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CURVES WITH SEVERAL THETA-CHARACTERISTICS WITH A PRESCRIBED NUMBER OF SECTIONS: A QUESTION LIST

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Abstract: Let ${}^{[s]}\mathcal{S}_g$, $s > 0$, $g \geq 2$, be the moduli scheme of all $(C, L_1, \dots, L_s)$, where C is a smooth curve of genus g and each L_i is a theta-characteristic. For all $r_i > 0$, $1 \leq i \leq s$, set ${}^{[s]}\mathcal{S}_g^{r_1,\dots,r_s} := \{(C, L_1, \dots, L_s) \in {}^{[s]}\mathcal{S}_g : h^0(L_i) = r_i + 1$ for all $i\}$. Here we raise several questions on these algebraic sets. We show that if T is a general element of a component of ${}^{[s]}\mathcal{S}_g^{r_1,\dots,r_s}$ with the expect codimension $\sum_{i=1}^{s} \binom{r_i+1}{2}$ and $(C, L_1, \dots, L_s)$ is general in T, then each L_i has no base points.

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1. Several Thetas

For all integers $g \geq 2$ let $\mathcal{S}_g$ be the moduli scheme of spin curves of genus g, i.e. the coarse moduli scheme of all pairs (C, L), where $C \in \mathcal{M}_g$, $L \in \mathrm{Pic}^{g-1}(C)$ and $L^{\otimes 2} \cong \omega_C$ ([2], [3], [4]). Let $\pi : \mathcal{S}_g \to \mathcal{M}_g$ be the forgetful map $(C, L) \mapsto C$. For each integer $r \geq 0$ set $\mathcal{S}_g^r := \{(C, L) \in \mathcal{S}_g : h^0(C, L) = r + 1\}$. For every irreducible component T of $\mathcal{S}_g^r$ the variety T has codimension $\leq \binom{r+1}{2}$ in $\mathcal{S}_g$ ([2, p. 616], [3, Corollary 2]). Indeed, $\binom{r+1}{2}$ is the " expected codimension " of an irreducible component of $\mathcal{S}_g^r$.

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The fiber product of s copies of the map π induces a finite map $\pi_s : {}^{[s]}\mathcal{S}_g \to \mathcal{M}_g$ (just the forgetful map $(C, L_1, \dots, L_s) \mapsto C$). Hence ${}^{[s]}\mathcal{S}_g$ parametrizes all $(s+1)$-ples $(C, L_1, \dots, L_s)$ with $C \in \mathcal{M}_g$, $L_i \in \mathrm{Pic}(C)$ and $L_i^{\otimes 2} \cong \omega_C$. For all positive integers s, r_i, $1 \le i \le s$, let ${}^{[s]}\mathcal{S}_g^{r_1,\dots,r_s}$ be the set of all $(s+1)$-ples $(C, L_1, \dots, L_s)$ with $(C, L_i) \in \mathcal{S}_g$ for all i, $L_i \neq L_j$ for all $i \neq j$, and $h^0(C, L_i) = r_i + 1$ for all i. An irreducible component T of ${}^{[s]}\mathcal{S}_g^{r_1,\dots,r_s}$ has the expected dimension or the expected codimension if $\pi_s(T)$ has dimension $3g-3-\sum_{i=1}^{s}\binom{r_i+1}{2}$, i.e. it has codimension $\sum_{i=1}^{s}\binom{r_i+1}{2}$. Since $\mathcal{M}_g$ has quotient singularities and each irreducible component of $\mathcal{S}_g^r$ has codimension at most $\binom{r+2}{2}$, the integer $\sum_{i=1}^{s}\binom{r_i+1}{2}$ is the minimal codimension of $\pi_s(A)$ for any irreducible component $A \subseteq {}^{[s]}\mathcal{S}_g^{r_1,\dots,r_s}$.

For any $L_1, \dots, L_s \in \mathrm{Pic}(C)$ such that $L_i^{\otimes 2} \cong L_j^{\otimes 2}$ for all i, j we have a joint Gaussian map

$$\gamma_{L_1,\dots,L_s} : \wedge^2(H^0(C, L_1)) \oplus \cdots \oplus \wedge^2(H^0(C, L_s)) \to H^0(C, \omega_C \otimes L_1^{\otimes 2})$$

taking the sum of each Gaussian map $\gamma_{L_i} : \wedge^2(H^0(C, L_i)) \to H^0(C, \omega_C \otimes L_1^{\otimes 2})$. When each L_i is a theta we get a linear map

$$\gamma_{L_1,\dots,L_s} : \wedge^2(H^0(C, L_1)) \oplus \cdots \oplus \wedge^2(H^0(C, L_s)) \to H^0(C, \omega_C^{\otimes 2})$$

From the classical case $s = 1$ ([3, Theorem 1]) we immediately get that to prove that ${}^{[s]}\mathcal{S}_g^{r_1,\dots,r_s}$ is smooth and of the expected dimension in a neighborhood of $(C, L_1, \dots, L_s)$ it is sufficient to prove that $\gamma_{L_1,\dots,L_s}$ is injective. Conversely, take any irreducible component Γ of ${}^{[s]}\mathcal{S}_g$ with the expected dimension $3g-3-\sum_{i=1}^{s}\binom{r_i+1}{2}$. Then $\gamma_{L_1,\dots,L_s}$ is injective for a general $(C, L_1, \dots, L_s) \in \Gamma$.

Question 1. Take $r_i = 1$ for all i and fix $s \ge 1$. Take any integer $g \ge 3s$ and any irreducible component $\Gamma \subseteq {}^{[s]}\mathcal{S}_g^{1,\dots,1}$ of the expected codimension, s. Is it true that for a general $(C, L_1, \dots, L_s) \in \Gamma$ the line bundles $L_1, \dots, L_s$ are the only theta-characteristics R on C with $h^0(C, R) \ge 2$?

Question 1 is true if $s = 1$ by [4, Corollary 2.15 and Theorem 2.16]. The following weaker question seems to be a safe bet.

Question 2. Fix an integer $s \ge 2$. Take $r_i = 1$ for all i. Prove the existence of an integer $g_0(s)$ such that for every $g \ge g_0(s)$ and every irreducible component $\Gamma \subseteq {}^{[s]}\mathcal{S}_g^{1,\dots,1}$ of the expected codimension, s, for a general $(C, L_1, \dots, L_s) \in \Gamma$ and any theta-characteristic R on C we have $h^0(C, R) \ge 2$ if and only if $R \cong L_i$ for some i.

We may even try the following far stronger conjecture.

Question 3. Fix integers $s \geq 2$ and $r_i > 0$, $1 \leq i \leq s$. Prove the existence of an integer $g_0(s; r_1, \dots, r_s)$ such that for every $g \geq g_0(s; r_1, \dots, r_s)$ and every irreducible component $\Gamma \subseteq {}^{[s]}\mathcal{S}_g^{r_1,\dots,r_s}$ of the expected codimension, s, such that for a general $(C, L_1, \dots, L_s) \in \Gamma$ and any theta-characteristic R on C we have $h^0(C, R) \geq 2$ if and only if $R \cong L_i$ for some i.

Question 4. Take $s = 2$ and $r_1 = r_2 = 1$. Is ${}^{[2]}\mathcal{S}_g^{1,1}$ irreducible?

Question 5. Fix an integer $s \geq 2$. If there an integer g_s such that ${}^{[s]}\mathcal{S}_g^{1,\dots,1}$ (or $\pi_s({}^{[s]}\mathcal{S}_g^{1,\dots,1})$) is irreducible for all $g \geq g_s$?

Theorem 1. *Fix positive integers g, s, r_i, $1 \leq i \leq s$, such that $\sum_{i=1}^{s} \binom{r_i+1}{2} \leq g-2$. Let T be an irreducible component ${}^{[s]}\mathcal{S}_g^{r_1,\dots,r_s}$ with the expected dimension and let $E \subseteq T$ be an irreducible family of dimension $\geq 2g-1$. Fix a general $(C, L_1, \dots, L_s) \in E$. Assume that $\gamma_{L_1,\dots,L_s}$ and $\gamma_{R_1,\dots,R_s}$ have the same rank, where $(D, R_1, \dots, R_s)$ is a general element of T. Then:*

1. *C is not a multiple covering of a curve of genus > 0.*

2. *Each L_i is base point free.*

3. *Fix $i \in \{1, \dots, s\}$ such that $r_i \geq 3$. Then $|L_i|$ is birational onto its image.*

Proof. Since $\dim(\pi_s(T)) = \dim(T) \geq 2g-1$ and C is a general in $\pi_s(T)$, C is not a multiple covering of a curve of positive genus. Fix $i \in \{1, \dots, s\}$. Since T has the expected dimension, $\gamma_{R_1,\dots,R_s}$ is injective. Hence $\gamma_{L_1,\dots,L_s}$ is injective. Hence γ_{L_i} is injective. Look at [4, Definition 2.9]. The linear map γ_{L_i} is just the restriction to $\langle G \rangle$ of the map m considered in [4, eq. (2.4)]. L_i has no base points by [4, Proposition 2.11] (recall that as in the proof of [4, Proposition 2.11] to get $H^1(N) = 0$ we use [4, Lemma 1.5] if $r_i \geq 2$ (we have $\dim(E) > g$ and hence we may apply it), while $H^1(N) = 0$ for every pencil, by the definition of the normal sheaf in [1]). Now assume $r_i \geq 3$ and that the morphism u induced by $|L_i|$ is not birational onto its image. Since C is not a multiple covering of a curve of positive genus, $u(C)$ is a rational curve, maybe singular. Since L_i has no base points, we get the existence of integers $m \geq 2$ and $t > 0$ and a degree m morphism $f : C \to \mathbb{P}^1$ such that $L_i \cong f^*(\mathcal{O}_{\mathbb{P}^1}(t))$. Since u is induced by the complete linear system $|L_i|$ and $h^0(C, L_i) = r_i + 1$, we have $t = r_i$. Since $t \geq 3$, we have $\binom{t+1}{2} > 2t-1$. Hence the map $\gamma_{\mathcal{O}_{\mathbb{P}^1}(t)} : \wedge^2(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t))) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t)))$, is not injective. Hence γ_{L_i} is not injective, a contradiction $\square$

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