete by Proposition 2.5, where its complete file is fh_{E_0}; there exists $E_0 \in \mathcal{E}$ such that $\{f \neq 0\} \subset E_0$. Note that E_0 depends on f. Now we define the integral $\int f\langle h\rangle := \int fh_{E_0}$. We can show that it does not depend the choice of $E_0 \in \mathcal{E}$ containing the carrer of f.

Proposition 3.1. *Let $\langle h\rangle$ be a folder. The mapping $P : \mathcal{H} \to \mathbb{R}$ with $P(f) = \int f\langle h\rangle$ is linear.*

Proof. Let $\varphi, \psi \in \mathcal{H}$. For any $E \in \mathcal{E}$, φh_E and ψh_E are finite almost everywhere. It follows $(\varphi+\psi)h_E = \varphi h_E+\psi h_E$ a.e., and this implies $(\varphi+\psi)\langle h\rangle = \varphi\langle h\rangle + \psi\langle h\rangle$ a.e. The additivity of P follows from that of $\int$. Homogeneity is obvious. $\square$

Proposition 3.2. *Let $\langle h\rangle$ be a density folder. Then every file of the prefolder $\langle h\rangle_{\mathcal{E}_0}$ is integrable, that is, function h_E is integrable for any $E \in \mathcal{E}_0$.*

Proof. For $E \in \mathcal{E}_0$, there exists $\varphi \in \mathcal{H}; E = \{\varphi > 1\}$, where φ may be assumed non-negative. By the Stone condition, we have $\varphi \wedge 1 \in \mathcal{H}$ and hence, from the definition of E, $(\varphi \wedge 1)I(E) = I(E)$. Then, it follows $(\varphi \wedge 1)h_E = h_E$ a.e. Since the left-hand-side is integrable, h_E is integrable. Indeed, by the Stone condition $(\varphi \wedge 1) \in \mathcal{H}$. Since $\langle h\rangle$ is a density folder, it follows that $(\varphi \wedge 1)h_E$ is integrable. It is also easy to prove that $g \in \mathcal{L}$ and that $\int f = \int g$ whenever $f = g$ a.e. and $f \in \mathcal{L}$. Thus, h_E is integrable. $\square$

Corollary 3.1. (1) *Every file of a density folder $\langle h\rangle$ is finite almost everywhere.* (2) *If $\varphi_n \in \mathcal{H}$; $\varphi_n \searrow 0$ then $P(\varphi_n) \to 0$, where P is a linear mapping from Proposition 3.1.*

Proof. (1) Let $E \in \mathcal{E}$. We aim to prove that h_E is finite a.e. By Proposition 2.1(4), we can choose $\mathcal{E}_0 \ni E_n \nearrow E$. Since $h_{E_n} = h_E I(E_n)$ a.e., we have $I(|h_{E_n}| = +\infty) = I(|h_E| = +\infty)I(E_n)$ a.e. and left-hand-side $= 0$ a.e. by Proposition 3.2. Letting $n \to \infty$, we obtain $I(|h_E| = +\infty)I(E) = I(|h_E| = +\infty) = 0$ a.e.

(2) Let $\varphi_n \in \mathcal{H}; \varphi_n \searrow 0$ and $E \in \mathcal{E}$. By (1), it follows $|\varphi_n h_E| \to 0$ $(n \to \infty)$ a.e. Since we have $|\varphi_n h_E| \leq |\varphi_1 h_E|$ a.e. and $\varphi_1 h_E \in \mathcal{L}$, the Dominated

Convergence Theorem gives $P(\varphi_n) = \int \varphi_n \langle h \rangle \to 0$. $\square$

Theorem 3.1. (1) *Let $\langle h \rangle$ be a non-negative density folder. If another folder $\langle g \rangle$ satisfies $\langle h \rangle = \langle g \rangle$ a.e. then $\langle g \rangle$ is density, and for any $f \in \mathcal{H}$ it follows that $\int f \langle h \rangle = \int f \langle g \rangle$.*

(2) *Conversely, if non-negative density folders $\langle h \rangle, \langle g \rangle$ satisfy $\int f \langle h \rangle = \int f \langle g \rangle$ for any $f \in \mathcal{H}$, then it follows $\langle h \rangle = \langle g \rangle$ a.e.*

Proof. (1) is clear. We will prove (2). Note that $\int f \langle h \rangle = \int f \langle g \rangle$ remains valid for $f \in \mathcal{H}^+$. By Proposition 3.2, for any $E \in \mathcal{E}_0$, h_E and g_E are integrable, then h_E and g_E vanish almost everywhere outside E, and this implies $\{h_E - g_E > 0\} \subset E$ a.e. Hence, $I(h_E - g_E > 0) \in \mathcal{L}$. By the definition of $\mathcal{L}$, there exists $0 \le f_n \in \mathcal{H}^+_{\text{int}}$ such that $f_n \searrow I(h_E - g_E > 0)$ a.e. and hence,

$$f_n \wedge I(E) \searrow I(h_E - g_E > 0)I(E) = I(h_E - g_E > 0) \ \ (a.e.).$$

We write $|\langle h \rangle| = (|h_E|)_{E \in \mathcal{E}}$. Since

$$|(f_n \wedge I(E))\langle h \rangle| = (f_n \wedge I(E))|\langle h \rangle| \le I(E)|\langle h \rangle| = |h_E \langle I \rangle| (\in \mathcal{L}) \ (a.e.),$$

we see $(f_n \wedge I(E))\langle h \rangle$ is integrable for all $n \in \mathbb{N}$. Since $f_n \wedge I(E) \in \mathcal{H}^+_{\text{int}}$,

$$\int f_n \wedge I(E) \langle h \rangle = \int f_n \wedge I(E) \langle g \rangle$$

holds by the first remark in this proof. Thus, the Dominated Convergence Theorem gives

$$\int I(h_E - g_E > 0) h_E = \int I(h_E - g_E > 0) g_E < \infty.$$

This implies $\int I(h_E - g_E > 0)(h_E - g_E) = 0$, and we obtain $h_E \le g_E$ a.e. By a similar argument, we have the opposite inequality. Hence it follows $h_E = g_E$ a.e. for any $E \in \mathcal{E}_0$. Therefore, we obtain $\langle h \rangle = \langle g \rangle$ a.e. by Proposition 2.3. $\square$

Combining Proposition 3.2 and Corollary 3.1, we can easily see the following lemma:

Lemma 3.1. *If the density $\langle h \rangle$ is non-negative, that is, for all $E \in \mathcal{E}$, $h_E \ge 0$ (a.e.), then $P: \mathcal{H} \to \mathbb{R}$ is a Daniell integral on $\mathcal{H}$.*

In the following we shall assume that the density folder $\langle h \rangle$ is non-negative. In particular, the indicator folder $\langle I \rangle$ is non-negative density, and the Daniell integral P induced by $\langle I \rangle$ is nothing else but $\int$. We shall say that $\langle h \rangle$ *is a non-negative density of* P, if the non-negative density $\langle h \rangle$ defines a Daniell integral in the sense just described.

Hereafter, we consider several integrals at the same time. The null sets and the integrabilities depend on each integral, and thereby we will use $\mathcal{H}^+_{\mathrm{int}}(P)$, P-null set, and P-*a.e.* and so on. For simplicity, we may use "a.e." for "$\int$-a.e.".

Proposition 3.3. *Let $f \in \mathcal{H}^+$.*
(1) *Then $f\langle h\rangle$ belongs to $\mathcal{L}^+$, and $P(f) = \int f\langle h\rangle$. remains valid for $f \in \mathcal{H}^+$.*
(2) *If $f \in \mathcal{H}^+_{\mathrm{int}}(P)$ if and only if $f\langle h\rangle \in \mathcal{L}$.*
(3) *If a set $Z \subset \Omega$ is null, then Z is P-null.*

Proof. (1) and (2) follow from definition and convergence theorems.
(3) Let Z be a null set. There exists $f \in \mathcal{H}^+_{\mathrm{int}}$ such that $Z \subset \{f = +\infty\}$.
We claim $\{f = +\infty\}$ is P-null. To do this, fix $E \in \mathcal{E}$ arbitrarily and choose $\mathcal{E}_0 \ni E_n \nearrow E$. We observe that $\mathcal{E} \ni \{f > m\} \searrow \{f = +\infty\} \in \mathcal{D}$. Then we have

$$\mathcal{E} \ni E_n \cap \{f > m\} \searrow E_n \cap \{f = +\infty\} \nearrow E \cap \{f = +\infty\},$$

and $I(E_n \cap \{f > m\})$ is integrable. We apply the Dominated Convergence Theorem to $P(I(E_n \cap \{f > m\})) = \int I(E_n \cap \{f > m\})\langle h\rangle$, and we obtain

$$P(I(E \cap \{f > m\})) = \int I(E \cap \{f > m\})\langle h\rangle.$$

We choose E containing $\{f = +\infty\}$, then the Monotone Convergence Theorem gives

$$P(I(f = +\infty)) = \int I(f = +\infty)\langle h\rangle = 0.$$

It follows $\{f = +\infty\}$ is P-null. □

4. Radon-Nikodym Theorem

Now we formulate the Radon-Nikodym Theorem. Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system satisfying the Stone condition. We consider another Daniell integral Q on $\mathcal{H}$.

Definition 4.1. A Daniell integral Q on $\mathcal{H}$ is said to be absolutely continuous (with respect to $\int$) if any null set is a Q-null set.

Proposition 3.3 (3) implies that the Daniell integral having non-negative density $\langle h\rangle$ is absolutely continuous. The Radon-Nikodym Theorem asserts its converse as follows:

Theorem 4.1. *Suppose that $\mathcal{H}$ satisfies the Stone condition and that Q is a Daniell integral on $\mathcal{H}$.*

(1) *If Q is absolutely continuous, then Q has a non-negative density $\langle h \rangle$.*
(2) *This density is unique in the a.e. sense.*

Let Q be a Daniell integral on $\mathcal{H}$ and suppose that absolutely continuous with respect to $\int$.

Proposition 4.1. (1) *If we define $\left(\int +Q\right)(f) := \int f + Q(f)$ for $f \in \mathcal{H}$, then $(\int +Q)$ is a Daniell integral on $\mathcal{H}$.*
(2) *$(\int +Q)(f) = \int f + Q(f)$ holds for $f \in \mathcal{H}^+$.*
(3) *Z is $(\int +Q)$-null set if and only if Z is null.*
(4) *$\mathcal{M}(\int +Q) = \mathcal{M}$.*
(5) *If $\varphi \in \mathcal{L}^+(\int +Q)$, then $\varphi \in \mathcal{L}^+ \cap \mathcal{L}^+(Q)$ and $(\int +Q)(\varphi) = \int \varphi + Q(\varphi)$.*

Proof. (1) and (2) are evident from the definition of $(\int +Q)$.

(3) ($\Rightarrow$) is clear. ($\Leftarrow$): if Z is null, then Z is Q-null, and hence there exist $f \in \mathcal{H}^+_{\text{int}}$ and $g \in \mathcal{H}^+_{\text{int}}(Q)$ such that $Z \subset \{f, g = +\infty\} = \{f \wedge g = +\infty\}$. Since $f \wedge g \in \mathcal{H}^+$ and

$$\left(\int +Q\right) f \wedge g \le \int f + Q(g) < \infty,$$

which proves Z is $(\int +Q)$-null.

(4) is clear by (3).

(5) If $\varphi \in \mathcal{L}^+(\int +Q)$, then $\varphi = f - g$ $((\int +Q)$-*a.e.*) for some $f \in \mathcal{H}^+$, $g \in \mathcal{H}^+_{\text{int}}(\int +Q)$. By (2), g is in $\mathcal{H}^+_{\text{int}} \cap \mathcal{H}^+_{\text{int}}(Q)$. This implies $\varphi = f - g$ a.e. $(g \in \mathcal{H}^+_{\text{int}})$ and $\varphi = f - g$ (Q-*a.e.*) $(g \in \mathcal{H}^+_{\text{int}}(Q))$ by (3). Thus, we see $\varphi \in \mathcal{L}^+$ and $\varphi \in \mathcal{L}^+(Q)$. The last equation easily follows from the definition of $(\int +Q)$. □

We denote by $\mathcal{L}^2$ the set of all measurable functions φ for which $|\varphi|^2 \in \mathcal{L}$. The set $\mathcal{L}^2$ is a Hilbert space with respect to $(f, g) = \int fg$.

Lemma 4.1. *Suppose that $\mathcal{H}$ satisfies Stone condition and Q is an absolutely continuous Daniell integral on $\mathcal{H}$. Let $E \in \mathcal{E}_0$.*
(1) *There exists a non-negative measurable function h_E such that $h_E = h_E I(E)$ a.e.*
(2) *For any $f \in \mathcal{L}^+(\int +Q)$, it follows that $fh_E \in \mathcal{L}^+$ and that $Q(fI(E)) = \int fh_E$. Furthermore, this h_E is unique in the a.e. sense.*

Proof. Let us fix $E \in \mathcal{E}_0$. For any $f \in \mathcal{L}^2(\int +Q)$, $fI(E)$ is measurable and $f^2 I(E) \le f^2$, and hence $fI(E) \in \mathcal{L}^2(\int +Q)$. By Proposition 4.1 (5), we see $fI(E) \in \mathcal{L}^2(Q)$. In general, we can show that $f \in \mathcal{L}^2(Q)$ if and only if $f \in \mathcal{M}(Q)$ and $Q(|f|^2) < \infty$. From this, f is Q-measurable, and hence by Schwarz's inequality,

$$|Q(|f|I(E))|^2 \le Q(f^2) \cdot Q(I(E)^2)$$

$$\begin{aligned} &\leq M \cdot \left(\int +Q\right) f^2 \quad (M := Q(I(E))) \\ &= M \cdot \|f\|^2_{\mathcal{L}^2(\int +Q)} < \infty. \end{aligned}$$

This implies $F(f) := Q(fI(E))$ is bounded linear functional on $\mathcal{L}^2(Q)$, and on $\mathcal{L}^2(\int +Q)$.

By Riesz's Representation Theorem there exists a unique $g_E \in \mathcal{L}^2(\int +Q)$ for which $fg_E \in \mathcal{L}(\int +Q)$ and

$$Q(fI(E)) = \left(\int +Q\right) fg_E \tag{4.1}$$

for all f in $\mathcal{L}^2(\int +Q)$. Replacing f with $fI(E) \in \mathcal{L}^2(\int +Q)$, we have

$$(\int +Q)fg_E = (\int +Q)fg_E I(E),$$

and hence we obtain $g_E = g_E I(E)$ a.e. by uniqueness.

We shall prove $0 \leq g_E < 1$ a.e. Since $\{g_E < 0\} \subset E$ a.e., $I(g_E < 0) \in \mathcal{L}^2(\int +Q)$. Replacing f with $I(g_E < 0)$ in (4.1), we have $Q(I(g_E < 0)) = \left(\int +Q\right) I(g_E < 0)g_E \leq 0$, and hence $I(g_E < 0) = 0$ Q-a.e. Substituting it for (4.1), we obtain $\int I(g_E < 0)g_E = 0$. This implies $I(g_E < 0)g_E = 0$ a.e., and hence it follows $I(g_E < 0) = 0$ a.e., that is, $g_E \geq 0$ a.e. Similarly, we obtain $I(g_E \geq 1) = 0$ a.e.

Thus, it follows $|fg_E| \leq |f|$ a.e. for any $f \in \mathcal{L}^2(\int +Q)$, and this implies fg_E is in $\mathcal{L}^2(\int +Q)$. By equation (4.1),

$$\begin{aligned} Q(fI(E)) &= \left(\int +Q\right) fg_E = \int fg_E + Q(fg_E) \\ &= \int fg_E + \left(\int +Q\right) fg_E^2 = \int f(g_E + g_E^2) + Q(fg_E^2). \end{aligned}$$

Repeating this procedure, we obtain

$$Q(fI(E)) = \int f(g_E + g_E^2 + \cdots + g_E^n) + Q(fg_E^n). \tag{4.2}$$

Since $0 \leq g_E < 1$ a.e.,

$$\lim_{n\to\infty} Q(fg_E^n) = 0$$

and $0 \leq g_E + g_E^2 + \cdots + g_E^n$ a.e. converges increasingly to a function assuming its value in $\mathbb{R}$ almost everywhere as $n \to \infty$, let h_E denote the limit function.

Observe that h_E is measurable and that $0 \leq h_E < \infty$ a.e. By definition, $h_E = h_E I(E)$ a.e.

To prove (2), we use the truncation argument. We first assume $f \in \mathcal{L}(\int +Q)$ is non-negative. Then $(f \wedge m)I(E)$ is in $\mathcal{L}^2(\int +Q)$ for any $m \in \mathbb{N}$. Noting

$$0 \leq (f \wedge m)(g_E + g_E^2 + \cdots + g_E^n) \nearrow_{n\to\infty} (f \wedge m)h_E \ \ (a.e.) \ \ (\in \mathcal{L}^+)$$
$$(f \wedge m)g_E^n \searrow_{n\to\infty} 0 \quad (Q\text{-}a.e.), \quad \text{(absolute continuity)}$$

we replace f with $(f \wedge m)I(E)$ in (4.2) and applying convergence theorem, it follows $Q((f \wedge m)I(E)) = \int (f \wedge m)h_E$. The Monotone Convergence Theorem gives, $fh_E \in \mathcal{L}^+$ and

$$Q(fI(E)) = \int fh_E. \tag{4.3}$$

This implies fh_E is integrable. For general $f \in \mathcal{L}(\int +Q)$, we apply the same argument to f^+, f^- separately. Since $f^\pm h_E \in \mathcal{L}$ and $Q(f^\pm I(E)) = \int f^\pm h_E$, it follows $Q(fI(E)) = \int fh_E$. If $f \in \mathcal{L}^+(\int +Q)$, there exists $f_n \in \mathcal{L}(\int +Q)$ such that $f_n \nearrow f$ a.e., and hence we obtain the desired equation as $n \to \infty$.

The uniqueness is proved by the same way as the proof of Theorem 3.1 (2). □

Lemma 4.2. *The set $(h_E)_{E\in\mathcal{E}_0}$ is a prefolder, where h_E is defined in Lemma 4.1.*

Proof. Let $E, F \in \mathcal{E}_0$. Replacing $I(E)$ with $I(E \cap F)$ in (4.3), it follows $Q(fI(E \cap F)) = \int fh_{E\cap F}$. On the other hand, replacing f with $fI(F)$ in (4.3) we have $Q(fI(E \cap F)) = \int fI(F)h_E$. By the uniqueness of Lemma 4.1 (2), we obtain $h_{E\cap F} = h_E I(F)$ a.e. □

By Proposition 2.3, we immediately obtain the following lemma:

Lemma 4.3. *Suppose that $\mathcal{H}$ satisfies the Stone condition and that Q is an absolutely continuous Daniell integral on $\mathcal{H}$.*
(1) *There exists a non-negative density $\langle h\rangle$ such that for any $f \in \mathcal{L}^+(\int +Q)$, it follows $f\langle h\rangle \in \mathcal{L}^+$ and*

$$Q(f) = \int f\langle h\rangle. \tag{4.4}$$

(2) *This $\langle h\rangle$ is unique in the a.e. sense.*

Finally, we may take $f \in \mathcal{H}$ in Lemma 4.3 (1) and obtain Theorem 4.1.

5. Applications

We will apply Theorem 4.1 to the classical measure theory. To do this, we introduce more comprehensive definition:

Definition 5.1. Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system with Stone condition.
(1) A function $f : \Omega \to \overline{\mathbb{R}}$ is said to be *locally* (*Daniell*) *measurable* if fh is Daniell measurable for all $h \in \mathcal{H}$.
(2) A folder $\langle h \rangle$ is said to be *weakly complete* if there exists a locally measurable function f_0 such that

$$\langle h \rangle = f_0 \langle I \rangle \ a.e.$$

By definition, all complete folders are weakly complete. We call f_0 *weakly complete file.*

Hereafter, we fix a complete measure space (Ω, Σ, μ). Put $\Sigma_0 := \{A \in \Sigma : \ \mu(A) < \infty\}$, and let $\mathcal{H}(\Sigma_0)$ be the set of all finite linear combinations of indicator functions of the sets of Σ_0. We define the functional $\int$ on $\mathcal{H}(\Sigma_0)$ by

$$\int h := \sum_{k=1}^{n} a_k \mu(A_k), \qquad \left(h = \sum_{k=1}^{n} a_k I(A_k) \right).$$

Then $(\Omega, \mathcal{H}(\Sigma_0), \int)$ is a Daniell system satisfying the Stone condition. Since the measure space is complete, each null set obtained by Daniell scheme is also μ-null set and the converse is true. We see that $\mathcal{E}_0 = \Sigma_0$, and $\mathcal{E} = \{$all countable unions of elements of $\Sigma_0\}$, i.e., $\mathcal{E}$ is the set of all σ-finite sets in Σ. Further, all Daniell measurable functions are Σ-measurable, and all Σ-measurable functions having σ-finite carrier are Daniell measurable. The set $\mathcal{D}$ of all the Daniell measurable sets is a σ-ring generated by the union of the elements of $\mathcal{E}$ and the null sets.

Let ν be a finite measure on Σ and absolutely continuous with respect to μ. If we put $Q(h) := \sum_{k=1}^{n} a_k \nu(A_k)$ for $h = \sum_{k=1}^{n} a_k I(A_k) \in \mathcal{H}(\Sigma_0)$, then Q is a Daniell integral on $\mathcal{H}(\Sigma_0)$ and $Q \ll \int$. By Theorem 4.1, there exists a density folder $\langle h \rangle$ such that

$$Q(f) = \int f \langle h \rangle, \qquad (f \in \mathcal{H}(\Sigma_0)).$$

Let $f = I(F) \in \mathcal{H}^{+}(\Sigma_0)$, we obtain

$$\nu(F) = \int_F h_E d\mu, \quad (F \in \mathcal{E} : \ F \subset E)$$

for $E \in \mathcal{E}$. This means that for any $E \in \mathcal{E}$, h_E plays the role of density function with respect to ν on E. This is nothing but for the measure-theoretic Radon-Nikodym theorem. Conversely, for the measure-theoretic Radon-Nikodym density on each σ-finite set, we can verify that these functions form a folder by the uniqueness.

But in general, there is no single Daniell measurable function (namely, complete file of folder) which connects all these h_E, that is to say, it is impossible to construct a Daniell measurable function h_0 defined on a certain subset of Ω agreeing with h_E on each $E \in \mathcal{E}$. We will consider the conditionnunder which such function h_0 exists.

5.1. σ-Finite Measure Space

Recall the remark of the definition 2.3 below, if the complete measure space (Ω, Σ, μ) is σ-finite, then Ω belongs to $\mathcal{E}$, and hence all folders are complete. This implies the classical Radon-Nikodym Theorem remains valid for the σ-fintie measure space.

5.2. Localizable Measure Space

We will characterize the localizable measure by means of the folders. Let (Ω, Σ, μ) is a complete localizable measure space (cf. [4, 9, 13, 15]). We induce the Daniell system $(\Omega, \mathcal{H}(\Sigma_0), \int)$ in the same way of the above. For any non-negative folder $\langle h \rangle = (h_E)_{E \in \mathcal{E}}$, let $\mathcal{F} := \{h_E : E \in \mathcal{E}\} \subset \mathcal{M}$. Since $\mathcal{F}$ is the subset of Σ-measurable functions, there exists an essential supremum f_0 for $\mathcal{F}$ by the localizability of μ (cf. [13, 15]). It is not difficult to verify that

$$h_E = f_0 I(E) \text{ a.e. for all } E \in \mathcal{E}.$$

The essential supremum f_0 is Σ-measurable but not Daniell measurable. However, we can obtain the following characterization:

Theorem 5.1. *Let (Ω, Σ, μ) be a complete measure space. Then the measure μ is localizable if and only if all non-negative folder $\langle h \rangle$ is weakly complete, and its weakly complete file f_0 is Σ-measurable*

The "if" part is shown as above. We will prove the "only if" part.

Lemma 5.1. *For any Σ-measurable non-negative subcollection $\{f_\lambda : \lambda \in \Lambda\}$, there exists a folder $\langle h \rangle$ such that*

$$f_\lambda \langle I \rangle \le \langle h \rangle \text{ a.e. for all } \lambda \in \Lambda. \tag{5.1}$$

Moreover, we can choose $\langle h\rangle$ *is of minimum, i.e., if there exists another folder* $\langle g\rangle$ *satisfying* (5.1), *then* $\langle h\rangle \le \langle g\rangle$ *holds a.e.*

Proof. Fix $E \in \mathcal{E}$. Then E is a σ-finite measure, so that $\{f_\lambda I(E) : \lambda \in \Lambda\}$ have an essential supremum h_E satisfying $f_\lambda I(E) \le h_E$ a.e., and h_E is Σ-measurable. Obviously, the carrier of h_E is contained in E of σ-finite measure, and this implies h_E is a Daniell measurable function.

We prove $(h_E)_{E\in\mathcal{E}}$ satisfies the folder condition. Indeed, let h_E and $h_{E\cap F}$ be supremums of $\{f_\lambda I(E), \lambda \in \Lambda\}$ and $\{f_\lambda I(E\cap F), \lambda \in \Lambda\}$, respectively. Since $h_E I(F) \ge f_\lambda I(E\cap F)$, $h_E I(F)$ is an upper bound of $\{f_\lambda I(E\cap F), \lambda \in \Lambda\}$. This implies $h_E I(F) \ge h_{E\cap F}$. We set

$$h'_E := h_{E\cap F} + h_E I(E \setminus F),$$

then h'_E is Daniell measurable and

$$\begin{aligned} f_\lambda I(E) &= f_\lambda I(E\cap F) + f_\lambda I(E \setminus F) \\ &\le h_{E\cap F} + h_E I(E\setminus F) = h'_E. \end{aligned}$$

This implies h'_E is an upper bound of $\{f_\lambda I(E), \lambda \in \Lambda\}$. Hence $h_E \le h'_E$ and $h_E I(F) \le h'_E I(F) = h_{E\cap F}$. This implies $h_E I(F) = h_{E\cap F}$ a.e.

The minimality of $\langle h\rangle$ is immediately obtained by the minimality of each h_E. $\square$

Proof of Theorem 5.1. We suffices to consider $\mathcal{A} \subset L^1(\mu)$ (cf. [13, 15]). By Lemma 5.1, there exists a folder $\langle h\rangle$ such that

$$f_\lambda \langle I\rangle \le \langle h\rangle \text{ a.e. for all } \lambda \in \Lambda,$$

and we choose a minimal $\langle h\rangle$. By assumption, there exists an $\mathcal{F}$-measurable non-negative complete file f_0 of the folder $\langle h\rangle$ such that

$$f_\lambda \langle I\rangle \le \langle h\rangle = f_0\langle I\rangle \text{ a.e. for all } \lambda \in \Lambda.$$

Let $\lambda \in \Lambda$. The carrier of $f_\lambda \in L^1(\mu)$ is σ-finite so that we can have

$$f_\lambda \le f_0 I(E) \le f_0 \text{ a.e. for all } \lambda \in \Lambda,$$

where E is containing the carrier of f_λ. This implies that f_0 is an upper bound of $\mathcal{A}$

We will show the minimality of f_0. If there exists an $\mathcal{F}$-measurable g such that $f_\lambda \le g$, then it follows that $f_\lambda\langle I\rangle \le g\langle I\rangle = \langle g\rangle$, where $\langle g\rangle = (gI(E))_{E\in\mathcal{E}}$. By the minimality of $\langle h\rangle$, we obtain $\langle h\rangle \le \langle g\rangle$. This implies $f_0 \le g$. $\square$

Corollary 5.1. *Let (Ω, Σ, μ) be a localizable measure space, and $\nu : \Sigma \to \mathbb{R}$ be a finite measure with $\nu \ll \mu$. Then there is an a.e.-unique Σ-measurable function f_0 such that*

$$\nu(E) = \int_E f_0 \, d\mu \text{ for all } E \in \Sigma_0.$$

5.3. Examples

By Theorem 5.1, a Radon-Nikodym density folder can be determined by a Σ-measurable function if and only if μ is localizable. However, we can find that Theorem 4.1 covers more general situations.
(1) Let $\Omega = [0, 1] \subset \mathbb{R}$, $\Sigma = \{A \subset \Omega : A \text{ is countable or } A^c \text{ is countable}\}$. Let μ be a counting measure on Σ. Then Σ_0 consists of all finite subsets in $[0, 1]$, and Daniell measurable functions $\mathcal{M}$, induced by Daniell system $(\Omega, \mathcal{H}(\Sigma_0), \int)$, is the set of all extended real-valued functions whose carriers are countable subsets of Ω. Therefore, an arbitary function on Ω is locally Daniell measurable, and hence an arbitrary folder $\langle h \rangle$ can be determined by some f_0 with $\langle h \rangle = f_0 \langle I \rangle$. To the contrary, the measure space (Ω, Σ, μ) is known to be non-localizable [4].
(2) In view of this, we consider the following two measures:

$$\mu_0(E) := \sum_{a \in \mathbb{R}} \delta_a(E), \quad \mu^*(E) = \sum_{a \in \mathbb{R}} \varphi(a)\delta_a(E)$$
$$\text{for arbitrary function } \varphi : \mathbb{R} \to (0, \infty)$$

where δ_a is a Dirac measure and E is an element of the countable-cocountable σ-algebra Σ on Ω. Note that the only μ_0-null set is an emptyset, then (Ω, Σ, μ_0) is a complete measure space. Moreover, it is non-localizable measure space and $\mu^* \ll \mu_0$.

We observe that

$$\Sigma_0 := \{B \in \Sigma : \mu_0(B), \mu^*(B) < \infty\} = \{A \subset \Omega : \text{ finite set}\},$$

and $\mathcal{E}$ consists of all countable subset of Ω. Then, Daniell measurable functions $\mathcal{M}(\Sigma_0)$ consists of all extended-real-valued functions having countable carrier.

By Theorem 4.1, we can find unique density folder $\langle h \rangle$ such that

$$\mu^*(E) = \int_E \langle h \rangle \, d\mu_0 \text{ for all } E \in \mathcal{E}.$$

Furthermore, a simple observation shows

$$\varphi I(E) = h_E \text{ for all } E \in \mathcal{E},$$

where φ is not Σ-measurable but locally Daniell measurable, so that we can obtain

$$\mu^*(E) = \int_E \varphi \, d\mu_0 \text{ for all } E \in \mathcal{E}.$$

This is an example showing that the Radon-Nikodym Theorem remains valid for non-localizable measure.

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