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WEAK-STAR CONVERGENCE OF MEASURES

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Abstract: Consider a lebesgue measure λ on $[0\,,\,1]$ and let $0 \leq h \in L^1(\lambda)$. We construct a measure μ such that $\|\mu\| \leq 2.\ \|h\|_{L^1(\lambda)}$, $\mu \perp \lambda$ and $\mu\,(E) = 0$, for every Carleson set E in $[0\,,1]$ and converges weak-star to $h\,d\lambda$.

AMS Subject Classification: Carleson set, weak-star convergence, Cantor set, $L^1(\lambda)$

1. Introduction

For the convenience of the reader, we introduce the following definition and example.

Definition 1.1. A subset E of ∂D is called a *Carleson set* if E is closed with Lebesgue measure zero and $\sum\limits_{v} |I_v| \log \frac{1}{|I_v|} < \infty$, where $\{I_v\}$ are the complementary arcs of E

Example. Let E be the Cantor set on $[0\,,\,1]$. It can be easily seen that:

1. E is closed,
2. E is of Lebesgue measure zero, and
3. $[0\,,1] \setminus E$ is the pairwise disjoint union $\bigcup\limits_{k=0}^{\infty} \left(\bigcup\limits_{n=1}^{2^k} I_{n\,,\,k} \right)$, where each $I_{n\,,\,k}$

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has length $\frac{1}{3^{k+1}}$. Therefore:

$$\sum m(I_k)\,\log\left(\frac{1}{m(I_k)}\right) = \sum_{k=0}^{\infty} \frac{2^k}{3^{k+1}}\,.\,\log\left(\frac{1}{\left(\frac{1}{3^{k+1}}\right)}\right)$$
$$= \frac{1}{3}\sum_{k=1}^{\infty}(k+1)\,.\,\left(\frac{2}{3}\right)^k \log(3) < \infty$$

by the Ratio Test. Consequently, E is a Carleson subset of $[0\,,\,1]$.

2. Main Result

Let λ denote Lebesgue measure on $[0\,,\,1]$ and suppose that $0 \le h \in L^1(\lambda)$. Then, for any $\varepsilon > 0$ and $C_\circ \ge 1$ there exists a measure μ such that $\|\mu\| \le 2.\,\|h\|_{L^1(\lambda)}$, $\mu \perp \lambda$, $\mu(E) = 0$ for every Carelson set E in $[0\,,1]$, and

$$\left|\int\limits_{[0\,,1]} (g\,d\mu - g\,h\,d\lambda)\right| \le \varepsilon,$$

whenever $0 \le g \in C^\infty([0,1])$ and $\|g\acute{}\|_\infty\,, \|g\|_\infty \le C_\circ$.

Proof. Choose ε, $0 < \varepsilon < 1$. Since $\|g\|_\infty \le C_\circ$, we can apply the Mean-Value Theorem and find a partition $\{x_i\,:\,i = 0, ..., M\}$ of $[0\,,\,1]$ that depends only on the constant $C_\circ$ and a step function $g^* = \sum_{i=1}^{M} \alpha_i\,\chi_{F_i}$ $(F_i := [x_{i-1}, x_i])$ such that

$$\|g - g^*\|_\infty \le \frac{\varepsilon}{4}, \tag{1}$$

and $|\alpha_i| \le C_\circ$ for $i = 1, ..., M$. Since h is integrable (we may assume that $\|h\|_{L^1(\lambda)} = 1$) there exists a step function $h^* = \sum_{j=1}^{N} \beta_j\,\chi_{B_j}$, where the $B_{j's}$ are intervals that form a partition of $[0, 1]$ such that

$$\left|\int\limits_{[0\,,1]} (h^*\,d\lambda - \,h\,d\lambda)\right| \le \frac{\varepsilon}{4\,M\;C_\circ}. \tag{2}$$

Also, there exists $\delta > 0$ such that whenever $A \subseteq [0,1]$ with $\lambda(A) \le \delta$, then

$$\int\limits_A h^* \, d\lambda \le \frac{\varepsilon}{16 \; M \; C_\circ}. \tag{3}$$

Start with a measure η on $[0,1]$ such that $\eta \perp \lambda$, $\eta([0,1]) = 1$ and $\eta(E) = 0$ for every Carleson set E in $[0,1]$ (see [1]). Let K be a large enough fixed integer such that $\frac{1}{K} \le \delta$ and let $A_k = \left[\frac{k-1}{K}, \frac{k}{K}\right]$, $k = 1, ..., K$. It is clear that $\bigcup\limits_{k=1}^{K} A_k = [0,1]$ and $\lambda(A_k) = \frac{1}{K}$. For $0 < t < 1$, define η_t on the Borel subsets B of $[0,1]$ by $\eta_t(B) = \eta(\frac{1}{t}B)$. Define μ_k on the Borel subsets B of A_k by $\mu_k(B) = \eta_{1/K}(B - (\frac{k-1}{K}))$ ($k = 1, ..., K$). Notice that $\mu_k(A_j) = 0$, whenever $k \ne j$ and $\mu_k(A_k) = 1$, for $k = 1, 2, ..., K$. Now, if E is a Carleson set, then so is $\alpha E - \beta$ for any constants α, β, and hence $\mu_k(E) = 0$. Let $\mu = \sum\limits_{k=1}^{K} c_k \, \mu_k$ where $c_k = \sum\limits_{j=1}^{N} \beta_j \, \lambda(B_j \cap A_k)$. Then $\mu(E) = 0$ for every Carleson set E in $[0,1]$ and

$$\begin{aligned}
\mu([0,1]) =& \sum_{k=1}^{K} c_k \, \mu_k([0,1]) = \sum_{k=1}^{K} \sum_{j=1}^{N} \beta_j \, \lambda(B_j \cap A_k) \\
=& \sum_{j=1}^{N} \beta_j (\sum_{k=1}^{K} \lambda(B_j \cap A_k)) = \sum_{k=1}^{K} \beta_j \, \lambda(B_j) = \int\limits_{[0,1]} h^* \, d\lambda \\
\le& 2 \, \|h\| \, L^1(\lambda).
\end{aligned}$$

Let F be any closed subinterval of $[0,1]$, and let

$$\Delta = \{ k \, : k = 1, ..., K \quad \text{and} \;\; A_k \subseteq F \} .$$

Now, there are at most two intervals $A_{k_\circ}$, A_{k_1} such that $F \cap A_{k_i} \ne \emptyset$ for $i = 0, 1$ and yet k_0, $k_1 \notin \Delta$. Then

$$\begin{aligned}
\int\limits_F (d\mu - h^* \, d\lambda) =& \mu(F) - \sum_{j=1}^{N} \beta_j \, \lambda(B_j \cap F) \\
=& \sum_{k=1}^{K} c_k \, \mu_k(A_k \cap F) - \sum_{j=1}^{N} \beta_j \, \lambda(B_j \cap \; F)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in\Delta}\sum_{j=1}^{N}\beta_j\,\lambda(B_j\cap A_k)\,.\,\mu_k(A_k\cap F) - \sum_{j=1}^{N}\beta_j\,\lambda(B_j\cap(\bigcup_{k\in\Delta}A_k))\\
&\quad + \sum_{j=1}^{N}\beta_j\,\lambda(B_j\cap A_{k_\circ})\,.\,\mu_{k_\circ}(A_{k_\circ}\cap F)\\
&\quad + \sum_{j=1}^{N}\beta_j\,\lambda(B_j\cap A_{k_1})\,.\,\mu_{k_1}(A_{k_1}\cap F)\\
&\quad - \sum_{j=1}^{N}\beta_j\,\lambda(B_j\cap F\cap A_{k_\circ}) - \sum_{j=1}^{N}\beta_j\,\lambda(B_j\cap F\cap A_{k_1})\\
&= \sum_{j=1}^{N}\beta j\,\lambda(B_j\cap A_{k_\circ})\,.\,\mu_{k_\circ}(A_{k_\circ}\cap F)\\
&\quad + \sum_{j=1}^{N}\beta_j\,\lambda(B_j\cap A_{k_1})\,.\,\mu_{k_1}(A_{k_1}\cap F)\\
&\quad - \sum_{j=1}^{N}\beta_j\,\lambda(B_j\cap F\cap A_{k_\circ}) - \sum_{j=1}^{N}\beta_j\,\lambda(B_j\cap F\cap A_{k_1}).
\end{aligned}
$$

Therefore

$$
\left|\int_F (d\mu - h^*\,d\lambda)\right| \le 2\left[\int_{A_{k_\circ}} h^*\,d\lambda + \int_{A_{k_1}} h^*\,d\lambda\right] \overset{by\,(3)}{\le} \frac{\varepsilon}{4\,M\,C_\circ}. \tag{4}
$$

Now

$$
\begin{aligned}
\left|\int_{[0\,,1]}(g\,d\mu - g\,h\,d\lambda)\right| &\le \int_{[[0,1]} |g-g^*|\;d\mu + \left|\int_{[0,1]} g^*(d\mu - hd\lambda)\right|\\
&\quad + \int_{[0,1]} |g-g^*|\,hd\lambda\\
&\overset{by\,(1)}{\le} \frac{\varepsilon}{2} + \sum_{i=1}^{M}|\alpha_i|\left|\int_{F_i} d\mu - h\,d\lambda\right| \le \frac{\varepsilon}{2}\\
&\quad + MC_\circ\left|\int_{F_i} d\mu - hd\lambda\right|
\end{aligned}
$$

$$\leq \frac{\varepsilon}{2} + MC_{\circ} \left[\left| \int\limits_{F_i} d\mu - h^* d\lambda \right| + \left| \int\limits_{F_i} (h^* d\lambda - h d\lambda) \right| \right]$$

$$\overset{by\,(2)\&(4)}{\leq} \varepsilon. \qquad \square$$

References

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