International Journal of Pure and Applied Mathematics
Volume 88 No. 4 2013, 475-482
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v88i4.3

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GEOMETRICAL HYPERCOMPLEX COUPLING BETWEEN ELECTRIC AND GRAVITATIONAL FIELDS

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Abstract: The present work shows a coupling of electrical and gravitational fields through Cauchy-Riemann conditions for quaternions present in a previous paper [1]. It is also obtained an extended version of the Laplace-like equations for quaternions, now written in terms of both electric and gravitational fields.

AMS Subject Classification: 30G99, 30E99
Key Words: quaternions, Laplace's equations, quate

1. Initial Provisions

Throughout this work, are considered quaternionic functions which follow the pattern $f_i(t,x,y,z)$, with $i = 1,2,3,4$, where t is the time and the coordinates x, y and z are considered the spatial coordinates. Thus, the quaternion q is written here as follows;

$$q = t + xi + yj + zk, \tag{1}$$

or

$$q = t + \vec{u}. \tag{2}$$

The next section based on a paper by Borges and Machado [2] shows a set

Received: May 7, 2013

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Cauchy-Riemann like relations for quaternionic functions. These equations will be adapted to the particular case where x_1 will be replaced by the time and the other coordinates x_2, x_3 and x_4 will be identified here for x, y and z, respectively.

2. Cauchy-Riemann Conditions for Quaternionic Functions

The conditions named here as Cauchy-Riemann like relations for quaternionic functions, are treated in detail in [1]. It follows the theorem:

Theorem 1. *For any pair pontis a and b and any path joining them simply conect subdomain of the four-dimmensional space, the integral $\int_a^b f dq$ is independent form the given path if and only if there is a function $F = F_1 + F_2 i + F_3 j + F_4 k$ such that $\int_a^b f dq = F(a)F(b)$, and satisfying the following relations:*

$$\frac{\partial F}{\partial t} = \frac{\partial F_2}{\partial x} = \frac{\partial F_3}{\partial y} = \frac{\partial F_4}{\partial z}, \tag{3}$$

$$\frac{\partial F_2}{\partial t} = -\frac{\partial F_1}{\partial x} = -\frac{\partial F_3}{\partial z} = \frac{\partial F_4}{\partial y}, \tag{4}$$

$$\frac{\partial F_3}{\partial t} = -\frac{\partial F_1}{\partial y} = -\frac{\partial F_2}{\partial z} = \frac{\partial F_4}{\partial x}, \tag{5}$$

$$\frac{\partial F_4}{\partial t} = \frac{\partial F_1}{\partial z} = -\frac{\partial F_2}{\partial y} = \frac{\partial F_3}{\partial x}. \tag{6}$$

Proof. The proof of this theorem can be analyzed in greater detail in [1]. □

3. The Laplace's Equations

In this section it will be determined that from the relations showed in Theorem 1, naturally follows a new set of quaternionic Laplacelike equations. Therefore, the functions that make up the quaternionic function depend on t, x, y and z and are supposed of class C^2.

The first step to obtain the Laplace equations is the derivation of equations (5), (6), (7) and (8) over t, x, y and z. That will be done as follows: Deriving the conditions of equation (5), we have that:

$$\begin{aligned}
\frac{\partial^2 F_1}{\partial y \partial t} = \frac{\partial^2 F_2}{\partial t \partial x} = \frac{\partial^2 F_3}{\partial t \partial y} = \frac{\partial^2 F_4}{\partial t \partial z} \\
\frac{\partial^2 F_1}{\partial t \partial x} = \frac{\partial^2 F_2}{\partial x^2} = \frac{\partial^2 F_3}{\partial x \partial y} = \frac{\partial^2 F_4}{\partial x \partial z} \\
\frac{\partial^2 F_1}{\partial y \partial t} = \frac{\partial^2 F_2}{\partial y \partial x} = \frac{\partial^2 F_3}{\partial y^2} = \frac{\partial^2 F_4}{\partial z \partial y} \\
\frac{\partial^2 F_2}{\partial t \partial z} = \frac{\partial^2 F_2}{\partial z \partial x} = \frac{\partial^2 F_3}{\partial z \partial y} = \frac{\partial^2 F_4}{\partial z^2}.
\end{aligned} \tag{7}$$

Deriving the conditions of equation (6), we obtain:

$$\begin{aligned}
\frac{\partial^2 F_2}{\partial t^2} = -\frac{\partial^2 F_1}{\partial t \partial x} = -\frac{\partial^2 F_3}{\partial t \partial z} = \frac{\partial^2 F_4}{\partial t \partial y} \\
\frac{\partial^2 F_2}{\partial t \partial x} = -\frac{\partial^2 F_1}{\partial x^2} = -\frac{\partial^2 F_3}{\partial x \partial z} = \frac{\partial^2 F_4}{\partial y \partial x} \\
\frac{\partial^2 F_2}{\partial y \partial t} = -\frac{\partial^2 F_1}{\partial y \partial x} = -\frac{\partial^2 F_3}{\partial y \partial z} = \frac{\partial^2 F_4}{\partial y^2} \\
\frac{\partial^2 F_2}{\partial z \partial t} = -\frac{\partial^2 F_1}{\partial z \partial x} = -\frac{\partial^2 F_3}{\partial z^2} = \frac{\partial^2 F_4}{\partial z \partial y}.
\end{aligned} \tag{8}$$

Deriving the conditions of equation (7), we obtain:

$$\begin{aligned}
\frac{\partial^2 F_3}{\partial t^2} = -\frac{\partial^2 F_1}{\partial t \partial y} = -\frac{\partial^2 F_2}{\partial t \partial z} = \frac{\partial^2 F_4}{\partial t \partial x} \\
\frac{\partial^2 F_3}{\partial t \partial x} = -\frac{\partial^2 F_1}{\partial x \partial y} = -\frac{\partial^2 F_2}{\partial x \partial z} = \frac{\partial^2 F_4}{\partial x^2} \\
\frac{\partial^2 F_3}{\partial y \partial t} = -\frac{\partial^2 F_1}{\partial y^2} = -\frac{\partial^2 F_2}{\partial z \partial y} = \frac{\partial^2 F_4}{\partial y \partial x} \\
\frac{\partial^2 F_3}{\partial t \partial z} = -\frac{\partial^2 F_1}{\partial z \partial y} = -\frac{\partial^2 F_2}{\partial z^2} = \frac{\partial^2 F_4}{\partial z \partial x}.
\end{aligned} \tag{9}$$

And finally, in deriving the conditions of equation (8), it follows that:

$$\begin{aligned}
\frac{\partial^2 F_4}{\partial t^2} = \frac{\partial^2 F_1}{\partial t \partial z} = -\frac{\partial^2 F_2}{\partial t \partial y} = -\frac{\partial^2 F_3}{\partial t \partial x} \\
\frac{\partial^2 F_4}{\partial t \partial x} = \frac{\partial^2 F_1}{\partial x \partial z} = -\frac{\partial^2 F_2}{\partial x \partial y} = -\frac{\partial^2 F_3}{\partial x^2} \\
\frac{\partial^2 F_4}{\partial y \partial t} = \frac{\partial^2 F_1}{\partial y \partial z} = -\frac{\partial^2 F_2}{\partial y^2} = -\frac{\partial^2 F_3}{\partial y \partial x} \\
\frac{\partial^2 F_4}{\partial t \partial z} = \frac{\partial^2 F_1}{\partial z^2} = -\frac{\partial^2 F_2}{\partial z \partial y} = -\frac{\partial^2 F_3}{\partial z \partial x}.
\end{aligned} \tag{10}$$

Correlating groups of partial derivatives in (9), (10), (11) and (12), then immediately follows the Laplace-like Equations:

$$\frac{\partial^2 F_1}{\partial^2 t^2} + \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} = 0 \tag{11}$$

$$\frac{\partial^2 F_2}{\partial t^2} + \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2} = 0 \tag{12}$$

$$\frac{\partial^2 F_3}{\partial t^2} + \frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} = 0 \tag{13}$$

and

$$\frac{\partial^2 F_4}{\partial t^2} + \frac{\partial^2 F_4}{\partial x^2} + \frac{\partial^2 F_4}{\partial y^2} + \frac{\partial^2 F_4}{\partial z^2} = 0 \tag{14}$$

Taking now into account the functions $F_3(t,x,y,z)$ and $F_4(t,x,y,z)$ at (13) and (14), and making the limit in these equations when t tends to zero, we have that:

$$\lim_{t\to 0}\Big[\frac{\partial^2 F_3(t,x,y,z)}{\partial t^2} + \frac{\partial^2 F_3(t,x,y,z)}{\partial x^2} + \frac{\partial^2 F_3(t,x,y,z)}{\partial y^2} + \frac{\partial^2 F_3(t,x,y,z)}{\partial z^2}\Big] = 0, \tag{15}$$

and

$$\lim_{t\to 0}\Big[\frac{\partial^2 F_4(t,x,y,z)}{\partial t^2} + \frac{\partial^2 F_4(t,x,y,z)}{\partial x^2} + \frac{\partial^2 F_4(t,x,y,z)}{\partial y^2} + \frac{\partial^2 F_4(t,x,y,z)}{\partial z^2}\Big] = 0. \tag{16}$$

As already mentioned earlier, the functions F_3 and F_4 are of class C^2 and thus making the limit as t tends to zero, these functions will depend only of x, y and z, and will be denoted by $\varphi(x,y,z)$ and $\Phi(x,y,z)$, respectively. Moreover, in the second set of partial derivatives respect to t in the limit as t tends to zero are allowed constants, and now will be made the following identifications:

$$\lim_{t\to 0}\frac{\partial^2 F_3(t,x,y,z)}{\partial t^2} = \frac{\rho_f}{\varepsilon} \tag{17}$$

and

$$\lim_{t\to 0}\frac{\partial^2 F_4(t,x,y,z)}{\partial t^2} = 4\pi G\rho, \tag{18}$$

where ρ_f is free charge density, ε is permittivity of the medium. Furthermore, ρ is density and G is gravitational constant. Soon, with the identifications and the limits indicated above, we have the following equations:

$$\frac{\partial^2\varphi(x,y,z)}{\partial x^2}+\frac{\partial^2\varphi(x,y,z)}{\partial y^2}+\frac{\partial^2\varphi(x,y,z)}{\partial z^2}=-\frac{\rho_f}{\varepsilon} \tag{19}$$

and

$$\frac{\partial^2\Phi(x,y,z)}{\partial x^2}+\frac{\partial^2\Phi(x,y,z)}{\partial y^2}+\frac{\partial^2\Phi(x,y,z)}{\partial z^2}=-4\pi G\rho \tag{20}$$

There is the possibility of determining the solutions of the equations above, but considering that they are related by the Cauchy-Riemann conditions, after the treatment is again considered the limit when t tends to zero. Therefore, a system of partial differential equations, which arise from the Riemann Cauchy like conditions, is presented only for the functions F_3 and F_4. It follows that:

$$\begin{aligned}
&\frac{\partial^2 F_3}{\partial t\partial y}=\frac{\partial^2 F_4}{\partial t\partial z}, && \frac{\partial^2 F_3}{\partial x\partial y}=\frac{\partial^2 F_4}{\partial x\partial z},\\
&\frac{\partial^2 F_3}{\partial y^2}=\frac{\partial^2 F_4}{\partial z\partial y}, && \frac{\partial^2 F_3}{\partial z\partial y}=\frac{\partial^2 F_4}{\partial z^2},\\
&-\frac{\partial^2 F_3}{\partial t\partial z}=\frac{\partial^2 F_4}{\partial t\partial y}, && -\frac{\partial^2 F_3}{\partial x\partial z}=\frac{\partial^2 F_4}{\partial y\partial x},\\
&-\frac{\partial^2 F_3}{\partial y\partial z}=\frac{\partial^2 F_4}{\partial y^2}, && -\frac{\partial^2 F_3}{\partial z^2}=\frac{\partial^2 F_4}{\partial z\partial y},\\
&\frac{\partial^2 F_3}{\partial t^2}=\frac{\partial^2 F_4}{\partial t\partial x}, && \frac{\partial^2 F_3}{\partial t\partial x}=\frac{\partial^2 F_4}{\partial x^2},\\
&\frac{\partial^2 F_3}{\partial y\partial t}=\frac{\partial^2 F_4}{\partial y\partial x}, && \frac{\partial^2 F_3}{\partial t\partial z}=\frac{\partial^2 F_4}{\partial z\partial x},\\
&\frac{\partial^2 F_4}{\partial t^2}=-\frac{\partial^2 F_3}{\partial t\partial x}, && \frac{\partial^2 F_4}{\partial t\partial x}=-\frac{\partial^2 F_3}{\partial x^2},\\
&\frac{\partial^2 F_4}{\partial y\partial t}=-\frac{\partial^2 F_3}{\partial y\partial x}, && \frac{\partial^2 F_4}{\partial t\partial z}=-\frac{\partial^2 F_3}{\partial z\partial x},
\end{aligned}$$

$$\begin{aligned}
&\lim_{t\to 0}\frac{\partial^2 F_4}{\partial t^2}=4\pi G\rho, && \lim_{t\to 0}\frac{\partial^2 F_3}{\partial t^2}=\frac{\rho_f}{\varepsilon},\\
&\frac{\partial F_3}{\partial y}=\frac{\partial F_4}{\partial z}, && -\frac{\partial F_3}{\partial z}=\frac{\partial F_4}{\partial y},\\
&\frac{\partial F_3}{\partial t}=\frac{\partial F_4}{\partial x}, && \frac{\partial F_4}{\partial t}=-\frac{\partial F_3}{\partial x}.
\end{aligned}\tag{21}$$

The above system has the following solution (solution that verifies the Laplace like Equation for F_3 and F_4), where C_1 and C_2 are constants. Hence it follows that:

$$\begin{aligned}
F_3(t,x,y,z)&=-\frac{1}{2}\Big(\frac{\rho_f}{\varepsilon}\Big)x^2-(4\pi\rho G t+f_1(z-yi)+f_2(z+yi))x\\
&+\frac{1}{2}\Big(\frac{\rho_f}{\varepsilon}\Big)t^2+(if_1(z-yi)-if_2(z+yi)+C_1)t+if_3(z-yi)\\
&\qquad -if_4(z+yi)+C_2,
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
F_4(t,x,y,z)&=\frac{1}{2}(4\pi\rho G)t^2-\Big(\Big(\frac{\rho_f}{\varepsilon}\Big)x+f_1(z-iy)+f_2(z+iy)\Big)t\\
&-\frac{1}{2}(4\pi\rho G)x^2+(if_1(z-iy)-if_2(z+iy)+C_1)x+f_3(z-y)\\
&\qquad +f_4(z+iy).
\end{aligned}\tag{23}$$

Making F_3-iF_4 the threshold t tends to zero, ie, $F_3(x,y,z)-iF_4(x,y,z)$ we have that:

$$\begin{aligned}
F_3(x,y,z)-iF_4(x,y,z)&=-\frac{1}{2}\Big(\frac{\rho_f}{\varepsilon}\Big)x^2+\frac{1}{2}(4\pi G\rho)x^2 i-2f_2(z+iy)x\\
&\qquad +C_2-C_1xi-2if_4(z+iy).
\end{aligned}\tag{24}$$

On the other hand, performing the partial derivatives of the above functions and taken to the limit when $t\to 0$, and making the appropriate identifications, we have that:

$$i\frac{\partial F_3}{\partial y}(x,y,z)=-Df_1(z-iy)x+Df_2(z+iy)x+iDf_3(z-iy)+iDf_4(z+iy),\tag{25}$$

$$\frac{\partial F_4}{\partial y}(x,y,z)=Df_1(z-iy)x+Df_2(z+iy)x-iDf_3(z-iy)+iDf_4(z+iy),\tag{26}$$

which together give us:

$$\frac{\partial F_4}{\partial y}(x,y,z) + i\frac{\partial F_3}{\partial y}(x,y,z) = 2Df_2(z+iy)x + 2iDf_4(z+iy)x. \tag{27}$$

Similarly,

$$i\frac{\partial F_3}{\partial z}(x,y,z) = -iDf_1(z-iy)x - iDf_2(z+iy)x - Df_3(z-iy) + Df_4(z+iy), \tag{28}$$

$$\frac{\partial F_4}{\partial z}(x,y,z) = iDf_1(z-iy)x - iDf_2(z+iy)x + Df_3(z-iy) + iDf_4(z+iy), \tag{29}$$

which together generate the following equality:

$$\frac{\partial F_4}{\partial z}(x,y,z) + i\frac{\partial F_3}{\partial z}(x,y,z) = -2iDf_2(z+iy)x + 2Df_4(z+iy). \tag{30}$$

Finally, by taking the sum:

$$\frac{\partial F_4}{\partial y}(x,y,z) + i\frac{\partial F_3}{\partial y}(x,y,z) - \frac{\partial F_3}{\partial z}(x,y,z) + i\frac{\partial F_4}{\partial z}(x,y,z) = 4Df_1(z+iy)x + 4iDf_4(z+iy), \tag{31}$$

or

$$\Big(\frac{\partial F_4}{\partial y}(x,y,z) - \frac{\partial F_3}{\partial z}(x,y,z)\Big) + i\Big(\frac{\partial F_3}{\partial y}(x,y,z) + \frac{\partial F_4}{\partial z}(x,y,z)\Big) = 4Df_1(z+iy)x + 4iDf_4(z+iy). \tag{32}$$

Integrating the terms of $f_1(z+iy)$ and $f_4(z+iy)$ we have:

$$4Df_1(z+iy)x = \Big(\frac{\partial F_4}{\partial y}(x,y,z) - \frac{\partial F_3}{\partial z}(x,y,z)\Big); \tag{33}$$

that is equal to

$$f_1(z+iy)x = -\frac{1}{2}(F_3(x,y,z) - F_3(x,y,z_0)) + \frac{i}{2}(F_4(x,y,z) - F_4(x,y_0,z)). \tag{34}$$

Similarly, we have:

$$f_4(z+iy) = \frac{1}{2}(F_4(x,y,z) - F_4(x,y,z_0)) + \frac{i}{2}(F_3(x,y,z) - F_3(x,y_0,z)). \tag{35}$$

Substituting the above results integrated y_0 to y and z_0 by z and substituting in equation (24) it follows that:

$$(-1)[F_3(x,y,z) - F_3(x,y,z_0) - F_3(x,y_0,z)] + i[F_4(x,y,z) - F_4(x,y,z_0) - F_4(x,y_0,z)] = \Big(-\frac{1}{2}\Big(\frac{\rho_f}{\varepsilon}\Big)x^2 + C_2\Big) + i\Big(\frac{1}{2}(4\pi G\rho)x^2 + C_1 x\Big), \tag{36}$$

or

$$[F_3(x,y,z)-F_3(x,y,z_0)-F_3(x,y_0,z)]^2+[F_4(x,y,z)-F_4(x,y,z_0)-F_4(x,y_0,z)]^2 = \Big(\frac{1}{2}\Big(\frac{\rho_f}{\varepsilon}\Big)x^2 - C_2\Big)^2 + \Big(\frac{1}{2}(4\pi G\rho)x^2 + C_1 x\Big)^2. \tag{37}$$

The results of this work can be summarized in the following theorem:

Theorem 2. *Let $f(q)$ quaternionic function that satisfies the Cauchy-Riemann conditions. If $f(q)$ is of class C^2, then it is possible to determine a relationship between gravitational and electrical potential as listed below:*

$$[F_3(x,y,z)-F_3(x,y,z_0)-F_3(x,y_0,z)]^2+[F_4(x,y,z)-F_4(x,y,z_0)-F_4(x,y_0,z)]^2 = \Big(\frac{1}{2}\Big(\frac{\rho_f}{\varepsilon}\Big)x^2 - C_2\Big)^2 + \Big(\frac{1}{2}(4\pi G\rho)x^2 + C_1 x\Big)^2. \tag{38}$$

4. Conclusion

The present results in the previous sections, showed the feasibility of obtaining the equations of Laplace through the Cauchy-Riemann like conditions for quaternions. This fact will allow the relationship between equations that can explain such physical phenomena e. g. the possibility of a geometrical coupling regarding gravitational and electric fields. You can also use the above equations as a way of stating the theorem for harmonic functions that satisfy the Cauchy conditions.

References

[1] J.M. Machado, M.F. Borges, New ramrks on the differenciability of hypercomplex functions, *International Journal of Pure and Applied Mathematics*, **8**, No. 1 (2002), 85-101.

[2] J. Marão, M.F. Borges, A note on the hypercomplex Riemann-Cauchy like relations for quaternions and laplace equations, In: *notes*.