International Journal of Pure and Applied Mathematics
Volume 88 No. 3 2013, 437-442
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v88i3.11
ijpam.eu

CHARACTERIZATION OF VOLTERRA SPACES

Milan Matejdes
Department of Mathematics and Descriptive Geometry
Technical University in Zvolen
T.G. Masaryka 24, SK-960 53 Zvolen, SLOVAKIA
and
Department of Mathematics
Faculty of Applied Informatics
Tomas Bata University in Zlín
Nad Stráněmi 4511, CZ-760 05 Zlín, CZECH REPUBLIC

Abstract: In this paper we deal with a characterization of weak Volterra and Volterra spaces which is based on a general concept derived from a given nonempty system $\mathcal{E}$ of nonempty subsets of a topological space. It is shown that a system of all preopen G_δ-sets is suitable for characterization of weak Volterra and Volterra spaces.

AMS Subject Classification: 54A05, 54E52
Key Words: Volterra space, weakly Volterra space, cluster system, ideal, $\mathcal{J}$-resolvable space, preopen G_δ-set, boundary F_σ-set, σ-nowhere dense set

1. Introduction

In the sequel, X is a nonempty topological space. By $\overline{A}$, A° we denote the closure, the interior of A (in space X), respectively and $\mathcal{U}(z)$ is a base of the open neighborhoods of a point z.

By [1], [5] a topological space X is weakly Volterra (Volterra), if for any

Received: July 23, 2013

G_δ-sets A_1 and A_2 which are dense in X, $A_1 \cap A_2$ is nonempty (dense in X) and a nonempty open subset G is weakly Volterra (Volterra), if G as a subspace of X is weakly Volterra (Volterra) what is equivalent with condition $A_1 \cap A_2 \neq \emptyset$ ($A_1 \cap A_2$ is dense in G) for any G_δ-sets (in space X) A_i such that $G \subset \overline{A_i}$, $i = 1, 2$.

Considering so called cluster system the next notions and definitions were introduced in [6].

Definition 1. A nonempty system $\mathcal{E} \subset 2^X \setminus \{\emptyset\}$ is called a cluster system. For a set A, by $\mathcal{E}(A)$ we denote the set of all points $x \in X$ such that for any $U \in \mathcal{U}(x)$, the intersection $U \cap A$ contains a set form $\mathcal{E}$. A set A is called weakly $\mathcal{E}$-Volterra if for any two sets A_1 and A_2, such that $\mathcal{E}(A) \subset \mathcal{E}(A_i)$, $i = 1, 2$, $A_1 \cap A_2$ is nonempty and A is $\mathcal{E}$-Volterra if $A \neq \emptyset$ and $A_1 \cap A_2$ is dense in A for any A_i for which $\mathcal{E}(A) \subset \mathcal{E}(A_i)$, $i = 1, 2$. Finally, if any nonempty open subset of a nonempty open set G contains a set from $\mathcal{E}$, then $\mathcal{E}$ is called a π-network in G.

In [1] a few results can be found which have a nice analogy with Banach category theorem. Namely, the union of any family of nonempty open non weakly Volterra subspaces is not weakly Volterra ([1, Lemma 3.5], [5]). Moreover, X is Volterra if and only if any nonempty open subspace is weakly Volterra [1, Lemma 3.3]. Similar results hold in $\mathcal{E}$-Volterra and weakly $\mathcal{E}$-Volterra setting.

Theorem 1. (see [6]) *Let $\mathcal{E}$ be a π-network in a nonempty open set $X_0 \subset X$.*

(1) X_0 is $\mathcal{E}$-Volterra if and only if any nonempty open subset of X_0 is weakly $\mathcal{E}$-Volterra.

(2) The union of any family of nonempty open non weakly $\mathcal{E}$-Volterra subsets of X_0 is not weakly $\mathcal{E}$-Volterra.

For further results concerning weakly $\mathcal{E}$-Volterra and $\mathcal{E}$-Volterra spaces see [6].

In [3] the notion of $\mathcal{J}$-resolvability was introduced, where $\mathcal{J}$ is an ideal in X. A set A is $\mathcal{J}$-dense if for any nonempty open set U the intersection $A \cap U$ is not from $\mathcal{J}$. A space X is $\mathcal{J}$-resolvable if there are two disjoint subsets of X which are $\mathcal{J}$-dense and X is resolvable if it is $\mathcal{J}$-resolvable, where $\mathcal{J} = \{\emptyset\}$ (i.e., if there are two disjoint dense sets in X). Put $\mathcal{E}_\mathcal{J} := \{A \subset X : A \text{ is not from } \mathcal{J}\}$. If X is $\mathcal{J}$-resolvable, then X is not weakly $\mathcal{E}_\mathcal{J}$-Volterra. The opposite implication holds, provided $\mathcal{J}$ is proper and $\mathcal{E}_\mathcal{J}(X) = X$. If $\mathcal{J} = 2^X$, then X is not $\mathcal{J}$-resolvable and $\mathcal{E}_\mathcal{J} = \emptyset$ what is by Definition 1 excluded. If $\mathcal{E}_\mathcal{J}(X) \neq X$, then X is not $\mathcal{J}$-resolvable (there is no $\mathcal{J}$-dense set). Consider $X = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

with the usual topology and $\mathcal{J} = \{A : A \text{ is finite}\}$. It is clear there are two infinite disjoint sets A_1, A_2 for which $\mathcal{E}_{\mathcal{J}}(A_1) = \mathcal{E}_{\mathcal{J}}(A_2) = \mathcal{E}_{\mathcal{J}}(X) = \{0\}$, so X is not weakly $\mathcal{E}_{\mathcal{J}}$-Volterra.

A set A is called a boundary F_σ-set (σ-nowhere dense set, see [2]) if $A^\circ = \emptyset$ and A is F_σ. The smallest ideal that contains all boundary F_σ-sets will be denoted by $\mathcal{J}_\sigma$. As is showed in [2], if X is $\mathcal{J}_\sigma$-resolvable (strong irresolvable, i.e., any nonempty open subspace of X is irresolvable), then X is Volterra. The converse is not true, see [2, Example 3.8]. That means, "Volterraness" is not characterized by the ideal $\mathcal{J}_\sigma$ (by $\mathcal{E}_{\mathcal{J}_\sigma}$). It seems the notion of $\mathcal{J}$-dense set stated above by condition $A \cap U \notin \mathcal{J}$ (equivalently $A \cap U \in \mathcal{E}_{\mathcal{J}}$) for any nonempty open set U is not adequate. A cluster system setting is more flexible and a suitable specification of $\mathcal{E}$ allows to characterize "Volterraness". Replacing the system $\mathcal{E}_{\mathcal{J}_\sigma}$ by a system of all sets which are preopen and G_δ, we can obtain a characterization of weakly Volterra and Volterra spaces as we will see in the next section.

2. Weakly Volterra and Volterra Spaces

In this paragraph we will deal with a special case of a cluster system which characterizes weakly Volterra and Volterra spaces. Consider a cluster system $\mathcal{E}^p_\delta$ of all sets which are nonempty preopen and G_δ (a set A is preopen if $A \subset (\overline{A})^\circ$, see [4]). More precisely, $A \in \mathcal{E}^p_\delta$ (we will refer A as a preopen G_δ-set) if and only if it is G_δ and there is a nonempty open set $G \supset A$ such that A is dense in G or equivalently, A is G_δ and there is a nonempty open set $G \supset A$ such that $G \setminus A$ is a subset of some boundary F_σ-set (namely $G \setminus A \subset \overline{G} \cap (X \setminus A)$). Evidently, $\mathcal{E}^p_\delta$ is not an ideal, but it is closed with respect to finite union.

Lemma 1. *Let $\{A_t\}_{t\in T}$ be a family of G_δ-sets and $\{G_t\}_{t\in T}$ be a family of nonempty open pairwise disjoint sets such that $A_t \subset G_t$. Them $\cup_{t\in T} A_t$ is a G_δ-set.*

Proof. Let $A_t = \cap_{n=1}^\infty H_t^n$, $H_t^n \subset G_t$ and H_t^n be open. Since $G_t \cap G_s = \emptyset$ for any $t \neq s$, $\cup_{t\in T} A_t = \cap_{n=1}^\infty (\cup_{t\in T} H_t^n)$ that is a G_δ-set. □

Theorem 2. *A topological space X is Volterra if and only if any nonempty open subset of X is weakly $\mathcal{E}^p_\delta$-Volterra.*

Proof. "⇒" Note that $\mathcal{E}^p_\delta$ is a π-network in any nonempty open set G, so $\mathcal{E}^p_\delta(G) = \overline{G}$.

Let X be Volterra and G be nonempty open. Suppose G is not weakly $\mathcal{E}^p_\delta$-Volterra. Then there are two sets A, B such that $\mathcal{E}^p_\delta(A) \supset \mathcal{E}^p_\delta(G), \mathcal{E}^p_\delta(B) \supset \mathcal{E}^p_\delta(G)$ and $A \cap B = \emptyset$. Since $\mathcal{E}^p_\delta(A) \supset \mathcal{E}^p_\delta(G) = \overline{G} \supset G$, there is A_0 from $\mathcal{E}^p_\delta$ and $A_0 \subset G \cap A$. Further, there is a nonempty open set G_0 (we can suppose $G_0 \subset G$) such that $A_0 \subset G_0$ and A_0 is dense in G_0.

Since $\mathcal{E}^p_\delta(B) \supset \mathcal{E}^p_\delta(G) = \overline{G} \supset G_0$, there is B_0 from $\mathcal{E}^p_\delta$ and $B_0 \subset G_0 \cap B$. Further there is a nonempty open set B_1 (we can suppose $B_1 \subset G_0$) such that $B_0 \subset B_1$ and B_0 is dense in B_1. Note $A_0 \cap B_0 = \emptyset$ ($A_0 \cap B_0 \subset A \cap B = \emptyset$).

Since A_0 and B_0 are G_δ and dense in B_1, $A_1 := (X \setminus \overline{B}_1) \cup (A_0 \cap B_1)$ and $A_2 := (X \setminus \overline{B}_1) \cup B_0$ are dense in X and G_δ. Since X is Volterra, then $A_1 \cap A_2 = X \setminus \overline{B}_1$ is dense in X, contradiction.

" $\Leftarrow$ ". Suppose any nonempty open subset of X is weakly $\mathcal{E}^p_\delta$-Volterra and A, B are G_δ and dense in X, that means $\mathcal{E}^p_\delta(A) = \mathcal{E}^p_\delta(B) = X$. Let G be a nonempty open set. Then $A \cap G, B \cap G$ are G_δ dense in G, so $\mathcal{E}^p_\delta(A \cap G) \supset \mathcal{E}^p_\delta(G)$ and $\mathcal{E}^p_\delta(B \cap G) \supset \mathcal{E}^p_\delta(G)$. Since G is weakly $\mathcal{E}^p_\delta$-Volterra, $(A \cap G) \cap (B \cap G) = A \cap B \cap G \neq \emptyset$. So $A \cap B$ is dense in X, that means X is Voltera. □

The next corollary follows from Theorem 2 and Theorem 1 item (1).

Corollary 1. *A topological space X is Volterra if and only if X is $\mathcal{E}^p_\delta$-Volterra.*

Theorem 3. *A nonempty open subset G of X is weakly Volterra if and only if G is weakly $\mathcal{E}^p_\delta$-Volterra.*

Proof. "$\Rightarrow$" Let G be nonempty open weakly Volterra and A^1, A^2 be such that $\mathcal{E}^p_\delta(A^i) \supset G$, $i = 1, 2$. Suppose contrary $A^1 \cap A^2 = \emptyset$.

From inclusion $\mathcal{E}^p_\delta(A^i) \supset G$ and from Zorn lemma there is a family $\{G_t\}_{t \in T}$ of pairwise disjoint nonempty open subsets of G and two families $\{A^i_t\}_{t \in T}$ ($i = 1, 2$) of sets from $\mathcal{E}^p_\delta$ such that for $i = 1, 2$

(1) $\cup_{t \in T} G_t$ is dense in G,

(2) $A^i_t \subset G_t \cap A^i$ and A^i_t is dense in G_t.

Put $C^i = \cup_{t \in T} A^i_t$, $i = 1, 2$. Since $A^1 \cap A^2 = \emptyset$ and $G_t \cap G_s = \emptyset$ ($t \neq s$), $C^1 \cap C^2 = \emptyset$. Moreover, C^i is dense in $\cup_{t \in T} G_t$ and by item (1), C^i is dense in G, $i = 1, 2$. By Lemma 1, C^i is a G_δ-set and since G is weakly Voltera, $\emptyset \neq C^1 \cap C^2$, a contradiction.

The opposite implication is clear. □

The next corollary follows from Theorem 2 and Theorem 3.

Corollary 2. (see [1, Lemma 3.3]) *A topological space X is Volterra if and only if any its nonempty open set is weakly Volterra.*

Theorem 4. *A nonempty open set G is not weakly Volterra if and only if $G \subset P \cup R$ where P, R are boundary F_σ-sets.*

Proof. " $\Rightarrow$ " Since G is not weakly Volterra, their are two disjoint G_δ-subsets G_1, G_2 of G which are dense in G. Then $G \subset [\,\overline{G} \cap (X \setminus G_1)\,] \cup [\,\overline{G} \cap (X \setminus G_2)\,]$. We will show that $A_i := \overline{G} \cap (X \setminus G_i)$ has empty interior, $i = 1, 2$. If not, there is a nonempty open set $H \subset \overline{G} \cap (X \setminus G_i)$. Since $H_0 := H \cap G \neq \emptyset$ and $H_0 \cap G_i = \emptyset$, we have a contradiction with the assumption that G_i is dense in G. So $P := [\,\overline{G} \cap (X \setminus G_1)\,]$ and $R := [\,\overline{G} \cap (X \setminus G_2)\,]$ are boundary F_σ-sets and $G \subset P \cup R$.

" $\Leftarrow$ " If $G \subset P \cup R$ and P, R are boundary F_σ-sets, then $[\,G \cap (X \setminus P)\,]$ and $[\,G \cap (X \setminus R)\,]$ are disjoint G_δ-subsets of G which are dense in G, so G is not weakly Volterra. □

Corollary 3. *A topological space X is weakly Volterra (Volterra) if and only if X (any nonempty open set $G \subset X$) is not (a subset of) the union of two boundary F_σ-sets.*

By [2, Theorem 2.5], any subset of X that is union of finite many σ-nowhere dense sets (i.e., boundary F_σ-sets) can be expressed as the union of exactly two σ-nowhere dense sets (two boundary F_σ-sets). So we have the following ([2, Proposition 3.3 and Lemma 3.4], [5]).

Corollary 4. *A topological space X is weakly Volterra (Volterra) if and only if $X \notin \mathcal{J}_\sigma$ ($G \notin \mathcal{J}_\sigma$ for any nonempty open set $G \subset X$).*

Acknowledgments

Partially supported by APVV-0269-11.

References

[1] J. Cao, D. Gauld, Volterra spaces revisited, *J. Aust. Math. Soc.*, **79** (2005), 61-76, **doi:** 10.1017/S1446788700009332.

[2] J. Cao, S. Greenwood, The ideal generated by σ-nowhere dense sets, *Applied General Topology*, **7(2)** (2006), 253-264.

[3] J. Dontchev, M. Ganster, D. Rose, Ideal Resolvability, *Topology Appl.*, **93** (1999), 1-16, **doi:** 10.1016/S0166-8641(97)00257-5.

[4] M. Ganster, Preopen sets and resolvable spaces, *Kyungpook Math. J.*, **27** (1987), 135-143.

[5] D. Gauld, S. Greenwood, Z. Piotrowski, On Volterra spaces III:topological operations, *Topology Proc.*, **23** (1998), 167-182.

[6] M. Matejdes, Generalized Volterra spaces, *Int. J. Pure Appl. Math.*, **85** (2013), 955-963, **doi:** 10.12732/ijpam.v85i5.13.