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UNIQUENESS OF FRACTIONAL DIFFERENTIAL EQUATION USING THE INEQUALITIES

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Abstract: In the present paper, we study the uniqueness theorem of solution of fractional differential equation with initial condition by using Bihari's and Medved inequality.

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1. Introduction

Recently, fractional differential equations have been used in physical science and engineering application [2, 3, 10]. Fractional differential equation is generalization of ordinary differential equation of fractional order (i.e non-integer order). Fractional differential equation have many applications in various fields of science and engineering such as electrochemistry, viscoelasticity, electromagnetic, control theory, porous media [2, 6, 10]. In the last few decades, many researcher have studied the existence and uniqueness of the initial value problem for fractional differential equation, see [2, 3, 5, 7, 10, 13]. In this paper we

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consider the initial valve problem of the type

$$D^{\alpha}u(t) = f(t, u(t)), \tag{1.1}$$

$$D^{\alpha-1}u(t) \mid_{t=0} = \eta, \tag{1.2}$$

where $0 < \alpha < 1$, $t \in [0, \infty)$, and f is real valued function on $D = I \times R$ into R; where R denote the real space and η is real constant, D^{α} denote the Riemann-Liouville derivative operator.

2. Preliminaries

The necessary details of fractional calculus are given in the book I. Podlubny [10], A.A.Kilbas [3], and in book of S.G.Samko et al.[11], here we present some notation and definitions of Riemann-Liouville derivative and integral as given in [10, 12].

Definition 2.1. The fractional derivative of order $0 < \alpha < 1$ of continuous function $f : R^{+} \longrightarrow R$, by

$$D^{\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{0}^{x}(x-t)^{-\alpha}f(t)dt, \tag{2.1}$$

provided that the right side is pointwise defined on R^{+}.

Definition 2.2. The fractional primitive of order $\alpha > 0$, of function $f : R^{+} \longrightarrow R$ is given as follows

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}f(t)dt. \tag{2.2}$$

provided that the right side is pointwise defined on R^{+}, where $\Gamma(\alpha) = \int_{0}^{\infty} e^{-u}u^{\alpha-1}du$.

In [10], the author have been studied the existence and uniqueness of the initial value problem (1.1)-(1.2), first, let us reduce the problem (1.1)-(1.2)to a fractional integral equation, we obtain

$$u(t) = \frac{\eta}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}f(t, u(\tau))d\tau. \tag{2.3}$$

It is clear that equation (2.3) is equivalent to the initial problem (1.1)-(1.2), see [10,pp129-128].

Also, in [9] Medved defined a special class of nonlinear function and developed a method to estimate solution for nonlinear integral inequalities with singular kernel. The class of function defined as follows:

Definition 2.3. [9] Let $q > 0$ be a real number and $0 < T \leq \infty$, we say that a function $w : R_+ \longrightarrow R$ satisfies a condition (q) if

$$e^{-qt}[w(u)]^q \leq R(t)w(e^{-qt}u^q) \qquad (q)$$

for all $u \in R_+$, $t \in [0, T)$, where $R(t)$ is continues nonnegative function.

Remark 2.1. [9] If $w(u) = u^m$, $m > o$ then

$$e^{-qt}[w(u)]^q = e^{(m-1)qt}w(e^{-qt}u^q).$$

for any $q > 1$. i.e the condition (q) is satisfies with $R(t) = e^{(m-1)qt}$. for $w(u) = u + au^m$, where $0 \leq a \leq 1$, $m \geq 1$ the function w satisfies the condition (q) with $q > 1$ and $R(t) = 2^{q-1}e^{qmt}$.

Theorem 2.1. [9] *Let $0 \geq T \leq \infty$, $u(t)$, $b(t)$, $a(t)$, $a'(t) \in C([0, T),R_+)$; $w \in C(R_+, R)$ be nondecreasing function, $w(0) = 0$, $w(u) > 0$ on $(0, T)$, and*

$$u(t) \leq a(t) + \int_0^t (t - s)^{\beta-1}b(s)w(u(s))ds, \qquad (2.4)$$

for $t \in [0, T)$ where $\beta > 0$ is constant. Then following hold:
(i) Suppose $\beta > \frac{1}{2}$, and w satisfies the condition (q) with $q = 2$, then

$$u(t) \leq e^t\{\Omega^{-1}[\Omega(2a(t)^2) + g_1(t)]\}^{\frac{1}{2}}, \qquad (2.5)$$

for $t \in [0, T]$, where

$$g_1(t) = \frac{\Gamma(2\beta - 1)}{4^{\beta-1}} \int_0^t R(t)b(s)^2ds,$$

where Γ is gamma function, $\Omega(v) = \int_{v_0}^v \frac{ds}{w(s)}$, $v_0 > 0$, Ω^{-1}, is the inverse of Ω, and $t \in R_+$ is such that $\Omega(2a(t)^2) + g_1(t) \in Dom(\Omega^{-1})$ for all $t \in [0, T]$. (ii)Let $\beta = (0, \frac{1}{2}]$ and w satisfies the condition (q) with $q = z = 2$, where $z = \frac{1-\beta}{\beta}$, i.e $\beta = \frac{1}{z+1}$. Let Ω, Ω^{-1} be as in part (i). Then

$$u(t) \leq e^t\{\Omega^{-1}[\Omega(2^{q-1}a(t)^2) + g_2(t)]\}^{\frac{1}{q}}, \qquad (2.6)$$

for $t \in [0, T]$, where

$$g_2(t) = 2^{q-1}K_z^q \int_0^t R(s)b(s)^2ds,$$

$$K_z = [\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}]^{\frac{1}{p}}, \qquad \alpha = \frac{z}{z+1}, \qquad p = \frac{z+2}{z+1}, \tag{2.7}$$

and $T_1 \in R_+$ *is such that* $\Omega(2^{q-1}a(t)^q) + g_2(t)$ *for all* $t \in [0, T]$.

Theorem 2.2. [9] *Let* $\geq T \leq \infty$, $u(t), b(t), a(t), a'(t)$ *be as in theorem (2.1) and*

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s)u(s)ds, \tag{2.8}$$

for $t \in [0, T)$ *where* $\beta > 0$ *is constant. Then following hold:*
(i) Suppose $\beta > \frac{1}{2}$, *then*

$$u(t) \leq (\sqrt{2})a(t)exp[\frac{2\Gamma(2\beta-1)}{4^\beta} \int_0^t b(s)^2 ds + t], \tag{2.9}$$

for $t \in [0, T]$, *(ii)If* $\beta = \frac{1}{z+1}$, *for some* $z \geq 1$. *Then*

$$u(t) = (2^{q-1})^{\frac{1}{q}} a(t)exp[\frac{2^{q-1}}{q} K_z^q \int_0^t b(s)^2 ds + t], \tag{2.10}$$

for $t \in [0, T]$, *where* K_z *is defined by (2.7),* $q = z + 2$.

For detail proof of above two theorem see [9].

Theorem 2.3. *(Bihari's inequality)*[1, 8] *Let* u *and* f *be nonnegative defined on* R_+, *let* $w(u)$ *be continuous nondecreasing function defined on* R_+, *and* $w(u) > 0$, *on* $(0, \infty)$. *If*

$$u(t) \leq k + \int_0^t f(s)w(u(s))ds, \tag{2.11}$$

for $t \in R_+$ *where* k *is nonnegative constant, for* $0 \leq t \leq T$

$$u(t) \leq G^{-1}[G(k) + \int_0^t f(s)ds], \tag{2.12}$$

where,

$$G(r) = \int_0^t \frac{ds}{w(s)}, r > 0, r_0 > 0$$

and G^{-1} *is the inverse function of* G *and* $t_1 \in R_+$ *is chosen so that*

$$G(k) + \int_0^t f(s)ds \in Dom(G^{-1}).$$

For all $t \in R_+$ *laying in the interval* $0 \leq t \leq t_1$.

3. Main Result

In this section, we establish the uniqueness of solution of the following initial value problem.

Theorem 3.1. *Consider the fractional differential equation*

$$D^{\alpha}u(t) = f(t, u(t)), \tag{3.1}$$

with initial condition

$$D^{\alpha-1}u(t)\mid_{t=0} = \eta, \tag{3.2}$$

where $f(t, u(t)) : [0, T) \times R \longrightarrow R$ is continuous on $I = [0, T)$ and satisfies Lipschitz condition with respect to second variable i.e

$$|f(t, u(t)) - f(t, v(t))| \leq M\Phi(|u(t) - v(t)|). \tag{3.3}$$

If $\Phi(u)$ is a continuous nondecreasing function on $0 < u \leq A$, with $\Phi(0) = 0$ and

$$\int_0^A \frac{du}{\Phi(u)}, \tag{3.4}$$

then the solution of problem (3.1)-(3.2) is unique.

Proof. Assume that there exists two solution $u(t)$ and $v(t)$ of (3.1)-(3.2), we have

$$u(t) = \frac{\eta}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1} f(t, u(\tau))d\tau. \tag{3.5}$$

$$v(t) = \frac{\eta}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1} f(t, v(\tau))d\tau. \tag{3.6}$$

Which lead easily to

$$\begin{aligned} |u(t) - v(t)| \leq & \frac{1}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}|f(t, u(\tau)) - f(t, v(\tau))|d\tau \\ & \leq \frac{1}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}\Phi(|u(\tau) - v(\tau)|)d\tau \end{aligned}$$

$$|u(t) - v(t)| < \epsilon + \frac{L}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}\Phi(|u(\tau) - v(\tau)|)d\tau \tag{3.7}$$

Now, we can apply Bihari's inequality to (3.7) getting,

$$\begin{aligned}|u(t)-v(t)| <&\Phi^{-1}[\Phi(\epsilon)+\frac{ML}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}d\tau]\\ &< \Phi^{-1}[\Phi(\epsilon)+\frac{ML}{\Gamma(\alpha)}(\frac{(t-\tau)^{\alpha}}{-\alpha})_0^T]\\ &< \Phi^{-1}[\Phi(\epsilon)+\frac{ML}{\Gamma(\alpha)}(\frac{T^{\alpha}}{\alpha}-\frac{(t-T)^{\alpha}}{\alpha})]\\ &< \Phi^{-1}[\Phi(\epsilon)+\frac{ML}{\Gamma(\alpha+1)}(T^{\alpha}-(t-T)^{\alpha})],\end{aligned} \tag{3.8}$$

where $\Phi(u)$ is primitive for $\frac{1}{\Phi(u)}$. We shall prove that the right-hand side of (3.8)tends toward zero. as $\varepsilon \to 0$. As $|u(t)-v(t)|$ is independent of ε, it follows that $u(t) = v(t)$, which we need. Let us remark that condition (3.4) implies $\Phi(\varepsilon) \to -\infty$ for $\varepsilon \to 0$, no matter how we choose the primitive of $\frac{1}{\Phi(u)}$. Thus $\Phi^{-1}(u) \to o$ as $u \to -\infty$. Consequently, $\varepsilon \to 0$ in the inequality (3.8), the right-hand side tends toward zero. Thus theorem is proved.

Theorem 3.2. *The initial value problem (3.1)-(3.2) has unique solution in the interval $t \in [0,T)$. If the function f is continuous and satisfies the Lipschitz condition with respect to second variable i.e*

$$|f(t,u(t))-f(t,v(t))| \le M|u(t)-v(t)|, \tag{3.9}$$

for some positive constant M.

Proof. Assume that $u(t)$ and $v(t)$ be two solution of initial value problem (3.1)-(3.2), we have

$$u(t) = \frac{\eta}{\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}f(t,u(\tau))d\tau. \tag{3.10}$$

$$v(t) = \frac{\eta}{\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}f(t,v(\tau))d\tau. \tag{3.11}$$

which gives,

$$\begin{aligned}|u(t)-v(t)| \le&\frac{1}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}|f(t,u(\tau))-f(t,v(\tau))|d\tau\\ &\le \frac{M}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}1|u(\tau)-v(\tau)|d\tau\\ &\le \varepsilon+\frac{M}{\Gamma(\alpha)}\int_0^T (t-\tau)^{\alpha-1}1|u(\tau)-v(\tau)|d\tau\end{aligned} \tag{3.12}$$

Now, (i) suppose that $\alpha > \frac{1}{2}$, then applying theorem (2.2) (i) to (3.12), we have

$$|u(t) - v(t)| < \frac{\sqrt{2\varepsilon M}}{\Gamma(\alpha)} exp[\frac{2\sqrt{2\alpha - 1}}{4^{\alpha}} t + t], \tag{3.13}$$

for $t \in [0, T)$ since ε is arbitrary, as $\varepsilon \to 0$ the inequality (3.13) implies that $u(t) = v(t)$ on $[0, T)$.

(ii) Let $\alpha > \frac{1}{z+1}$ for some $z \geq 1$, then apply the theorem (2.2)(ii) to (3.13), again we have,

$$|u(t) - v(t)| < \frac{(2^{q-1})^{\frac{1}{q}} \varepsilon M}{\Gamma(\alpha)} exp[\frac{2^{q-1}}{q} K_z^q t + t], \tag{3.14}$$

for $t \in [0, T)$ where K_z is defined by (2.7). Since ε is arbitrary in (3.14) implies that $u(t) = v(t)$. Hence the theorem is proved.

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