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PARTITIONING OF LOOP-FREE SPERNER HYPERGRAPHS INTO TRANSVERSALS

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Abstract: This article explores possibilities of partitioning the vertex set of a given simple loop-free Sperner hypergraph into a union of transversals. Studies are done on the possible number of transversals in such partitions, followed by forming a hypergraph (on the vertex set of the given hypergraph) that consists of transversals for hyperedges.

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1. Introduction

The set of all subsets (including the empty set ϕ) of a nonempty set V is denoted by 2^V and is called the *power set* [6] of V. The set 2^{V*} denotes the set of all nonempty subsets of V; that is, $2^{V*} = 2^V - \{\phi\}$. A *hypergraph* [1] on a nonempty finite set V is an ordered couple $H = (V, E)$ where E is a family of nonempty subsets of V such that $\bigcup_{X \in E} X = V$. The set V is the *vertex set* of

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H and each member of E is a *hyperedge* of H.

The cardinality of a finite set V is denoted by $\mid V \mid$. A hyperedge X in H with $\mid X \mid = 1$ is a *loop*; more specifically, a hyperedge $X = \{a\}$ is a loop at the vertex a. A hypergraph is *loop-free* if $\mid X \mid \geq 2$ for every hyperedge X. If no hyperedge in H equals all of V then H is *non-trivial.* If the members of E are all distinct (that is, no two members coincide as subsets of V; or, $E \subseteq 2^{V*}$; or, E has no repeated hyperedges) then H is *simple.* If no member of E is a subset (proper or otherwise) of another, then H is a *Sperner* hypergraph. In some instances ([1] and [3]) Sperner hypergraphs and simple ones are deemed same, but there is a distinction [5] between the two: Sperner hypergraphs are simple but not conversely (1.1 of [2]).

A hypergraph $H = (V, E)$ is *r-uniform* (for some positive integer r) if $\mid X \mid = r$ for every hyperedge X. Two vertices x and y are *adjacent* if $x \in X$ and $y \in X$ for some hyperedge X. H is *complete* if every vertex is adjacent to every other vertex. H is a *partitioned hypergraph* if the hyperedges form a partition [6] (or, a set partition) of V. A partitioned hypergraph is necessarily Sperner.

A *Helly hypergraph* is a Sperner hypergraph $H = (V, E)$, with $\mid V \mid > \mid E \mid = k$, that has the following properties (with $E = \{X_1, \ldots, X_k\}$):

(i) The set $n(H) = \bigcap_{j=1tok} X_j$ is nonempty;

(ii) each $X_j \in E$ can be partitioned as $X_j = S_j \cup n(H)$ (that is, $S_j \cap n(H) = \phi$) with $S_i \cap S_j = \phi$ whenever $i \neq j$; in other words, the intersection of two distinct hyperedges is precisely $n(H)$.

In a Helly hypergraph, each S_j as above is a *cabal* and the set $n(H)$ is the *nucleus* of the hypergraph.

This research work is a result of theoretical interest. Motivating ideas are (i) hypergraph colouring, treated in [1] in substantial detail, and (ii) bipartite graphs [4]. All the hypergraphs in this article are assumed non-trivial, loop-free, Sperner and having at least three vertices unless some unambiguous indication to the contrary is provided. If H is a hypergraph, then $H = (V, E)$ unless another couple takes the place of (V, E) explicitly.

2. Partitioning by Transversals

A *transversal* [1] in a hypergraph H is a nonempty subset T of the vertex set V (i.e., $T \in 2^{V*}$) such that T intersects each hyperedge in H (i.e., $T \cap X \neq \phi$ for each hyperedge X).

A *2-transversal partition* of H is a partition $V = T_1 \cup T_2$ such that T_1 and T_2 are transversals in H (though it is not required that T_1 or T_2 be a hyperedge in H). If such a partition of V exists then write $H \in 2-TP$; else $H \notin 2-TP$.

More generally, an *r-transversal partition* ($r \in N$, $r > 1$, with N being the set of positive integers) of H is a partition $V = T_1 \cup \ldots \cup T_r$ such that each T_j ($j = 1$ through r) is a transversal in H. Write $H \in r-TP$ if such a partition of V exists; else $H \notin r-TP$. If $H \in r-TP$ with a partitioning $V = T_1 \cup \ldots \cup T_r$ by transversals, then each T_j is a *component transversal*.

Let $\tau^*(H) = \{r \in N | H \in r-TP\}$. Then $\tau^*(H)$ is a finite subset of N, $1 \notin \tau^*(H)$, and $\tau^*(H)$ may be empty. If $\tau^*(H) \neq \phi$ then let $\Omega(H) = \max \tau^*(H)$; and if $\tau^*(H) = \phi$, then set $\Omega(H) = 0$. $\Omega(H)$ is the *transversal partition number* of H. Clearly $\Omega(H) \geq 2$ if $\tau^*(H) \neq \phi$.

Proposition 2.1. *If T_1 and T_2 are transversals in H then so is $T_1 \cup T_2$.* (The proof is elementary.)

Proposition 2.2. *If $H \in r-TP$ and if $r \geq 3$ then $H \in k-TP$ for each k with $2 \leq k < r$.*

Proof. Let $V = T_1 \cup \ldots \cup T_r$ be a partition of V such that each T_j ($j = 1$ through r) is a transversal in H. Then $V = T_1 \cup \ldots \cup T_{r-2} \cup M$, where $M = T_{r-1} \cup T_r$, is clearly a partition of V by $r-1$ transversals, whence $H \in (r-1)-TP$, from which the conclusion follows for any $k \in N$ with $2 \leq k < r$. □

Proposition 2.3. *Let $r \in N$, $r \geq 2$. If $H \in r-TP$ then $| X | \geq r$ for each hyperedge X.*

The proof of 2.3 is straightforward. The converse is not true in the general case, as 2.4 shows. But the converse does hold for all partitioned hypergraphs (2.5) and all Helly hypergraphs (2.6), so long as these are loop-free.

Example 2.4. Let $H_{n,r}$ ($n > r \geq 2$) denote the complete r-uniform hypergraph on a finite set V where $n = | V |$; that is, the hyperedges of $H_{n,r}$ are precisely all the subsets X of V satisfying $| X | = r$. If T is any proper transversal in $H_{n,2}$ then T has at least two elements. Let $x, y \in T$. Then $V - T$ does not meet the hyperedge $\{x, y\}$. Consequently $H_{n,2} \notin 2-TP$ though $| X | = 2$ for each hyperedge X.

Proposition 2.5. *Let $r \in N, r \geq 2$ and $H = (V, E)$ be a partitioned hypergraph. Then $H \in r-TP$ if and only if $| X | \geq r$ for each hyperedge X. Moreover, $\tau^*(H) = min | X |$as X runs over E.*

Proof. ($\Longrightarrow$) follows from 2.3.

($\Longleftarrow$) Since H is partitioned, $| X | \geq r$ ensures the presence of r pair wise disjoint transversals whose union is V. The last part, namely $\tau^*(H) = min | X |$ as X runs over E, is then evident, □

Proposition 2.6. *Let $r \in N, r \geq 2$ and $H = (V, E)$ be a Helly hypergraph. Then $H \in r - TP$ if and only if $\mid X \mid \geq r$ for each hyperedge X. Moreover, $\tau^*(H) = min \mid X \mid$as X runs over E.* xms

Proof. ($\Longrightarrow$) follows from 2.3.

($\Longleftarrow$) Write $E = \{X_1, \ldots, X_k\}$ and let $n(H) = \{z_1, \ldots, z_q\}$. Set $T_j = \{z_j\}$ for $j = 1$ through q. Then each T_j is a transversal in H. Further, $r > q$, and so let $p = r - q$. Writing $X_j = S_j \cup n(H)$ (with $S_j \cap n(H) = \phi$ and $S_i \cap S_j = \phi$ whenever $i \neq j$) for $j = 1$ through k, we at once have $\mid S_j \mid \geq p$ for each j, using which we come up with p transversals - say U_1 through U_p – that are pair wise disjoint and that cover $V - n(H)$. Then, the $q + p$ $(= r)$ transversals $T_1, \ldots, T_q, U_1, \ldots, U_p$ are pair wise disjoint and their set union is V, whence $H \in r - TP$. The part about $\tau^*(H)$ is then straightforward. □

Proposition 2.7. *If $H \in r - TP$, then $\mid T \cap X \mid < \mid X \mid$ for each component transversal T and for every hyperedge X.*

Proof. Let $V = T_1 \cup \ldots \cup T_r$ be a transversal partition of H. Were it to happen $\mid T_j \cap X \mid = \mid X \mid$ for some hyperedge X and some component T_j then it would lead to $X \subseteq T_j$, a contradiction. □

For an arbitrary loop-free Sperner hypergraph, is there a necessary and sufficient condition for $H \in r - TP$ for a given positive integer $r \geq 2$? For $r = 2$, there is one (2.8) but then it might not be easy to implement (in an algorithm) when the number of hyperedges is very large (not to speak of the number of vertices). Moreover, 2.8 cannot be extended to $r > 2$, as 2.9 shows.

Proposition 2.8. *$H \in 2-TP$ if and only if (i) $\mid X \mid \geq 2$ for each hyperedge X and (ii) there is a transversal T in H such that $\mid T \cap X \mid < \mid X \mid$ for every hyperedge X.*

Proof. ($\Longrightarrow$) If $H \in 2 - TP$, then (i) holds by 2.3 and (ii) by 2.7.

($\Longleftarrow$) Assume (i) and (ii) hold. Let T be a transversal in H with $\mid T \cap X \mid < \mid X \mid$ for every hyperedge X. Then T is a proper subset of V. Let $V - T = W$. Were $W \cap Y = \phi$ for some hyperedge Y, then $T \cap Y = Y$, running contrary to $\mid T \cap Y \mid < \mid Y \mid$. So W is a transversal in H, whence $H \in 2 - TP$. □

Example 2.9. Let $V = \{1, 2, 3, 4\}$ and let $H_{4,3}$ denote the complete 3-uniform hypergraph on V. $Y = \{1, 2\}$ is a transversal such that $\mid Y \cap X \mid < \mid X \mid$ for every hyperedge X. If T is a transversal in $H_{4,3}$ then $\mid T \mid \geq 2$. But this rules out the intersection, for instance, of the hyperedge $\{1, 2, 3\}$ with more than two disjoint transversals. Consequently $H_{4,3} \notin 3-TP$. Still, $H_{4,3} \in 2-TP$ because $\{1, 2\}$ and $\{3, 4\}$ are transversals in $H_{4,3}$ with $V = \{1, 2\} \cup \{3, 4\}$.

3. Transversal Conjugates

Let H_1 and H_2 be two hypergraphs on the same set V. If each hyperedge in H_1 is a transversal in H_2 then H_1 is a *transversal conjugate* of H_2, written $H_1(=\tau)H_2$; else $H_1(\neq\tau)H_2$.

Proposition 3.1. *Let H_1, H_2 and H be hypergraphs on the same set V.*

(i) If $H_1(=\tau)H_2$ then $H_2(=\tau)H_1$; for this reason, H_1 and H_2 are also called transversal conjugates.

(ii) $H(=\tau)H$ if and only if the hyperedges of H are pair wise intersecting.

Example 3.2. If $H_1(=\tau)H_2$ and $H_2(=\tau)H_3$ then it is not necessary that $H_1(=\tau)H_3$. Consider the hypergraphs $H_1=(V,E_1)$, $H_2=(V,E_2)$ and $H_3=(V,E_3)$ on $V=\{a,b,c,d\}$, where: $X_1=\{a,b\}$, $X_2=\{c,d\}$, $X_3=\{a,d\}$, $X_4=\{b,c\}$, $X_5=\{a,b,c\}$ and $X_6=\{c,d\}$; $E_1=\{X_1,X_2\}$, $E_2=\{X_3,X_4\}$ and $E_3=\{X_5,X_6\}$. Then $H_1(=\tau)H_2$ and $H_2(=\tau)H_3$ but $H_1(\neq\tau)H_3$.

Proposition 3.3. *Let $H=(V,E)$ with $E=\{X_1,\ldots,X_k\}$. Let $V=T_1\cup\ldots\cup T_r$ be an r-transversal partition of V and let $E_{T(r)}=\{T_1,\ldots,T_r\}$. Then $H_{T(r)}=(V,E_{T(r)})$ is a hypergraph on V. Also, $H(=\tau)H_{T(r)}$.*

The proofs of (3.1) and (3.3) are straightforward.

4. Resume

Partitioning a loop-free Sperner hypergraph into two transversals is relatively simpler than the general case where the number of transversals required is arbitrary. The authors are working on the latter case, especially for a large number of vertices and hyperedges. This kind of partitioning could have close associations with hypergraph colouring.

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