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ON POINT SPECTRUM OF SUBSPACE-HYPERCYCLIC OPERATORS

Mansooreh Moosapoor
Farhangian University
Postcode 4166616711, Rasht, IRAN

Abstract: It is well known that if T is a hypercyclic operator, then $\sigma_p(T^*) = \phi$. We prove in this paper that this is not true for subspace-hypercyclic operators. We show that if T is subspace-hypercyclic, then $\sigma_p(T^*)$ may be empty or not. Moreover we show that for every scalar λ with $|\lambda| > 1$, there exists a subspace-hypercyclic operator T such that $||T|| = |\lambda|$ and $\sigma_p(T^*) \neq \phi$.

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1. Introduction and Preliminaries

Let X be a Banach space. An operator T on X is hypercyclic if there exists a vector $x \in X$ whose orbit under T, $orb(T, x) = \{x, Tx, T^2x, ...\}$, is dense in X. Such a vector x is called a hypercyclic vector for T.

By Ansari's theorem in [1], hypercyclic operators exist on every separable and infinite dimensional Frechet space. Hypercyclic operators have been actively studied for more than twenty years. One can refer to [2-5] for more information.

Recently, B. F. Madore and R. A. Martnez-Avendano in [8] introduced the concept of subspace-hypercyclicity for an operator. We recall some preliminaries from [8] in this paper.

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In all of paper H always denotes a separable Hilbert space over C, the field of complex numbers and M is always a non-zero topologically closed subspace of H. $B(H)$ is the space of bounded linear operators acting on H and we say its elements, operators.

Let us recall some definitions and theorems from [8].

Definition 1.1. Let $T \in B(H)$ and M be a closed subspace of H. We say T is M-hypercyclic, if there exists $x \in H$ such that $orb(T, x) \cap M$ is dense in M. Such a vector x is called a M-hypercyclic vector for T.

Definition 1.2. Let $T \in B(H)$ and M be a closed subspace of H. We say T is M-transitive, if for any non-empty open sets $U, V \subseteq M$,both relatively open, there exists $n \in N_0$ such that $U \cap T^{-n}(V)$ contains a relatively open non-empty subset of M.

Theorem 1.3. *Let $T \in B(H)$. The following conditions are equivalent:*

(i) T is subspace-transitive with respect to M.

(ii) for any non-empty sets $U \subseteq M$, $V \subseteq M$ both relatively open, there exists $n \in N_0$ such that $T^{-n}(U) \cap V$ is a relatively open non-empty subset of M.

(iii) for any non-empty sets $U \subseteq M$, $V \subseteq M$ both relatively open, there exists $n \in N_0$ such that $T^{-n}(U) \cap V$ is non-empty and $T^n(M) \subseteq M$.

Lemma 1.4. *Let $T \in B(H)$ and M be a closed subspace of H. If T is M-transitive, then $HC(T, M) = (\bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} T^{-n}(B_j))$ is a dense subset of M, where $HC(T, M)$ is the set of M-hypercyclic vectors for T and $\{B_j\}$ is a countable open basis for the relative topology of M as a subspace of H.*

An immediate corollary of Lemma 1.4 is that subspace-transitive operators are subspace-hypercyclic.

Theorem 1.5. *Let $T \in B(H)$ and M be a nonzero subspace of H. If T is is subspace-hypercyclic for M, then T is subspace-hypercyclic for M.*

The following theorem shows that subspace-hypercyclicity, like hypercyclicity, is a purely infinite dimensional concept.

Theorem 1.6. *Let H be finite-dimensional. If $T \in B(H)$, then T is not subspace-hypercyclic for any M.*

Theorem 1.7. *Let $T \in B(H)$. If T is subspace-hypercyclic for M, then M is not finite dimensional.*

Theorem 1.8. *Let $T \in B(H)$. If T is subspace-hypercyclic for some subspace, then $\sigma(T) \cap S^1 \neq \phi$.*

By Theorem 1.8, if an operator is subspace-hypercyclic, then its spectral radius must be greater or equal to 1.

2. Main Results

If $T \in B(H)$ is a hypercyclic operator, then the point spectrum of T^*, is empty by [6]. That means we have:

$$\sigma_p(T^*) = \{\lambda \in C : T - \lambda I \text{ is not one to one } \} = \phi.$$

However this is not true for subspace-hypercyclic operators as the following example shows.

Example 2.1. Let $A \in B(H)$ be a hypercyclic operator and x be a hypercyclic vector for A and let $T = A \oplus \lambda_1 I \oplus \lambda_2 I \oplus ... \oplus \lambda_n I$, where $\lambda_1, \lambda_2, ..., \lambda_n$ are complex numbers and I is the identity operator on H. Then it is clear that T is a subspace-hypercyclic operator with respect to $M := H \oplus \{0\} \oplus ... \oplus \{0\}$ and $x \oplus \{0\} \oplus ... \oplus \{0\}$ is a M-hypercyclic vector for it. Also it is clear that $\{\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_n\} \subseteq \sigma_p(T^*)$.

Furthermore for every bounded sequence of scalars, we can construct an operator T such that $\sigma_p(T^*)$ contains that sequence.

Corollary 2.2. *For every bounded sequence $\{\lambda_n\}$ of scalars, we can find a subspace-hypercyclic operator T such that $\{\lambda_n\} \subseteq \sigma_p(T^*)$.*

Proof. It is sufficient to consider $T = A \oplus \bar{\lambda}_1 I \oplus \bar{\lambda}_2 I \oplus ... \oplus \bar{\lambda}_n I \oplus ...$, where $A \in B(H)$ is hypercyclic. □

Remark 2.3. In [8], it is proved that if $T \in B(H)$ is subspace-hypercyclic for M, then $ker(T^* - \lambda)^p \subseteq M^\perp$ for all $\lambda \in C$ and all $p \in N$. By Example 2.1, we can say that in some cases $ker(T^* - \lambda)^p$ is also non-empty.

Let B be the backward shift on l^2. So for every $(x_0, x_1, x_2, ...) \in l^2$,

$$B(x_0, x_1, x_2, ...) = (x_1, x_2, ...).$$

Rolweicz showed in [10] that for every scalar λ with $|\lambda| > 1$, λB is hypercyclic. We use this fact in the proof of next theorem.

Theorem 2.4. *For every scalar $\lambda \in C$ with $|\lambda| > 1$, there exists a subspace-hypercyclic operator T such that $||T|| = |\lambda|$ and $\sigma_p(T^*) \neq \phi$.*

Proof. Let $A = \lambda B$, where B is the backward shift on l^2 and $|\lambda| > 1$. Let λ_1 be a complex number such that $|\lambda_1| \leq |\lambda|$. Then $T = A \oplus \lambda_1 I \in B(l^2 \oplus l^2)$ is a subspace-hypercyclic operator with respect to $M = l^2 \oplus \{0\}$. Moreover

$$||T|| = max\{|\lambda|, |\lambda_1|\} = |\lambda|.$$

Like Example 2.1, $\sigma_p(T^*) \neq \phi$ and similar to Example 2.2 in [8], T is not hypercyclic.

Clearly T is not hypercyclic, since $\sigma_p(T^*) \neq \phi$. □

Theorem 2.5. *For every scalar α with $|\alpha| > 1$, there exists a subspace-hypercyclic operator T such that, T is not hypercyclic, $\rho(T) = |\alpha|$ and $\sigma_p(T^*) \neq \phi$.*

Proof. Let $T = \alpha B \oplus \lambda I$ where B is the backward shift on l^2 and $1 < |\lambda| < |\alpha|$. As we know T is subspace- hypercyclic operator but not hypercyclic. Also we have:

$$\rho(\alpha B \oplus \lambda I) = \lim_{n\to\infty} ||(\alpha B \oplus \lambda I)^n||^{\frac{1}{n}} = \lim_{n\to\infty} ||(\alpha^n B^n \oplus \lambda^n I^n)||^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} (\max\{(|\alpha^n|, |\lambda^n|)\})^{\frac{1}{n}} = \lim_{n\to\infty} (|\alpha|^n)^{\frac{1}{n}} = |\alpha|.$$

□

Lemma 2.6. (see [9]) *Let $T \in B(H)$ and $\lambda_1, \lambda_2, ..., \lambda_n$ be distict non-zero eigenvalues of T, and let $x_1, x_2, ..., x_n$ be corresponding non-zero eigenvectors. Then the vectors $x_i (i \in N)$ are linearly independent.*

Theorem 2.7. *Let T be a subspace-hypercyclic operator of the form $T = A \oplus \lambda_1 I \oplus \lambda_2 I \oplus ... \oplus \lambda_{n-1} I \in B(H \oplus H \oplus ... \oplus H)$, where $A \in B(H)$ is hypercyclic, λ_i's $(1 \leq i \leq n-1)$ are scalars and I is the identity operator on H. Then T^* has an invariant and non-trivial subspace of $H \oplus H \oplus ... \oplus H$.*

Proof. First we note that $\{\bar{\lambda}_1, ..., \bar{\lambda}_{n-1}\} \subseteq \sigma_p(T^*)$. Let x_i be corresponding non-zero eigenvector for $\bar{\lambda}_i$ where $1 \leq i \leq n-1$.

If we consider $M := span\{x_1, x_2, ..., x_{n-1}\}$, then M is closed and invariant under T^*. Clearly $M \neq H \oplus H \oplus ... \oplus H$, since M is finite dimensional. □

Remark 2.8. Note that if we replace all $\lambda_i I$'s $(1 \leq i \leq n-2)$ with hypercyclic operators on H, the theorem remains true.

In the following example, we show that the point spectrum of the adjoint of a subspace-hypercyclic operator may be empty.

Example 2.9. Let $A, B \in B(H)$ be hypercyclic operators. Then $T = A \oplus B$ is a subspace-hypercyclic operator with respect to $M := H \oplus \{0\}$.

Let $\lambda \in \sigma_p(T^*)$ and $x \oplus y$ be an eigenvector correspondence to λ and without lose of generality we can assume that x and y are non-zero. So

$$T^*(x \oplus y) = (A^*x \oplus B^*y) = \lambda(x \oplus y).$$

Hence $A^*x = \lambda x$ and $B^*y = \lambda y$. That means $\sigma_p(A^*) \neq \phi$ and $\sigma_p(B^*) \neq \phi$. But this is a contradiction. So $\sigma_p(T^*)$ must be empty.

From the above example, one can obtain subspace-hypercyclic operators with empty point spectrum.

For example let λ_1, λ_2 be two scalars such that $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and let B be the backward shift on l^2. Then if we consider $T = \lambda_1 B \oplus \lambda_2 B$, then it is clear that T is subspace-hypercyclic with respect to $M := l^2 \oplus \{0\}$. Note that the point spectrum of T^* is empty by Example 2.9, since $\lambda_1 B$ and $\lambda_2 B$ are hypercyclic operators on l^2.

Now we can say the following lemma:

Lemma 2.10. *There is a subspace-hypercyclic operator T such that, $\sigma_p(T^*)$ is empty.*

In fact we can find many of these operators by Example 2.9.

Theorem 2.11. *If $T \in B(H)$ is a subspace-hypercyclic operator, then $\sigma_p(T^*)$ may be empty or not.*

Proof. The proof is clear by Theorem 2.4 and Lemma 2.10. □

By Theorem 1.5, subspace-transitive operators are subspace-hypercyclic. So we have the following corollary:

Corollary 2.12. *If $T \in B(H)$ is a subspace-transitive operator, then $\sigma_p(T^*)$ may be empty or not.*

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