WEAK-STAR CONVERGENCE OF MEASURES

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Abstract: Consider a lebesgue measure \( \lambda \) on \([0, 1]\) and let \( 0 \leq h \in L^1(\lambda) \). We construct a measure \( \mu \) such that \( \|\mu\| \leq 2 \|h\|_{L^1(\lambda)} \), \( \mu \perp \lambda \) and \( \mu(E) = 0 \), for every Carleson set \( E \) in \([0, 1]\) and converges weak-star to \( h \, d\lambda \).

AMS Subject Classification: Carleson set, weak-star convergence, Cantor set, \( L^1(\lambda) \)

1. Introduction

For the convenience of the reader, we introduce the following definition and example.

Definition 1.1. A subset \( E \) of \( \partial D \) is called a Carleson set if \( E \) is closed with Lebesgue measure zero and \( \sum_v |I_v| \log \frac{1}{|I_v|} < \infty \), where \( \{I_v\} \) are the complementary arcs of \( E \).

Example. Let \( E \) be the Cantor set on \([0, 1]\). It can be easily seen that:

1. \( E \) is closed,

2. \( E \) is of Lebesgue measure zero, and

3. \([0, 1] \setminus E \) is the pairwise disjoint union \( \bigcup_{k=0}^{\infty} \bigcup_{n=1}^{2^k} I_{n,k} \), where each \( I_{n,k} \)
has length $\frac{1}{3^k+1}$. Therefore:

$$\sum m(I_k) \log \left(\frac{1}{m(I_k)}\right) = \sum_{k=0}^{\infty} \frac{2^k}{3^k+1} \cdot \log \left(\frac{1}{\frac{1}{3^k+1}}\right)$$

$$= \frac{1}{3} \sum_{k=1}^{\infty} (k + 1) \cdot \left(\frac{2}{3}\right)^k \log(3) < \infty$$

by the Ratio Test. Consequently, $E$ is a Carleson subset of $[0, 1]$.

2. Main Result

Let $\lambda$ denote Lebesgue measure on $[0, 1]$ and suppose that $0 \leq h \in L^1(\lambda)$. Then, for any $\varepsilon > 0$ and $C_0 \geq 1$ there exists a measure $\mu$ such that $\|\mu\| \leq 2 \cdot \|h\|_{L^1(\lambda)}$, $\mu \perp \lambda$, $\mu(E) = 0$ for every Carelson set $E$ in $[0, 1]$, and

$$\left|\int_{[0,1]} (g \, d\mu - gh \, d\lambda)\right| \leq \varepsilon,$$

whenever $0 \leq g \in C^\infty([0, 1])$ and $\|g\|_\infty, \|g\|_\infty \leq C_0$.

Proof. Choose $\varepsilon, \varepsilon < 1$. Since $\|g\|_\infty \leq C_0$, we can apply the Mean-Value Theorem and find a partition $\{x_i : i = 0, ..., M\}$ of $[0, 1]$ that depends only on the constant $C_0$ and a step function $g^* = \sum_{i=1}^{M} \alpha_i \chi_{F_i}$ ($F_i := [x_{i-1}, x_i]$) such that

$$\|g - g^*\|_\infty \leq \frac{\varepsilon}{4}, \quad (1)$$

and $|\alpha_i| \leq C_0$ for $i = 1, ..., M$. Since $h$ is integrable (we may assume that $\|h\|_{L^1(\lambda)} = 1$) there exists a step function $h^* = \sum_{j=1}^{N} \beta_j \chi_{B_j}$, where the $B_j$'s are intervals that form a partition of $[0, 1]$ such that

$$\left|\int_{[0,1]} (h^* \, d\lambda - h \, d\lambda)\right| \leq \frac{\varepsilon}{4M \cdot C_0}.$$

(2)
Also, there exists $\delta > 0$ such that whenever $A \subseteq [0, 1]$ with $\lambda (A) \leq \delta$, then

$$\int_A h^* \, d\lambda \leq \frac{\varepsilon}{16 M C}.$$  \hfill (3)

Start with a measure $\eta$ on $[0, 1]$ such that $\eta \perp \lambda$, $\eta ([0, 1]) = 1$ and $\eta (E) = 0$ for every Carleson set $E$ in $[0, 1]$ (see [1]). Let $K$ be a large enough fixed integer such that $\frac{1}{K} \leq \delta$ and let $A_k = \left[ \frac{k-1}{K}, \frac{k}{K} \right]$, $k = 1, ..., K$. It is clear that

$$\bigcup_{k=1}^K A_k = [0, 1] \text{ and } \lambda (A_k) = \frac{1}{K}. \quad \text{For } 0 < t < 1, \text{ define } \eta_t \text{ on the Borel subsets } B \text{ of } [0, 1] \text{ by } \eta_t (B) = \eta (\frac{1}{t} B). \text{ Define } \mu_k \text{ on the Borel subsets } B \text{ of } A_k \text{ by } \mu_k (B) = \eta (\frac{1}{K} (B - \left( \frac{k-1}{K} \right))), \quad (k = 1, ..., K). \text{ Notice that } \mu_k (A_j) = 0, \text{ whenever } k \neq j \text{ and } \mu_k (A_k) = 1, \text{ for } k = 1, 2, ..., K. \text{ Now, if } E \text{ is a Carleson set, then so is } \alpha E - \beta \text{ for any constants } \alpha, \beta, \text{ and hence } \mu_k (E) = 0. \text{ Let } \mu = \sum_{k=1}^K c_k \mu_k \text{ where } c_k = \sum_{j=1}^N \beta_j \lambda (B_j \cap A_k). \text{ Then } \mu (E) = 0 \text{ for every Carleson set } E \text{ in } [0, 1] \text{ and}

$$\mu ([0, 1]) = \sum_{k=1}^K c_k \mu_k ([0, 1]) = \sum_{k=1}^K \sum_{j=1}^N \beta_j \lambda (B_j \cap A_k)$$

$$= \sum_{j=1}^N \beta_j \left( \sum_{k=1}^K \lambda (B_j \cap A_k) \right) = \sum_{k=1}^K \beta_j \lambda (B_j) = \int_{[0,1]} h^* \, d\lambda$$

$$\leq 2 \| h \| L^1 (\lambda).$$

Let $F$ be any closed subinterval of $[0, 1]$, and let

$$\Delta = \{ k : k = 1, ..., K \text{ and } A_k \subseteq F \}. \quad \text{Now, there are at most two intervals } A_{k_0}, A_{k_1} \text{ such that } F \cap A_{k_i} \neq \emptyset \text{ for } i = 0, 1 \text{ and yet } k_0, k_1 \notin \Delta. \text{ Then}

$$\int_F (d\mu - h^* \, d\lambda) = \mu (F) - \sum_{j=1}^N \beta_j \lambda (B_j \cap F)$$

$$= \sum_{k=1}^K c_k \mu_k (A_k \cap F) - \sum_{j=1}^N \beta_j \lambda (B_j \cap F)$$
\[
\sum_{k \in \Delta} \sum_{j=1}^{N} \beta_j \lambda(B_j \cap A_k) \cdot \mu_k(A_k \cap F) - \sum_{j=1}^{N} \beta_j \lambda(B_j \cap (\bigcup_{k \in \Delta} A_k))
\]
\[
+ \sum_{j=1}^{N} \beta_j \lambda(B_j \cap A_{k_0}) \cdot \mu_{k_0}(A_{k_0} \cap F)
\]
\[
+ \sum_{j=1}^{N} \beta_j \lambda(B_j \cap A_{k_1}) \cdot \mu_{k_1}(A_{k_1} \cap F)
\]
\[
- \sum_{j=1}^{N} \beta_j \lambda(B_j \cap F \cap A_{k_0}) - \sum_{j=1}^{N} \beta_j \lambda(B_j \cap F \cap A_{k_1})
\]
\[
= \sum_{j=1}^{N} \beta_j \lambda(B_j \cap A_{k_0}) \cdot \mu_{k_0}(A_{k_0} \cap F)
\]
\[
+ \sum_{j=1}^{N} \beta_j \lambda(B_j \cap A_{k_1}) \cdot \mu_{k_1}(A_{k_1} \cap F)
\]
\[
- \sum_{j=1}^{N} \beta_j \lambda(B_j \cap F \cap A_{k_0}) - \sum_{j=1}^{N} \beta_j \lambda(B_j \cap F \cap A_{k_1}).
\]

Therefore
\[
\left| \int_{F} (d\mu - h^* d\lambda) \right| \leq 2 \left[ \int_{A_{k_0}} h^* d\lambda + \int_{A_{k_1}} h^* d\lambda \right] \leq \frac{\varepsilon}{4 M C_o}. \tag{4}
\]

Now
\[
\left| \int_{[0,1]} (g d\mu - g h d\lambda) \right| \leq \int_{[0,1]} |g - g^*| d\mu + \int_{[0,1]} g^* (d\mu - h d\lambda)
\]
\[
+ \int_{[0,1]} |g - g^*| h d\lambda
\]
\[
\leq \frac{\varepsilon}{2} + \sum_{i=1}^{M} |\alpha_i| \left| \int_{F_i} d\mu - h d\lambda \right| \leq \frac{\varepsilon}{2}
\]
\[
+ M C_o \left| \int_{F_i} d\mu - h d\lambda \right|
\]
\[ \leq \frac{\varepsilon}{2} + MC_0 \left[ \left| \int_{F_i} d\mu - h^* d\lambda \right| + \left| \int_{F_i} (h^* d\lambda - h d\lambda) \right| \right] \]

by (2)\&(4)

\[ \leq \varepsilon. \]

References
