International Journal of Pure and Applied Mathematics
Volume 88 No. 4 2013, 483-497
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v88i4.4

ijpam.eu

THE δ-SQUARED PROCESS AND FOURIER SERIES OF FUNCTIONS WITH MULTIPLE JUMPS

Emily Jennings[1], Charles N. Moore[2 §], Daniel Muñiz[3], Ashley Toth[4]

[1]Department of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332, USA

[2]Department of Mathematics
Kansas State University
Manhattan, KS 66506, USA

[3]Department of Mathematics
University of Florida
Gainsville, FL 32611, USA

[4]Department of Mathematics
Rollins College
Winter Park, FL 32789, USA

Abstract: We investigate the effects of the δ^2 transform on the partial sums of Fourier series for functions with a finite number of jumps, which in general, converge slowly. Although the δ^2 process is known to accelerate convergence for many sequences, we prove that in this case, the transformed series will usually fail to converge to the original function.

AMS Subject Classification: 65B10, 65T40, 42A20
Key Words: Fourier series, delta-squared process, convergence acceleration

Received: June 9, 2013

[§]Correspondence author

1. Introduction

If $f \in L^1([-\pi, \pi])$, define the Fourier coefficients of f by

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

for each integer n, and the N^{th} partial sum of the Fourier series by

$$S_N f(x) := \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}, \tag{1.1}$$

where N is a positive integer and $x \in [-\pi, \pi]$.

When $f \in L^2$, $S_N f$ converges to f in L^2. Theorems of Dini-Lipschitz, Lebesgue, and Dirichlet-Jordan give conditions for pointwise convergence (see e.g. Zygmund [19]). Fourier series are useful if they converge rapidly, which is often not the case, say, for a function with jump discontinuities. For such functions it is typical that the decay of the Fourier coefficients is $\mathcal{O}(\frac{1}{n})$; in particular, these Fourier series do not converge absolutely. In this paper we investigate the possibility of applying a well-known sequence acceleration method to the partial sums of certain slowly converging Fourier series.

Given a sequence s_n, the δ^2 process transforms it to

$$t_n := s_n - \frac{(s_{n+1} - s_n)(s_n - s_{n-1})}{(s_{n+1} - s_n) - (s_n - s_{n-1})}, \tag{1.2}$$

where we set $t_n = s_n$ if the denominator of the fraction is zero.

This transform is usually attributed to Aitken [1], although the idea had appeared earlier. This transform and generalizations were studied extensively by Shanks [12], and it is for this reason this is sometimes called the Shanks transform. The δ^2 process and many related transforms are discussed in Brezinski and Redivo-Zaglia [7] or Sidi [14]. Tucker [17] gives conditions on a convergent s_n so that the resulting t_n converge to the same limit but more quickly.

We will investigate what happens when the δ^2 process is applied pointwise to the sequence of partial sums $S_n f(x)$ of a function which is smooth except for a finite number of jumps. Usually such functions have Fourier series which converge slowly. In Abebe, Graber, and Moore [2] the authors investigate the application of the δ^2 process to the partial sums of a function which has a single jump discontinuity. The methods needed in this paper to analyze the situation of multiple jumps are much more elaborate.

Smith and Ford [16] used numerical tests to compare different methods of convergence acceleration to the partial sums of Fourier series. They tested slowly and rapidly converging Fourier series and found, on the average, the transformation (1.2) improves convergence of the slowly converging series but slightly degrades convergence in the rapidly convergent case. Drummond [8] discusses many methods of convergence acceleration and includes discussion of their application to Fourier series. Sidi [13] discusses the effects of nonlinear sequence transformations on Fourier series. The d-transformation of Levin and Sidi [10] has been shown to be an effective means of acceleration of Fourier series in some situations. Sidi [15] has shown that the Shanks transformation is effective in accelerating the convergence of a class of infinite sequences which contain certain cases of Fourier series of piecewise smooth functions.

The δ^2 process is same as $\varepsilon_2^{(n)}$ in family of transforms $\varepsilon_k^{(n)}$ known as the epsilon algorithm. Brezinski [6] (see also Wynn [18]) proposed the following procedure, called the complex ε algorithm: To $S_n f$ add its conjugate function $\widetilde{S_n} f$ to create an analytic function on the unit disk $G_n f(z)$. Apply the epsilon algorithm to $G_n f(e^{i\theta})$ and then take the real part. Brezinski gives numerical experiments to demonstrate that this procedure reduces the Gibbs phenomenon. Beckermann, Matos, and Wielonsky [3] show that this method accelerates convergence for functions of the form $f = f_1 + f_2$, where f_1 has prescribed discontinuities but is smooth elsewhere, f_2 has quickly decaying Fourier coefficients, and $G(f_1) = \lim_{n\to\infty} G_n(f_1)$ is a certain type of hypergeometric function. However, there are analytic functions on the unit disk, continuous on the boundary of the disk (that is, as a function of $e^{i\theta}$, $\theta \in [-\pi, \pi]$) for which the δ^2 process destroys convergence. (See [11].) As we will see, the real $\varepsilon_2^{(n)}$ algorithm, i.e. the δ^2 process, performs badly on a large class of functions with jumps.

2. Main Results

Lemma 1. *Let f be a 2π periodic function having a finite number of jump discontinuities in $(-\pi, \pi)$ at $a_1 < a_2 < \dots < a_m$. Suppose that for each j, $\lim_{x\to a_j^\pm} f(x) = f(a_j^\pm)$ and $\lim_{x\to a_j^\pm} f'(x) = f'(a_j^\pm)$ exist and are finite. Suppose that f'' is continuous except at the a_j, and bounded. For each j set $d_j = f(a_j^+) - f(a_j^-)$ and $d_j^* = f'(a_j^-) - f'(a_j^+)$. Then the N^{th} partial sum of the*

Fourier series is

$$S_N f(x) = \hat{f}(0) + \sum_{k=1}^{N} \left[\frac{1}{k\pi} \sum_{j=1}^{m} [d_j \sin k(x - a_j)] + \epsilon_k \right], \tag{2.1}$$

where

$$\epsilon_k = \frac{1}{k^2 \pi} \sum_{j=1}^{m} \left[d_j^* \cos k(x - a_j) \right] - \frac{\widehat{f''}(k) e^{ikx} + \widehat{f''}(-k) e^{-ikx}}{k^2}. \tag{2.2}$$

Proof. Integration by parts gives the n^{th} Fourier coefficient,

$$\begin{aligned} \hat{f}(n) = & \frac{1}{-2in\pi} \sum_{j=1}^{m} \left[(f(a_j^-) - f(a_j^+)) e^{-ina_j} \right] \\ & + \frac{1}{2n^2\pi} \sum_{j=1}^{m} \left[(f'(a_j^-) - f'(a_j^+)) e^{-ina_j} \right] - \frac{1}{n^2} \widehat{f''}(n). \end{aligned}$$

By (1.1), $S_N f(x) = \hat{f}(0) + \sum_{k=1}^{N} \hat{f}(-k) e^{-ikx} + \sum_{k=1}^{n} \hat{f}(k) e^{ikx}$, which simplifies to (2.1) and (2.2). □

Applying the δ^2 process pointwise to $\{S_N f(x)\}$ yields

$$\begin{aligned} T_N f(x) &= S_N f(x) - \frac{(S_{N+1} f(x) - S_N f(x))(S_N f(x) - S_{N-1} f(x))}{(S_{N+1} f(x) - S_N f(x)) - (S_N f(x) - S_{N-1} f(x))} \\ &= S_N f(x) - \\ & \frac{\left[\sum_{j=1}^{m} \left[d_j \frac{\sin(N+1)(x-a_j)}{(N+1)\pi} \right] + \epsilon_{N+1} \right] \left[\sum_{j=1}^{m} \left[d_j \frac{\sin N(x-a_j)}{N\pi} \right] + \epsilon_N \right]}{\sum_{j=1}^{m} [d_j \frac{\sin(N+1)(x-a_j)}{(N+1)\pi}] - \sum_{j=1}^{m} [d_j \frac{\sin N(x-a_j)}{N\pi}] + \epsilon_{N+1} - \epsilon_N}, \end{aligned} \tag{2.3}$$

where ϵ_N is as defined in (2.2). Multiplying the numerator and denominator in the fraction on the right above by $N(N+1)\pi^2$, we obtain the expression.

$$\frac{\left[\sum_{j=1}^{m} [d_j \sin(N+1)(x - a_j)] + (N+1)\pi \epsilon_{N+1} \right] \left[\sum_{j=1}^{m} [d_j \sin N(x - a_j)] + N\pi \epsilon_N \right]}{N\pi \sum_{j=1}^{m} [d_j \sin(N+1)(x - a_j)] - (N+1)\pi \sum_{j=1}^{m} [d_j \sin N(x - a_j)] + N(N+1)\pi^2 (\epsilon_{N+1} - \epsilon_N))}. \tag{2.4}$$

We note that $\epsilon_N \in \mathcal{O}(\frac{1}{N^2})$. We need estimates of both the numerator and denominator of (2.4). We begin with two lemmas which estimate the denominator, first for N fixed, then for x fixed.

Lemma 2. *For x_0 fixed,*

$$g(x) = \sum_{j=1}^{m} d_j(\sin((N+1)(x-a_j)) - \sin N(x-a_j))$$

satisfies $|g(x)| \le \frac{4\pi}{N} \sum_{j=1}^{m} |d_j|$ on a subinterval of $\left[x_0, x_0 + \frac{\pi}{N+1}\right]$.

Proof. Note that

$$g\left(x_0 + \frac{\pi}{N+1}\right) =$$
$$\sum_{j=1}^{m} d_j \left(\sin((N+1)(x_0-a_j)+\pi) - \sin(N(x_0-a_j) + \frac{N\pi}{N+1})\right)$$
$$= \sum_{j=1}^{m} d_j \left(-\sin(N+1)(x_0-a_j) + \sin(N(x_0-a_j) - \frac{\pi}{N+1})\right) =$$
$$\sum_{j=1}^{m} d_j \big(-\sin(N+1)(x_0-a_j) + \sin N(x_0-a_j)\cos\frac{\pi}{N+1}$$
$$- \cos N(x_0-a_j)\sin\frac{\pi}{N+1})$$
$$= -\sum_{j=1}^{m} d_j (\sin(N+1)(x_0-a_j) - \sin N(x_0-a_j)) +$$
$$\sum_{j=1}^{m} d_j \left[\sin N(x_0-a_j)\cos(\frac{\pi}{N+1} - 1) - \cos N(x_0-a_j)\sin\frac{\pi}{N+1}\right]$$
$$= -g(x_0) + e(x_0, N),$$

where

$$e(x_0,N) = \sum_{j=1}^{m} d_j \Bigg[(\sin N(x_0-a_j))(\cos\frac{\pi}{N+1} - 1)$$
$$- \cos N(x_0-a_j)\sin\frac{\pi}{N+1}\Bigg]$$

Note $|e(x_0,N)| \le \sum_{j=1}^{m} |d_j| \left(\frac{2\pi}{N+1}\right) \le \frac{2\pi}{N}\sum_{j=1}^{m} |d_j|$.

Case 1: If $|g(x_0)| \le \frac{2\pi}{N} \sum_{j=1}^{m} |d_j|$, then since

$|g'| \le (\sum_{j=1}^{m} |d_j|)(2N+1)$, $|g| \le 2\left(\frac{2\pi}{N}\right) \sum_{j=1}^{m} |d_j|$ on a subinterval of $[x_0, x_0 + \frac{\pi}{N+1}]$ of length approximately $\frac{1}{N^2}$.

Case 2: If $|g(x_0)| > \frac{2\pi}{N}\sum_{j=1}^{m}|d_j|$, we claim $g(x_0)$ and $g\left(x_0+\frac{\pi}{N+1}\right)$ have opposite signs: Say, without loss of generality, that $g(x_0)>0$. Then $g(x_0) > \frac{2\pi}{N}\sum_{j=1}^{m}|d_j|$, so that $-g(x_0) < -\frac{2\pi}{N}\sum_{j=1}^{m}|d_j|$. Then

$$\begin{aligned} g(x_0+\frac{\pi}{N+1}) &= -g(x_0)+e\,(x_0,N) \\ &< -\frac{2\pi}{N}\sum_{j=1}^{m}|d_j| + \frac{2\pi}{N}\sum_{j=1}^{m}|d_j| = 0. \end{aligned}$$

and thus, $g(x)=0$ somewhere on the interval. As in Case 1, this implies that g is bounded above by $|g| \le 2\left(\frac{2\pi}{N}\right)\sum_{j=1}^{m}|d_j|$ on an interval of length approximately $\frac{1}{N^2}$. □

Now set $\alpha_N = \sum_{j=1}^{m} d_j \cos Na_j$, $\beta_N = \sum_{j=1}^{m} d_j \sin Na_j$, and $\phi_N = \arctan\left(\frac{\beta_N}{\alpha_N}\right)$. (If $\alpha_N=0$, $\beta_N \ge 0$, set $\phi_N = \frac{\pi}{2}$, and if $\alpha_N = 0$, $\beta_N<0$, set $\phi_N = -\frac{\pi}{2}$.) Set $A_N = \operatorname{sgn}(\alpha_N)\sqrt{{\alpha_N}^2+{\beta_N}^2}$. With these definitions it follows that for $x \in [-\pi,\pi]$,

$$\begin{aligned} &\sum_{j=1}^{m} d_j \sin N(x-a_j) = \sum_{j=1}^{m} d_j(\sin Nx\cos Na_j - \cos Nx \sin Na_j) \\ &= \alpha_N \sin Nx - \beta_N \cos Nx = A_N \sin(Nx-\phi_N). \end{aligned} \tag{2.5}$$

In Lemma 4 below, we will assume that for each j, $a_j = \frac{p_j}{q_j}\pi$ with $\frac{p_j}{q_j}$ in lowest terms. Then for $L = 2\,\mathrm{lcm}\{q_j\}$, the sequences α_n, β_n, A_n and ϕ_n have period L, that is, $\alpha_{n+L}=\alpha_n$ for all n, etc. We first need a result from elementary number theory.

Lemma 3. *(Chebyshev; see [9], pg. 39) For an arbitrary irrational a and arbitrary real μ, there exists an infinite sequence of positive integers k and m such that $|ka-m-\mu| < \frac{3}{k}$.*

Lemma 4. *Suppose that each a_j is a rational multiple of π, that is, $a_j = \frac{p_j}{q_j}\pi$ which we assume to be in lowest terms. Suppose $x = 2a\pi$, where $a \in [-.5,.5]$ is an irrational number. Fix $J \in \{1,...,L\}$ which has $A_J A_{J+1} \neq 0$. Then there are an infinite number of integers $n = kL+J$ such that*

$$|A_{n+1}\sin\left((n+1)x-\phi_{n+1}\right) - A_n \sin\left(nx-\phi_n\right)| \le \max\{|A_J|,|A_{J+1}|\}\frac{24L^2\pi}{n}.$$

Proof. Note that for $n = kL+J$, we have $A_n = A_J$, $A_{n+1} = A_{J+1}$, $\phi_n = \phi_J$, and $\phi_{n+1} = \phi_{J+1}$. Thus, we must show that

$$\begin{aligned} &|A_{J+1}\sin((kL+J+1)x-\phi_{J+1}) - A_J\sin((kL+J)x-\phi_J)| \\ &\qquad \leq \frac{24L^2\pi\max\{|A_J|,|A_{J+1}|\}}{kL+J} \end{aligned} \tag{2.6}$$

for an infinite number of k. Set $h(\theta) = A_{J+1}\sin(\theta + x - \phi_{J+1}) - A_J\sin(\theta - \phi_J)$. Note $\|h'\|_\infty \leq 2\max\{|A_J|,|A_{J+1}|\}$. Since $\int_{-\pi}^{\pi} h(\theta)d\theta = 0$ and h is continuous, there exists $\theta^* \in [-\pi,\pi)$ with $h(\theta^*) = 0$.

By Lemma 3 with $\mu = \dfrac{\theta^* - Jx}{2L\pi}$, there are an infinite number of positive integers k and m such that $\left|ka - m - \dfrac{\theta^* - Jx}{2L\pi}\right| < \dfrac{3}{k}$. Multiplying by $2L\pi$ and substituting $x = 2\pi a$, we obtain

$$|(kL+J)x - \theta^* - 2Lm\pi| < \frac{6L\pi}{k} \leq \left(\frac{6L\pi}{kL+J}\right)\left(\frac{L(k+1)}{k}\right) \leq \frac{12L^2\pi}{kL+J}.$$

Hence, since h is 2π periodic,

$$\begin{aligned} |h((kL+J)x)| &= |h((kL+J)x) - h(\theta^*)| \\ &= |h((kL+J)x) - h(\theta^* + 2Lm\pi)| \\ &\leq \left(\frac{||h'||_\infty 12L^2\pi}{kL+J}\right) \leq \frac{24L^2\pi\max\{|A_J|,|A_{J+1}|\}}{kL+J}, \end{aligned}$$

which gives (2.6). □

Lemma 5. *Suppose f is as in Lemma 1 and suppose that $A_N A_{N+1} \neq 0$. Let $\eta > 0$. Then if N is sufficiently large, there is a set $E_{N,\eta}$, such that for any $x \in [-\pi,\pi] \setminus E_{N,\eta}$ where $|g(x)| \leq \frac{4\pi}{N}\sum_{j=1}^{m}|d_j|$ (g as in Lemma 2), the absolute value of the numerator of (2.4) is bounded below by $c\min\{A_N^2, A_{N+1}^2\} - \frac{C}{N}$. Here c and C do not depend on x or N. $E_{N,\eta}$ is comprised of two or three intervals and has $|E_{N,\eta} \cap [-\pi,\pi]| = 8\eta$.*

Proof. We estimate the numerator of (2.4), rewriting as in (2.5):

$$\begin{aligned}
&|A_N \sin(Nx-\phi_N)A_{N+1}\sin((N+1)x-\phi_{N+1})\\
&+(N+1)\epsilon_{N+1}\pi A_N\sin(Nx-\phi_N)\\
&+N\epsilon_N\pi A_{N+1}\sin((N+1)x-\phi_{N+1})+N(N+1)\pi^2\epsilon_N\epsilon_{N+1}|\\
&\geq |A_N\sin(Nx-\phi_N)A_{N+1}\sin((N+1)x-\phi_{N+1})|\\
&\qquad -\frac{c_o}{N+1}-\frac{c_o}{N}-\frac{{c_o}^2}{N(N+1)}
\end{aligned}\tag{2.7}$$

for some $c_o > 0$, which (from Lemma 1) depends only on the d_j, a_j and $\|f''\|_\infty$. Here we have used the fact that $\epsilon_N \in \mathcal{O}(\frac{1}{N^2})$ to estimate the last three terms.

Suppose x has $|A_N\sin(Nx-\phi_N)-A_{N+1}\sin((N+1)x-\phi_{N+1})| = |g(x)| \leq \frac{4\pi}{N}\sum_{j=1}^m |d_j|$. Squaring both sides and rearranging yields

$$\begin{aligned}
&A_NA_{N+1}\sin((N+1)x-\phi_{N+1})\sin(Nx-\phi_N)\\
&\geq \frac{\min\{A_N^2,A_{N+1}^2\}}{2}\left[\sin^2((N+1)x-\phi_{N+1})+\sin^2(Nx-\phi_N)\right]\\
&\qquad -\frac{1}{2}\left(\frac{4\pi}{N}\sum_{j=1}^m|d_j|\right)^2.
\end{aligned}\tag{2.8}$$

Set $y = x - \dfrac{\phi_{N+1}+\phi_N}{2N+1}$ and $\Phi_N = \dfrac{N\phi_{N+1}-(N+1)\phi_N}{2N+1}$. Note that $-\frac{\pi}{2} \leq \Phi_N \leq \frac{\pi}{2}$. Then, using trigonometric identities,

$$\begin{aligned}
&\sin^2((N+1)x-\phi_{N+1})+\sin^2(Nx-\phi_N)\\
&=\sin^2((N+1)y-\Phi_N)+\sin^2(Ny+\Phi_N)\\
&=\left[\sin\left(Ny+\frac{y}{2}\right)\cos\left(\frac{y}{2}-\Phi_N\right)+\cos\left(Ny+\frac{y}{2}\right)\sin\left(\frac{y}{2}-\Phi_N\right)\right]^2\\
&+\left[\sin\left(Ny+\frac{y}{2}\right)\cos\left(\frac{y}{2}-\Phi_N\right)-\cos\left(Ny+\frac{y}{2}\right)\sin\left(\frac{y}{2}-\Phi_N\right)\right]^2\\
&=\sin^2\left(Ny+\frac{y}{2}\right)\cos^2\left(\frac{y}{2}-\Phi_N\right)+\cos^2\left(Ny+\frac{y}{2}\right)\sin^2\left(\frac{y}{2}-\Phi_N\right).
\end{aligned}$$

Let $\eta > 0$, but small, say $\eta < \frac{1}{10}$. Consider the set

$$\begin{aligned}
F_{N,\eta} &= (2\Phi_N-\eta,2\Phi_N+\eta)\cup(2\Phi_N-\pi-\eta,2\Phi_N-\pi+\eta)\\
&\quad\cup(2\Phi_N+\pi-\eta,2\Phi_N+\pi+\eta).
\end{aligned}$$

Then $\gamma = \sin^2\left(\pm\frac{\eta}{2}\right) = \cos^2\left(\frac{\pi}{2} \pm \frac{\eta}{2}\right)$ is a lower bound of $\sin^2\left(\frac{y}{2} - \Phi_N\right)$ and $\cos^2\left(\frac{y}{2} - \Phi_N\right)$, on $[-\pi, \pi] \setminus F_{N,\eta}$, and hence on $[-\pi, \pi] \setminus F_{N,\eta}$

$$\sin^2\left(Ny + \frac{y}{2}\right)\cos^2\left(\frac{y}{2} - \Phi_N\right) + \cos^2\left(Ny + \frac{y}{2}\right)\sin^2\left(\frac{y}{2} - \Phi_N\right)$$
$$\geq \sin^2\left(Ny + \frac{y}{2}\right)\gamma + \cos^2\left(Ny + \frac{y}{2}\right)\gamma = \gamma.$$

Put $\varphi_N = \phi_{N+1} - \phi_N$. For large enough N, independent of x, y, ϕ_N and ϕ_{N+1}, $|x-y| \leq \frac{\eta}{2}$ and $|2\Phi_N - \varphi_N| \leq \frac{\eta}{2}$. Since $|x-\varphi_N| \leq |y-x|+|y-2\Phi_N|+|2\Phi_N-\varphi_N|$, then $|x - \varphi_N| > 2\eta$, implies $|y - 2\Phi_N| > \eta$. Then for $x \in [-\pi, \pi] \setminus E_{N,\eta}$, where

$$E_{N,\eta} = (\varphi_N - 2\eta, \varphi_N + 2\eta) \cup (\varphi_N - \pi - 2\eta, \varphi_N - \pi + 2\eta)$$
$$\cup\, (\varphi_N + \pi - 2\eta, \varphi_N + \pi + 2\eta),$$

$y \in [-\pi, \pi] \setminus F_{N,\eta}$ and so we have $\sin^2((N+1)x - \phi_{N+1}) + \sin^2(Nx - \phi_N) \geq \gamma$. Note that when considered on the unit circle, $E_{N,\eta}$ consists of the circle minus two intervals: an interval centered at φ_N of length 4η, and an interval centered at $\varphi_N + \pi$ of length 4η. Thus, $|E_{N,\eta} \cap [-\pi, \pi]| = 8\eta$. Combining this, (2.7) and (2.8) gives, for large enough N, a lower bound of $\frac{\gamma}{2}\min\{A_N^2, A_{N+1}^2\} - \frac{C}{N}$ for (2.7) at any $x \in [-\pi, \pi] \setminus E_{N,\eta}$ at which $|g(x)| \leq \frac{4\pi}{N}\sum_{j=1}^m |d_j|$. □

Theorem 6. *Let f be a 2π periodic function having a finite number of jump discontinuities in $(-\pi, \pi)$ at $a_1 < a_2 < ... < a_m$. Suppose that for each j, $\lim_{x\to a_j^\pm} f(x) = f(a_j^\pm)$ and $\lim_{x\to a_j^\pm} f'(x) = f'(a_j^\pm)$ exist and are finite. Suppose that f'' is continuous except at the a_j, and bounded.*

(a) Suppose N is a sufficiently large positive integer such that $A_N A_{N+1} \neq 0$. Then there exists intervals on which $|T_N f(x) - S_N f(x)| \geq m\min\{A_N^2, A_{N+1}^2\} - \frac{c_1}{N}$, where m and c_1 are constants which do not depend on N or x.

(b) Suppose that the a_j are rational multiples of π, $a_j = \frac{p_j}{q_j}\pi$ (in lowest terms), so that the sequence A_n has period $L = 2\,\mathrm{lcm}\{q_j\}$. Suppose that there exists a $J \in \{1, \ldots, L\}$ such that $A_J A_{J+1} \neq 0$. Then $T_N f$ does not converge to $S_N f$ uniformly.

Proof. Using Lemma 2, we find for each interval $[-\pi+\frac{\pi j}{N+1}, -\pi+\frac{\pi(j+1)}{N+1}]$, $j = 0, 1, \ldots, 2N+1$, a subinterval where $|g(x)| \leq \frac{4\pi}{N}\sum_{j=1}^m |d_j|$, so that for x in these intervals the denominator of (2.4) can be estimated by $N\pi|g(x)| + \pi\sum_{j=1}^m |d_j| + C_* \leq (4\pi+1)\sum_{j=1}^m |d_j| + C_*$. Here we have written $N(N+1)\pi^2(\epsilon_{N+1} - \epsilon_N) \leq C_*$, where C_* is an absolute constant. (Recall $\epsilon_N \in \mathcal{O}(\frac{1}{N^2})$.) By Lemma 5, on these intervals intersected with $[-\pi, \pi] \setminus E_{N,\eta}$, the numerator is bounded below by

$c\min\{A_N^2, A_{N+1}^2\} - \frac{C}{N}$ if N is large enough. Thus, on these intervals, we have a lower bound for the size of (2.4) and (a) follows.

For (b), note that $A_{kL+J} = A_J$, $A_{kL+J+1} = A_{J+1}$. Then for $k = 1, 2, \ldots$, (a) gives a constant m and points x (depending on k) at which $|T_{kL+J}f(x) - S_{kL+J}f(x)| \geq m\min\{A_J^2, A_{J+1}^2\} - \frac{C}{kL+J}$, and hence when k is large enough shows (b). □

Theorem 7. *Assume f is as in Theorem 6. Suppose that the a_j are rational multiples of π, $a_j = \frac{p_j}{q_j}\pi$ (in lowest terms), so that the sequence A_n has period $L = 2\,\mathrm{lcm}\{q_j\}$. Suppose $x = 2a\pi$, where a is an irrational number, and suppose that there exists $J \in \{1, ..., L\}$ such that $A_J A_{J+1} \neq 0$. Then $\{T_N f(x)\}$ fails to converge.*

Proof. First note that the ϕ_N, and hence the φ_N, and therefore the sets $E_{N,\eta}$ are periodic of period L, so fixing a small $\eta > 0$ we have that the measure of $E = \cup_{N=1}^{\infty} E_{N,\eta}$ is less than $8\eta L$. For $x = 2\pi a$, a irrational, Lemma 4 and estimations as in the previous proof, immediately give an infinite number of k (which depend on x) for which the denominator of (2.4) with $N = kL + J$ is bounded. Lemma 5 gives a lower bound for the numerator at such x which are not in E. Since $|E|$ can be made small, the theorem follows. □

Remarks. Thus, we have shown that under the hypotheses of the Theorems, a transformed partial sum will always have "spikes", and that at each $x = 2\pi a$ with a irrational, these spikes will occur at x for an infinite number of N as $N \to \infty$.

Notice that if just one of A_N and A_{N+1} is 0 then in (2.4) the numerator is at most $\mathcal{O}(N^{-1})$, whereas the denominator seems to be either $\mathcal{O}(N)$, or bounded, or maybe smaller with sufficient cancellation. If both A_N and A_{N+1} are 0, then the numerator of (2.4) (see equation (2.7)) seems to be $\mathcal{O}(N^{-2})$ and the denominator seems to be bounded, or maybe smaller with sufficient cancellation. This is very imprecise, but in either case it seems likely that the numerator of (2.4) is much smaller than the denominator and thus, in these cases, we would expect that graphs of the $S_N f$ and $T_N f$ to look nearly identical for large N.

The condition $A_N \neq 0$ is equivalent to $\sum_{j=1}^{m} d_j e^{ina_j} \neq 0$. For a fixed m, fix $\theta \in [-\pi, \pi]$, place all the a_j at the mth roots of $e^{i\theta}$, and set all d_j equal so that $\sum_{j=1}^{m} d_j e^{ina_j} = 0$ for all n except multiples of m. Thus, for a function with equal jumps at these a_j, the Theorems will not apply. Other constructions having $A_N A_{N+1} = 0$ for every N are also possible.

If there exists a jump, say d_1, such that $\sum_{j=2}^{M} |d_j| < |d_1|$, then $A_J \neq 0$ for every J. This is in particular true if there is one jump or two unequal jumps. Thus, if f is as in the theorems, and the a_j are all rational multiples of π, then Theorem 7 applies to show that $\{T_N f(x)\}$ fails to converge at any $x = 2\pi a$, where a is irrational.

Consider the function f_1 which is given by $f(x) = 1$, if $-1 \leq x \leq 1$ and 0 otherwise. Figure 1 shows the 20th and 30th terms of the transformed partial sums, that is, $T_{20}f_1$ and $T_{30}f_1$. Define f_2 by

$$f_2(x) = \begin{cases} \frac{3}{\pi}x + 1 & \text{if } \ x \in [-\pi, \frac{-2\pi}{3}) \\ \frac{3}{2\pi}x + 1 & \text{if } \ x \in (\frac{-2\pi}{3}, 0) \\ 0 & \text{if } \ x \in (0, \frac{2\pi}{3}) \\ \frac{3}{\pi}x - 1 & \text{if } \ \in (\frac{2\pi}{3}, \pi]. \end{cases}$$

which is a function with jumps of 1 at points γ, where $e^{i\gamma}$ is a cube root of 1. The graphs of $S_{20}f_2$ and $T_{20}f_2$ are shown in figure 2. Here, as discussed above, our theorems do not apply, and indeed it is difficult to distinguish the graphs. Finally, define f_3 by

$$f_3(x) = \begin{cases} \frac{3}{\pi}x + 1 & \text{if } \ x \in [-\pi, \frac{-2\pi}{3}) \\ \frac{3}{4\pi}x + \frac{1}{2} & \text{if } \ x \in (\frac{-2\pi}{3}, 0) \\ 0 & \text{if } \ x \in (0, \frac{2\pi}{3}) \\ \frac{3}{\pi}x - 1 & \text{if } \ \in (\frac{2\pi}{3}, \pi]. \end{cases}$$

This has jumps at the same points as f_2, but the jumps are not equal, and it is easy to check that Theorem 7 does apply. Figure 3 gives the graphs of $S_{20}f_3$ and $T_{20}f_3$.

3. Conclusions

Regrettably, the methods in this paper seem specific to the δ^2 process, so we have not investigated the behavior of series transformed using other algorithms. In [4] the authors show the failure of the Lubkin transform when applied to the Fourier series of a function with one jump. We suspect this holds for functions with multiple jumps, but the computations become unwieldy. We also suspect that many other transformations also fail in the situation of multiple jumps. A more coherent theory is needed.

In most cases, the δ^2 process destroys the convergence of the partial sums of the Fourier series of functions with jumps at rational multiples of π. Only if

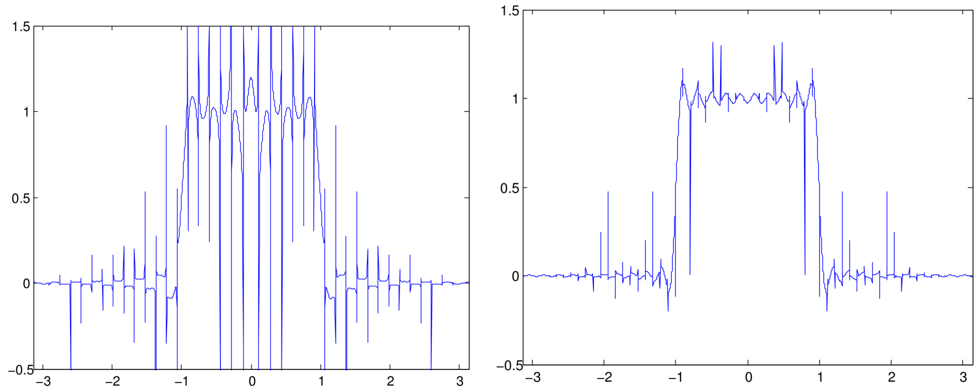

Figure 1: $T_{20}f_1$ and $T_{30}f_1$.

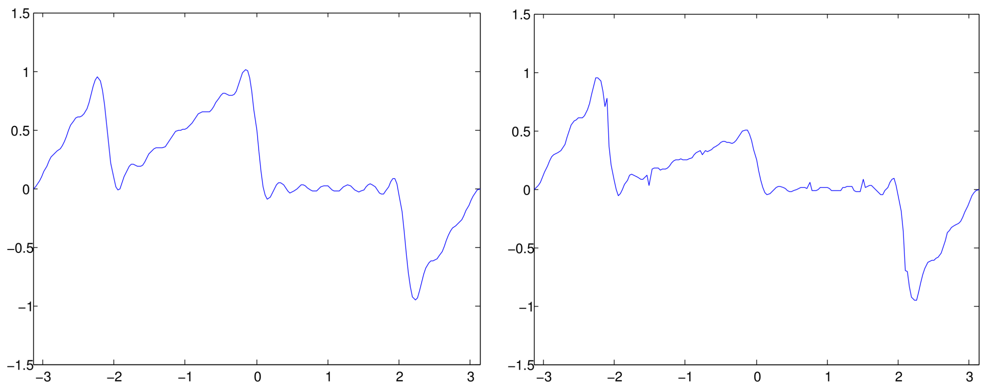

Figure 2: $S_{20}f_2$ and $T_{20}f_2$.

$A_N A_{N+1} = 0$ for every N does the δ^2 process not destroy convergence, and we have no evidence it improves convergence in this case. Applying the δ squared process to an analytic function results in a Padé approximation (see e.g. [5]); these can be produced using continued fractions, which is one approach to approximation theorems such as that of Chebyshev, and many authors have illuminated these connections. We used Chebyshev's theorem in a different way, and it would be interesting to see how these ideas are connected, and to have a deeper understanding of this circle of ideas. Although there has been some success accelerating Fourier series of some functions using the complex ε algorithm, that theory is far from complete, and we hope continued investigation will continue to develop a more complete theory.

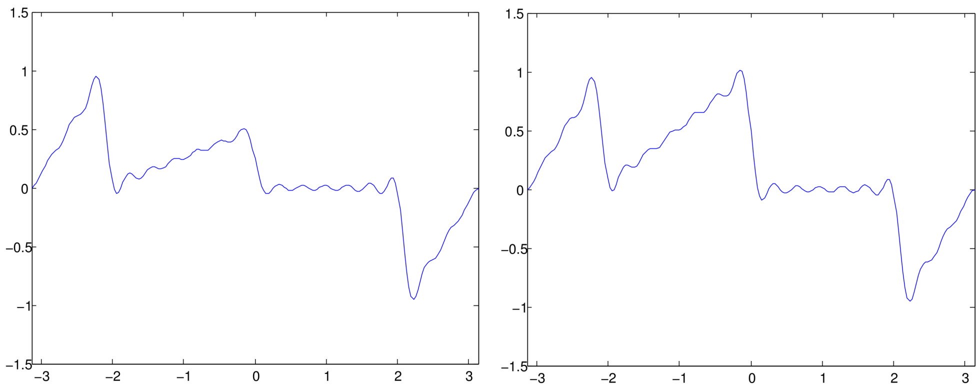

Figure 3: $S_{20}f_3$ and $T_{20}f_3$.

Acknowledgments

The authors were partially supported by the Kansas State University REU and NSF grant GOMT 1004336.

References

[1] A.C. Aitken, On Bernoulli's numerical solution of algebraic equations, *Proc. Roy. Soc. Edinburgh*, **46** (1926), 289-305.

[2] E. Abebe, J. Graber, C. N. Moore, Fourier series and the δ^2 process, *J. Comput. Appl. Math.*, **224**, No. 1 (2009), 146-151. http://dx.doi.org/10.1016/j.cam.2008.04.020

[3] B. Beckermann, A. Matos, F. Wielonsky, Reduction of the Gibbs phenomenon for smooth functions with jumps by the ε-algorithm, *J. Comput. Appl. Math.*, **219** (2008), 329-349. http://dx.doi.org/10.1016

[4] J. Boggess, E. Bunch, C.N. Moore, Fourier Series and the Lubkin W-transform, *Numer. Algorithms*, **47** (2008), 133 -142. http://dx.doi.org/10.1007/s11075-007-9151-x

[5] C. Brezinski, *Accélération de la Convergence en Analyse Numérique*, Springer-Verlag, Germany (1977).

[6] C. Brezinski, Extrapolation algorithms for filtering series of functions, and treating the Gibbs phenomenon, *Numer. Algorithms*, **36** (2004), 309-329. http://dx.doi.org/10.1007/s11075-004-2843-6

[7] C. Brezinski, M. Redivo-Zaglia, *Extrapolation Methods, Theory and Practice*, North-Holland, Netherlands (1991).

[8] J. E. Drummond, Convergence speeding, convergence and summability, *J. Comput. Appl. Math.*, **11** (1984), 145-159. http://dx.doi.org/10.1016/0377-0427(84)90017-7

[9] A. Ya. Khinchin, *Continued Fractions*, The University of Chicago Press, USA (1964).

[10] D. Levin, A. Sidi, Two new classes of non-linear transformations for accelerating the convergence of infinite integrals and series, *Appl. Math. Comp.*, **9** (1981), 175-215. doi: 10.1016/0096-3003(81)90028-X

[11] C. N. Moore, Acceleration of Fourier Series, *J. Anal.*, **17** (2009), 1-20.

[12] D. Shanks, Non-linear transformations of divergent and slowly convergent sequences, *J. Math. Phys.*, **34** (1955), 1-42.

[13] A. Sidi, Acceleration of convergence of (generalized) Fourier series by the d-transformation, *Annals of Numer. Math.*, **2** (1995), 381 -406.

[14] A. Sidi, *Practical Extrapolation Methods: Theory and Applications*, Cambridge University Press, England (2003).

[15] A. Sidi, Acceleration of convergence of general linear sequences by the Shanks transformation, *Numer. Math.*, **199** (2011), 725 -764.

[16] D. A. Smith and W. F. Ford, Numerical Comparisons of Nonlinear Convergence Accelerators, *Math. Comp.*, **38**, No. 158 (1982), 481-499. http://dx.doi.org/10.1090

[17] R. Tucker, The δ^2-Process and related topics, *Pacific J. Math.*, **22**, No. 2 (1967), 349-359. http://dx.doi.org/10.2140/pjm.1967.22.349

[18] P. Wynn, Transformations to accelerate the convergence of Fourier series, in: *Gertrude Blanch Anniversary Volume, Wright Patterson Airforce Base, Aeropspace Research Laboratories, Office of Aerospace Research, United States Air Force*, 1967, pp. 339-379.

[19] A. Zygmund, *Trigonometric Series, second ed.*, Cambridge University Press, England (1959).