THE $\delta$-SQUARED PROCESS AND
FOURIER SERIES OF FUNCTIONS WITH MULTIPLE JUMPS

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Abstract: We investigate the effects of the $\delta^2$ transform on the partial sums of Fourier series for functions with a finite number of jumps, which in general, converge slowly. Although the $\delta^2$ process is known to accelerate convergence for many sequences, we prove that in this case, the transformed series will usually fail to converge to the original function.

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1. Introduction

If \( f \in L^1([-\pi, \pi]) \), define the Fourier coefficients of \( f \) by

\[
\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx
\]

for each integer \( n \), and the \( N^{th} \) partial sum of the Fourier series by

\[
S_N f(x) := \sum_{k=-N}^{N} \hat{f}(k) e^{ikx},
\]

where \( N \) is a positive integer and \( x \in [-\pi, \pi] \).

When \( f \in L^2 \), \( S_N f \) converges to \( f \) in \( L^2 \). Theorems of Dini-Lipschitz, Lebesgue, and Dirichlet-Jordan give conditions for pointwise convergence (see e.g. Zygmund [19]). Fourier series are useful if they converge rapidly, which is often not the case, say, for a function with jump discontinuities. For such functions it is typical that the decay of the Fourier coefficients is \( O\left(\frac{1}{n}\right) \); in particular, these Fourier series do not converge absolutely. In this paper we investigate the possibility of applying a well-known sequence acceleration method to the partial sums of certain slowly converging Fourier series.

Given a sequence \( s_n \), the \( \delta^2 \) process transforms it to

\[
t_n := s_n - \frac{(s_{n+1} - s_n)(s_n - s_{n-1})}{(s_{n+1} - s_n) - (s_n - s_{n-1})},
\]

where we set \( t_n = s_n \) if the denominator of the fraction is zero.

This transform is usually attributed to Aitken [1], although the idea had appeared earlier. This transform and generalizations were studied extensively by Shanks [12], and it is for this reason this is sometimes called the Shanks transform. The \( \delta^2 \) process and many related transforms are discussed in Brezinski and Redivo-Zaglia [7] or Sidi [14]. Tucker [17] gives conditions on a convergent \( s_n \) so that the resulting \( t_n \) converge to the same limit but more quickly.

We will investigate what happens when the \( \delta^2 \) process is applied pointwise to the sequence of partial sums \( S_n f(x) \) of a function which is smooth except for a finite number of jumps. Usually such functions have Fourier series which converge slowly. In Abebe, Graber, and Moore [2] the authors investigate the application of the \( \delta^2 \) process to the partial sums of a function which has a single jump discontinuity. The methods needed in this paper to analyze the situation of multiple jumps are much more elaborate.
Smith and Ford [16] used numerical tests to compare different methods of convergence acceleration to the partial sums of Fourier series. They tested slowly and rapidly converging Fourier series and found, on the average, the transformation (1.2) improves convergence of the slowly converging series but slightly degrades convergence in the rapidly convergent case. Drummond [8] discusses many methods of convergence acceleration and includes discussion of their application to Fourier series. Sidi [13] discusses the effects of nonlinear sequence transformations on Fourier series. The d-transformation of Levin and Sidi [10] has been shown to be an effective means of acceleration of Fourier series in some situations. Sidi [15] has shown that the Shanks transformation is effective in accelerating the convergence of a class of infinite sequences which contain certain cases of Fourier series of piecewise smooth functions.

The \( \delta^2 \) process is same as \( \varepsilon_2^{(n)} \) in family of transforms \( \varepsilon_k^{(n)} \) known as the epsilon algorithm. Brezinski [6] (see also Wynn [18]) proposed the following procedure, called the complex \( \varepsilon \) algorithm: To \( S_n f \) add its conjugate function \( \overline{S}_n f \) to create an analytic function on the unit disk \( G_n f(z) \). Apply the epsilon algorithm to \( G_n f(e^{i\theta}) \) and then take the real part. Brezinski gives numerical experiments to demonstrate that this procedure reduces the Gibbs phenomenon. Beckermann, Matos, and Wielonsky [3] show that this method accelerates convergence for functions of the form \( f = f_1 + f_2 \), where \( f_1 \) has prescribed discontinuities but is smooth elsewhere, \( f_2 \) has quickly decaying Fourier coefficients, and \( G(f_1) = \lim_{n \to \infty} G_n(f_1) \) is a certain type of hypergeometric function. However, there are analytic functions on the unit disk, continuous on the boundary of the disk (that is, as a function of \( e^{i\theta}, \theta \in [−\pi, \pi] \)) for which the \( \delta^2 \) process destroys convergence. (See [11].) As we will see, the real \( \varepsilon_2^{(n)} \) algorithm, i.e. the \( \delta^2 \) process, performs badly on a large class of functions with jumps.

2. Main Results

Lemma 1. Let \( f \) be a \( 2\pi \) periodic function having a finite number of jump discontinuities in \((−\pi, \pi)\) at \( a_1 < a_2 < \ldots < a_m \). Suppose that for each \( j \), \( \lim_{x \to a_j^\pm} f(x) = f(a_j^\pm) \) and \( \lim_{x \to a_j^\pm} f'(x) = f'(a_j^\pm) \) exist and are finite. Suppose that \( f'' \) is continuous except at the \( a_j \), and bounded. For each \( j \) set \( d_j = f(a_j^+) - f(a_j^-) \) and \( d_j^* = f'(a_j^-) - f'(a_j^+) \). Then the \( N^{th} \) partial sum of the
Fourier series is

\[ S_N f(x) = \hat{f}(0) + \sum_{k=1}^{N} \left[ \frac{1}{k\pi} \sum_{j=1}^{m} [d_j \sin k(x - a_j)] + \epsilon_k \right], \quad (2.1) \]

where

\[ \epsilon_k = \frac{1}{k^2 \pi} \sum_{j=1}^{m} [d_j^* \cos k(x - a_j)] - \frac{\hat{f}''(k) e^{ikx} + \hat{f}''(-k) e^{-ikx}}{k^2}. \quad (2.2) \]

**Proof.** Integration by parts gives the \( n^{th} \) Fourier coefficient,

\[ \hat{f}(n) = \frac{1}{-2in\pi} \sum_{j=1}^{m} [(f(a_j^-) - f(a_j^+)) e^{-ina_j}] \]

\[ + \frac{1}{2n^2 \pi} \sum_{j=1}^{m} [(f'(a_j^-) - f'(a_j^+)) e^{-ina_j}] - \frac{1}{n^2} \hat{f}''(n). \]

By (1.1), \( S_N f(x) = \hat{f}(0) + \sum_{k=1}^{N} \hat{f}(-k) e^{-ikx} + \sum_{k=1}^{n} \hat{f}(k) e^{ikx} \), which simplifies to (2.1) and (2.2). \( \square \)

Applying the \( \delta^2 \) process pointwise to \( \{S_N f(x)\} \) yields

\[ T_N f(x) = S_N f(x) - \frac{(S_{N+1} f(x) - S_N f(x))(S_N f(x) - S_{N-1} f(x))}{(S_{N+1} f(x) - S_N f(x)) - (S_N f(x) - S_{N-1} f(x))} \]

\[ = S_N f(x) - \left( \frac{m}{\sum_{j=1}^{m} [d_j \sin\frac{(N+1)(x-a_j)}{(N+1)\pi}] + \epsilon_{N+1}} \right) \left[ \frac{m}{\sum_{j=1}^{m} [d_j \sin\frac{N(x-a_j)}{N\pi}] + \epsilon_N} \right], \quad (2.3) \]

where \( \epsilon_N \) is as defined in (2.2). Multiplying the numerator and denominator in the fraction on the right above by \( N(N+1)\pi^2 \), we obtain the expression.

\[ \frac{m}{\sum_{j=1}^{m} [d_j \sin (N+1)(x-a_j)] + (N+1)\pi \epsilon_{N+1}} \left[ \frac{m}{\sum_{j=1}^{m} [d_j \sin N(x-a_j)] + N \pi \epsilon_N} \right] = N\pi \sum_{j=1}^{m} [d_j \sin (N+1)(x-a_j)] - (N+1)\pi \sum_{j=1}^{m} [d_j \sin N(x-a_j)] + N(N+1)\pi^2 (\epsilon_{N+1} - \epsilon_N), \quad (2.4) \]

We note that \( \epsilon_N \in \mathcal{O}(\frac{1}{N^2}) \). We need estimates of both the numerator and denominator of (2.4). We begin with two lemmas which estimate the denominator, first for \( N \) fixed, then for \( x \) fixed.
Lemma 2. For $x_0$ fixed,

$$g(x) = \sum_{j=1}^{m} d_j (\sin ((N + 1)(x - a_j)) - \sin N(x - a_j))$$

satisfies $|g(x)| \leq \frac{4\pi}{N} \sum_{j=1}^{m} |d_j|$ on a subinterval of $[x_0, x_0 + \frac{\pi}{N+1}]$.

Proof. Note that

$$g\left(x_0 + \frac{\pi}{N+1}\right) = \sum_{j=1}^{m} d_j \left(\sin((N + 1)(x_0 - a_j) + \pi) - \sin(N(x_0 - a_j) + \frac{N\pi}{N+1})\right)$$

$$= \sum_{j=1}^{m} d_j \left(-\sin (N + 1)(x_0 - a_j) + \sin (N(x_0 - a_j) - \frac{\pi}{N+1})\right) = \sum_{j=1}^{m} d_j \left(-\sin (N + 1)(x_0 - a_j) + \sin N(x_0 - a_j) \cos \frac{\pi}{N+1} \right.$$

$$\left. - \cos N(x_0 - a_j) \sin \frac{\pi}{N+1}\right)$$

$$= -\sum_{j=1}^{m} d_j (\sin(N + 1)(x_0 - a_j) - \sin N(x_0 - a_j)) +$$

$$\sum_{j=1}^{m} d_j \left[\sin N(x_0 - a_j) \cos \left(\frac{\pi}{N+1} - 1\right) - \cos N(x_0 - a_j) \sin \frac{\pi}{N+1}\right]$$

$$= -g(x_0) + e(x_0, N),$$

where

$$e(x_0, N) = \sum_{j=1}^{m} d_j \left[\sin N(x_0 - a_j) \cos \left(\frac{\pi}{N+1} - 1\right) \right.$$ 

$$\left. - \cos N(x_0 - a_j) \sin \frac{\pi}{N+1}\right]$$

Note $|e(x_0, N)| \leq \sum_{j=1}^{m} |d_j| \left(\frac{2\pi}{N+1}\right) \leq \frac{2\pi}{N} \sum_{j=1}^{m} |d_j|$.

Case 1: If $|g(x_0)| \leq \frac{2\pi}{N} \sum_{j=1}^{m} |d_j|$, then since

$$|g'| \leq (\sum_{j=1}^{m} |d_j|)(2N + 1), \quad |g| \leq 2 \left(\frac{2\pi}{N}\right) \sum_{j=1}^{m} |d_j|$$

on a subinterval of $[x_0, x_0 + \frac{\pi}{N+1}]$ of length approximately $\frac{1}{N^\pi}$.
Case 2: If $|g(x_0)| > \frac{2\pi}{N} \sum_{j=1}^{m} |d_j|$, we claim $g(x_0)$ and $g\left(x_0 + \frac{\pi}{N+1}\right)$ have opposite signs: Say, without loss of generality, that $g(x_0) > 0$. Then $g(x_0) > \frac{2\pi}{N} \sum_{j=1}^{m} |d_j|$, so that $-g(x_0) < -\frac{2\pi}{N} \sum_{j=1}^{m} |d_j|$. Then

$$g(x_0 + \frac{\pi}{N+1}) = -g(x_0) + e(x_0, N) < -\frac{2\pi}{N} \sum_{j=1}^{m} |d_j| + \frac{2\pi}{N} \sum_{j=1}^{m} |d_j| = 0.$$ 

and thus, $g(x) = 0$ somewhere on the interval. As in Case 1, this implies that $g$ is bounded above by $|g| \leq 2 \left(\frac{2\pi}{N}\right) \sum_{j=1}^{m} |d_j|$ on an interval of length approximately $\frac{1}{N^2}$.

Now set $\alpha_N = \sum_{j=1}^{m} d_j \cos N\alpha_j$, $\beta_N = \sum_{j=1}^{m} d_j \sin N\alpha_j$, and $\phi_N = \arctan \left(\frac{\beta_N}{\alpha_N}\right)$. (If $\alpha_N = 0$, $\beta_N \geq 0$, set $\phi_N = \frac{\pi}{2}$, and if $\alpha_N = 0$, $\beta_N < 0$, set $\phi_N = -\frac{\pi}{2}$.) Set $A_N = \text{sgn}(\alpha_N)\sqrt{\alpha_N^2 + \beta_N^2}$. With these definitions it follows that for $x \in [-\pi, \pi]$,

$$\sum_{j=1}^{m} d_j \sin N(x - a_j) = \sum_{j=1}^{m} d_j \left(\sin N x \cos N\alpha_j - \cos N x \sin N\alpha_j\right)$$

$$= \alpha_N \sin N x - \beta_N \cos N x = A_N \sin(N x - \phi_N).$$

In Lemma 4 below, we will assume that for each $j$, $a_j = \frac{p_j}{q_j} \pi$ with $\frac{p_j}{q_j}$ in lowest terms. Then for $L = 2 \text{lcm}\{q_j\}$, the sequences $\alpha_n$, $\beta_n$, $A_n$ and $\phi_n$ have period $L$, that is, $\alpha_{n+L} = \alpha_n$ for all $n$, etc. We first need a result from elementary number theory.

**Lemma 3.** (Chebyshev; see [9], pg. 39) For an arbitrary irrational $a$ and arbitrary real $\mu$, there exists an infinite sequence of positive integers $k$ and $m$ such that $|ka - m - \mu| < \frac{3}{k}$.  

**Lemma 4.** Suppose that each $a_j$ is a rational multiple of $\pi$, that is, $a_j = \frac{p_j}{q_j} \pi$ which we assume to be in lowest terms. Suppose $x = 2a\pi$, where $a \in [-.5,.5]$ is an irrational number. Fix $J \in \{1, \ldots, L\}$ which has $A_J A_{J+1} \neq 0$. Then there are an infinite number of integers $n = kL + J$ such that

$$|A_{n+1} \sin ((n+1)x - \phi_{n+1}) - A_n \sin (nx - \phi_n)| \leq \max\{|A_J|, |A_{J+1}|\} \frac{24L^2\pi}{n}.$$
Proof. Note that for \( n = kL + J \), we have \( A_n = A_J, A_{n+1} = A_{J+1} \), \( \phi_n = \phi_J \), and \( \phi_{n+1} = \phi_{J+1} \). Thus, we must show that

\[
|A_{J+1} \sin ((kL + J+1)x - \phi_{J+1}) - A_J \sin ((kL + J)x - \phi_J)|
\leq \frac{24L^2\pi \max\{|A_J|, |A_{J+1}|\}}{kL + J}
\]

(2.6)

for an infinite number of \( k \). Set \( h(\theta) = A_{J+1} \sin (\theta + x - \phi_{J+1}) - A_J \sin (\theta - \phi_J) \).

Note \( \|h'\|_{\infty} \leq 2 \max\{|A_J|, |A_{J+1}|\} \). Since \( \int_{-\pi}^{\pi} h(\theta)d\theta = 0 \) and \( h \) is continuous, there exists \( \theta^* \in [-\pi, \pi) \) with \( h(\theta^*) = 0 \).

By Lemma 3 with \( \mu = \frac{\theta^* - Jx}{2L\pi} \), there are an infinite number of positive integers \( k \) and \( m \) such that \( \left| ka - m - \frac{\theta^* - Jx}{2L\pi} \right| < \frac{3}{k} \). Multiplying by \( 2L\pi \) and substituting \( x = 2\pi a \), we obtain

\[
|(kL + J)x - \theta^* - 2Lm\pi| < \frac{6L\pi}{k} \leq \left( \frac{6L\pi}{kL + J} \right) \left( \frac{L(k + 1)}{k} \right) \leq \frac{12L^2\pi}{kL + J}.
\]

Hence, since \( h \) is \( 2\pi \) periodic,

\[
|h((kL + J)x)| = |h((kL + J)x) - h(\theta^*)|
\leq \left( \frac{\|h'\|_{\infty} 12L^2\pi}{kL + J} \right) \leq \frac{24L^2\pi \max\{|A_J|, |A_{J+1}|\}}{kL + J},
\]

which gives (2.6).

Lemma 5. Suppose \( f \) is as in Lemma 1 and suppose that \( A_N A_{N+1} \neq 0 \). Let \( \eta > 0 \). Then if \( N \) is sufficiently large, there is a set \( E_{N,\eta} \), such that for any \( x \in [-\pi, \pi] \setminus E_{N,\eta} \) where \( |g(x)| \leq \frac{4\pi}{N} \sum_{j=1}^{m} |d_j| \) (\( g \) as in Lemma 2), the absolute value of the numerator of (2.4) is bounded below by \( c \min\{A_N^2, A_{N+1}^2\} - \frac{C}{N} \).

Here \( c \) and \( C \) do not depend on \( x \) or \( N \). \( E_{N,\eta} \) is comprised of two or three intervals and has \( |E_{N,\eta} \cap [-\pi, \pi]| = 8\eta \).
Proof. We estimate the numerator of (2.4), rewriting as in (2.5):

\[
\begin{align*}
|A_N \sin (Nx - \phi_N)A_{N+1} \sin ((N+1)x - \phi_{N+1}) & \quad + (N+1)\epsilon_{N+1} \pi A_N \sin (Nx - \phi_N) \\
& \quad + N\epsilon_N \pi A_{N+1} \sin ((N+1)x - \phi_{N+1}) + N(N+1)\pi^2 \epsilon_N \epsilon_{N+1}| \\
& \geq |A_N \sin (Nx - \phi_N)A_{N+1} \sin ((N+1)x - \phi_{N+1})| \\
& \quad - \frac{c_o}{N+1} - \frac{c_o}{N(N+1)} \\
& \geq |A_N \sin (N x - \phi_N)A_{N+1} \sin ((N+1)x - \phi_{N+1})| \\
& \quad - \frac{c_o}{N+1} - \frac{c_o}{N(N+1)} \quad \text{(2.7)}
\end{align*}
\]

for some \(c_o > 0\), which (from Lemma 1) depends only on the \(d_j, a_j\) and \(\|f''\|_{\infty}\). Here we have used the fact that \(\epsilon_N \in O\left(\frac{1}{N^2}\right)\) to estimate the last three terms.

Suppose \(x\) has

\[
|A_N \sin (Nx - \phi_N) - A_{N+1} \sin ((N+1)x - \phi_{N+1})| = |g(x)| \leq \frac{4\pi}{N} \sum_{j=1}^{m} |d_j|.
\]

Squaring both sides and rearranging yields

\[
\begin{align*}
A_N A_{N+1} \sin ((N+1)x - \phi_{N+1}) \sin (Nx - \phi_N) \\
& \geq \min \{A_N^2, A_{N+1}^2\} \left[ \sin^2 ((N+1)x - \phi_{N+1}) + \sin^2 (Nx - \phi_N) \right] \\
& \quad - \frac{1}{2} \left( \frac{4\pi}{N} \sum_{j=1}^{m} |d_j| \right)^2 \\
& \quad \text{(2.8)}
\end{align*}
\]

Set \(y = x - \frac{\phi_{N+1}}{2N+1} + \frac{\phi_N}{2N+1}\) and \(\Phi_N = \frac{N\phi_{N+1} - (N+1)\phi_N}{2N+1}\). Note that \(-\frac{\pi}{2} \leq \Phi_N \leq \frac{\pi}{2}\). Then, using trigonometric identities,

\[
\begin{align*}
\sin^2 ((N+1)x - \phi_{N+1}) + \sin^2 (Nx - \phi_N) \\
& = \sin^2 ((N+1)y - \Phi_N) + \sin^2 (Ny + \Phi_N) \\
& = \left[ \sin \left( Ny + \frac{y}{2} \right) \cos \left( \frac{y}{2} - \Phi_N \right) + \cos \left( Ny + \frac{y}{2} \right) \sin \left( \frac{y}{2} - \Phi_N \right) \right]^2 \\
& \quad + \left[ \sin \left( Ny + \frac{y}{2} \right) \cos \left( \frac{y}{2} - \Phi_N \right) - \cos \left( Ny + \frac{y}{2} \right) \sin \left( \frac{y}{2} - \Phi_N \right) \right]^2 \\
& = \sin^2 \left( Ny + \frac{y}{2} \right) \cos^2 \left( \frac{y}{2} - \Phi_N \right) + \cos^2 \left( Ny + \frac{y}{2} \right) \sin^2 \left( \frac{y}{2} - \Phi_N \right).
\end{align*}
\]

Let \(\eta > 0\), but small, say \(\eta < \frac{1}{10}\). Consider the set

\[
F_{N,\eta} = (2\Phi_N - \eta, 2\Phi_N + \eta) \cup (2\Phi_N - \pi - \eta, 2\Phi_N - \pi + \eta) \\
\cup (2\Phi_N + \pi - \eta, 2\Phi_N + \pi + \eta).
\]
Then \( \gamma = \sin^2\left(\pm \frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{2} \pm \frac{\pi}{2}\right) \) is a lower bound of \( \sin^2\left(\frac{\pi}{2} - \Phi_N\right) \) and \( \cos^2\left(\frac{\pi}{2} - \Phi_N\right) \), on \([−\pi, \pi] \setminus F_{N,\eta}\), and hence on \([−\pi, \pi] \setminus F_{N,\eta}\

\[
\sin^2\left(Ny + \frac{y}{2}\right) \cos^2\left(\frac{y}{2} - \Phi_N\right) + \cos^2\left(Ny + \frac{y}{2}\right) \sin^2\left(\frac{y}{2} - \Phi_N\right)
\geq \sin^2\left(Ny + \frac{y}{2}\right)\gamma + \cos^2\left(Ny + \frac{y}{2}\right)\gamma = \gamma.
\]

Put \( \varphi_N = \phi_{N+1} - \phi_N \). For large enough \( N \), independent of \( x, y, \phi_N \) and \( \phi_{N+1} \), \( |x-y| \leq \frac{\eta}{2} \) and \( |2\Phi_N - \varphi_N| \leq \frac{\eta}{2} \). Since \( |x-\varphi_N| \leq |y-x| + |y-2\Phi_N| + |2\Phi_N - \varphi_N| \), then \( |x-\varphi_N| > \eta \), implies \( |y - 2\Phi_N| > \eta \). Then for \( x \in [−\pi, \pi] \setminus E_{N,\eta} \), where

\[
E_{N,\eta} = (\varphi_N - 2\eta, \varphi_N + 2\eta) \cup (\varphi_N - \pi - 2\eta, \varphi_N - \pi + 2\eta) \\
\cup (\varphi_N + \pi - 2\eta, \varphi_N + \pi + 2\eta),
\]

\( y \in [−\pi, \pi] \setminus F_{N,\eta} \) and so we have \( \sin^2((N+1)x - \phi_{N+1}) + \sin^2(Nx - \phi_N) \geq \gamma \).

Note that when considered on the unit circle, \( E_{N,\eta} \) consists of the circle minus two intervals: an interval centered at \( \varphi_N \) of length \( 4\eta \), and an interval centered at \( \varphi_N + \pi \) of length \( 4\eta \). Thus, \( |E_{N,\eta} \cap [−\pi, \pi]| = 8\eta \). Combining this, (2.7) and (2.8) gives, for large enough \( N \), a lower bound of \( \frac{\eta}{2}\min\{A_N^2, A_{N+1}^2\} - \frac{C\eta}{N} \) for (2.7) at any \( x \in [−\pi, \pi] \setminus E_{N,\eta} \) at which \( |g(x)| \leq \frac{4\pi}{N} \sum_{j=1}^{m} |d_j| \).

**Theorem 6.** Let \( f \) be a \( 2\pi \) periodic function having a finite number of jump discontinuities in \((−\pi, \pi)\) at \( a_1 < a_2 < ... < a_m \). Suppose that for each \( j \), \( \lim_{x \to a_j^\pm} f(x) = f(a_j^\pm) \) and \( \lim_{x \to a_j^\pm} f'(x) = f'(a_j^\pm) \) exist and are finite. Suppose that \( f'' \) is continuous except at the \( a_j \), and bounded.

(a) Suppose \( N \) is a sufficiently large positive integer such that \( A_N A_{N+1} \neq 0 \). Then there exists intervals on which \( |T_N f(x) - S_N f(x)| \geq m \min\{A_N^2, A_{N+1}^2\} - \frac{C\eta}{N} \), where \( m \) and \( c_1 \) are constants which do not depend on \( N \) or \( x \).

(b) Suppose that the \( a_j \) are rational multiples of \( \pi \), \( a_j = \frac{p_j}{q_j} \pi \) (in lowest terms), so that the sequence \( A_n \) has period \( L = 2 \text{lcm}\{q_j\} \). Suppose that there exists a \( J \in \{1, \ldots, L\} \) such that \( A_J A_{J+1} \neq 0 \). Then \( T_N f \) does not converge to \( S_N f \) uniformly.

**Proof.** Using Lemma 2, we find for each interval \([−\pi + \frac{\pi j}{N+1}, −\pi + \frac{\pi (j+1)}{N+1}]\), \( j = 0, 1, \ldots, 2N+1 \), a subinterval where \( |g(x)| \leq \frac{4\pi}{N} \sum_{j=1}^{m} |d_j| \), so that for \( x \) in these intervals the denominator of (2.4) can be estimated by \( N\pi |g(x)| + \pi \sum_{j=1}^{m} |d_j| + C_* \leq (4\pi+1) \sum_{j=1}^{m} |d_j| + C_* \). Here we have written \( N(N+1)\pi^2(\epsilon_{N+1} - \epsilon_N) \leq C_* \), where \( C_* \) is an absolute constant. (Recall \( \epsilon_N \in \mathcal{O}(\frac{1}{N^2}) \).) By Lemma 5, on these intervals intersected with \([−\pi, \pi] \setminus E_{N,\eta} \), the numerator is bounded below by
c \min \{A_N^2, A_{N+1}^2\} - \frac{C}{N} \text{ if } N \text{ is large enough. Thus, on these intervals, we have}

a lower bound for the size of (2.4) and (a) follows.

For (b), note that \( A_{kL+J} = A_J, A_{kL+J+1} = A_{J+1} \). Then for \( k = 1, 2, \ldots, \)

(a) gives a constant \( m \) and points \( x \) (depending on \( k \)) at which \( |T_{kL+J}f(x) - S_{kL+J}f(x)| \geq m \min \{A_J^2, A_{J+1}^2\} - \frac{C}{kL+J} \), and hence when \( k \) is large enough shows (b).

\[ \square \]

**Theorem 7.** Assume \( f \) is as in Theorem 6. Suppose that the \( a_j \) are rational multiples of \( \pi, a_j = \frac{p_j}{q_j} \pi \) (in lowest terms), so that the sequence \( A_n \) has period \( L = 2 \lcm \{q_j\} \). Suppose \( x = 2a\pi, \) where \( a \) is an irrational number, and suppose that there exists \( J \in \{1, \ldots, L\} \) such that \( A_JA_{J+1} \neq 0 \). Then \( \{T_Nf(x)\} \) fails to converge.

**Proof.** First note that the \( \phi_N, \) and hence the \( \varphi_N, \) and therefore the sets \( E_{N,\eta} \) are periodic of period \( L, \) so fixing a small \( \eta > 0 \) we have that the measure of \( E = \cup_{N=1}^{\infty} E_{N,\eta} \) is less than \( 8\eta L. \) For \( x = 2\pi a, \) a irrational, Lemma 4 and estimations as in the previous proof, immediately give an infinite number of \( k \) (which depend on \( x \)) for which the denominator of (2.4) with \( N = kL + J \) is bounded. Lemma 5 gives a lower bound for the numerator at such \( x \) which are not in \( E. \) Since \( |E| \) can be made small, the theorem follows. \[ \square \]

**Remarks.** Thus, we have shown that under the hypotheses of the Theorems, a transformed partial sum will always have “spikes”, and that at each \( x = 2\pi a \) with \( a \) irrational, these spikes will occur at \( x \) for an infinite number of \( N \) as \( N \to \infty. \)

Notice that if just one of \( A_N \) and \( A_{N+1} \) is 0 then in (2.4) the numerator is at most \( \mathcal{O}(N^{-1}), \) whereas the denominator seems to be either \( \mathcal{O}(N), \) or bounded, or maybe smaller with sufficient cancellation. If both \( A_N \) and \( A_{N+1} \) are 0, then the numerator of (2.4) (see equation (2.7)) seems to be \( \mathcal{O}(N^{-2}) \) and the denominator seems to be bounded, or maybe smaller with sufficient cancellation. This is very imprecise, but in either case it seems likely that the numerator of (2.4) is much smaller than the denominator and thus, in these cases, we would expect that graphs of the \( S_Nf \) and \( T_Nf \) to look nearly identical for large \( N. \)

The condition \( A_N \neq 0 \) is equivalent to \( \sum_{j=1}^{m} d_j e^{ina_j} \neq 0. \) For a fixed \( m, \) fix \( \theta \in [-\pi, \pi], \) place all the \( a_j \) at the \( m \)th roots of \( e^{i\theta}, \) and set all \( d_j \) equal so that \( \sum_{j=1}^{m} d_j e^{ina_j} = 0 \) for all \( n \) except multiples of \( m. \) Thus, for a function with equal jumps at these \( a_j, \) the Theorems will not apply. Other constructions having \( A_NA_{N+1} = 0 \) for every \( N \) are also possible.
If there exists a jump, say \( d_1 \), such that \( \sum_{j=2}^{M} |d_j| < |d_1| \), then \( A_J \neq 0 \) for every \( J \). This is in particular true if there is one jump or two unequal jumps. Thus, if \( f \) is as in the theorems, and the \( a_j \) are all rational multiples of \( \pi \), then Theorem 7 applies to show that \( \{ T_N f(x) \} \) fails to converge at any \( x = 2\pi a \), where \( a \) is irrational.

Consider the function \( f_1 \) which is given by \( f(x) = 1 \), if \(-1 \leq x \leq 1\) and 0 otherwise. Figure 1 shows the 20th and 30th terms of the transformed partial sums, that is, \( T_{20} f_1 \) and \( T_{30} f_1 \). Define \( f_2 \) by

\[
 f_2(x) = \begin{cases} 
 3\pi x + 1 & \text{if } x \in [-\pi, -2\pi/3) \\
 3\pi x + 1 & \text{if } x \in (-2\pi/3, 0) \\
 0 & \text{if } x \in (0, 2\pi/3) \\
 3\pi x - 1 & \text{if } x \in (2\pi/3, \pi]. 
\end{cases}
\]

which is a function with jumps of 1 at points \( \gamma \), where \( e^{i\gamma} \) is a cube root of 1. The graphs of \( S_{20} f_2 \) and \( T_{20} f_2 \) are shown in figure 2. Here, as discussed above, our theorems do not apply, and indeed it is difficult to distinguish the graphs. Finally, define \( f_3 \) by

\[
 f_3(x) = \begin{cases} 
 3\pi x + 1 & \text{if } x \in [-\pi, -2\pi/3) \\
 3\pi x + \frac{1}{2} & \text{if } x \in (-2\pi/3, 0) \\
 0 & \text{if } x \in (0, 2\pi/3) \\
 3\pi x - 1 & \text{if } x \in (2\pi/3, \pi]. 
\end{cases}
\]

This has jumps at the same points as \( f_2 \), but the jumps are not equal, and it is easy to check that Theorem 7 does apply. Figure 3 gives the graphs of \( S_{20} f_3 \) and \( T_{20} f_3 \).

3. Conclusions

Regrettably, the methods in this paper seem specific to the \( \delta^2 \) process, so we have not investigated the behavior of series transformed using other algorithms. In [4] the authors show the failure of the Lubkin transform when applied to the Fourier series of a function with one jump. We suspect this holds for functions with multiple jumps, but the computations become unwieldy. We also suspect that many other transformations also fail in the situation of multiple jumps. A more coherent theory is needed.

In most cases, the \( \delta^2 \) process destroys the convergence of the partial sums of the Fourier series of functions with jumps at rational multiples of \( \pi \). Only if
$A_N A_{N+1} = 0$ for every $N$ does the $\delta^2$ process not destroy convergence, and we have no evidence it improves convergence in this case. Applying the $\delta$ squared process to an analytic function results in a Padé approximation (see e.g. [5]); these can be produced using continued fractions, which is one approach to approximation theorems such as that of Chebyshev, and many authors have illuminated these connections. We used Chebyshev’s theorem in a different way, and it would be interesting to see how these ideas are connected, and to have a deeper understanding of this circle of ideas. Although there has been some success accelerating Fourier series of some functions using the complex $\varepsilon$ algorithm, that theory is far from complete, and we hope continued investigation will continue to develop a more complete theory.
Figure 3: $S_{20}f_3$ and $T_{20}f_3$.

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References


