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EXPLICIT BOND OPTION IN HEATH JARROW MORTON MODEL WITH CONSTANT VOLATILITY

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Abstract: The purpose of this paper is to investigate the Heath Jarrow Morton model (HJM) with constant volatility to provide an explicit formula for European option on zero-coupon bonds.

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1. Introduction

There has been an accelerated growth in the use of interest rate instruments in fixed income portfolios and risk management systems over the past two decades. With this proliferation has emerged the necessity to formulate reliable models that accurately price and hedge such interest rate sensitive cash flows.

The literature on interest rate modelling is dominated by two main groups of models; the short rate and forward rate model. The Vasicek model [8] and the Cox-Ingersoll-Ross model [1] relies in the fact that prices are explicit functions of the instantaneous 'spot' interest rate so that these models are unable to take

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the whole yield curve observed on the market into account in the price structure [6]. Some authors have resorted to a two-dimensional analysis to improve the models in terms of discrepancies between short and long rates, e.g. Scaefer and Schwartz [7]. These more complex models do not lead to explicit formulae and require the solution of partial differential equations [5].

More recently, Ho and Lee [3] have proposed a discrete time model describing the behaviour of the whole yield curve. The continuous time model we present is based on the same idea and has been introduced by Heath, Jarrow and Morton [2]. These models are extremely popular due to their simplicity and mathematical tractability. The HJM one factor model essentially assumes that the term structure is affected by a single source of uncertainly.

The aim of this study is to present a Heath, Jarrow, and Morton model where the Volatility is a constant. Furthermore, it is to give an explicit formula for European option on zero-coupon bonds.

2. Generalities

We take as given a Brownian motion $\{W(t); 0 \leq t \leq T^*\}$ an some probability space $(\Omega, \mathcal{F}, P)$, and $\{F(t); 0 \leq t \leq T^*\}$ is the filtration generated by W. The market we will study is mainly the market of zero coupon bonds

Definition 1. A zero coupon bond with maturity date T, also called a T-bond, is contract which guarantees the holder 1 \$ to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $B(t,T)$. Obviously we have $B(T,T) = 1$.

We now make an assumption to guarantee the existence of a sufficiently rich bond market

Definition 2. 1. The forward rate for $[S,T]$ contracted at t is defined as

$$R(t; S, T) = -\frac{\log B(t,T) - \log B(t,S)}{T - S}. \tag{1}$$

2. The spot rate, $R(S,T)$, for the period $[S,T]$ is defined as

$$R(S,T) = R(S; S, T). \tag{2}$$

3. The instantaneous forward rate with maturity T, contracted at t is defined by

$$f(t,T) = -\frac{\partial \log B(t,T)}{\partial T}. \tag{3}$$

4. The instantaneous short rate at time t is defined by

$$r(t) = f(t,t). \tag{4}$$

As an immediate consequence of the definitions we have the following useful formulas

Lemma 3. *For $t \leq s \leq T$ we have*

$$B(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}. \tag{5}$$

We note this as an alternative representation for $B(t,T)$ by contrast with its expression in terms of the short rate process r :

$$B(t,T) = E\left[\exp\left(-\int_t^T r(s)ds\right) \mid \mathcal{F}_t\right]. \tag{6}$$

3. The Heath-Jarrow-Morton Model

The Heath-Jarrow-Morton (HJM) model for term structure considers stochastic differential equations for the evolution of the forward rate $f(t,T)$.

For each $T \in]0,T^*]$ suppose the dynamics of f are given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t). \tag{7}$$

Here the coefficients $\alpha(u,T)$ and $\sigma(u,T)$, for $0 \leq u \leq T$, are measurable (in (u,w)) and adapted. The integral form of (7) is

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T)dW(u). \tag{8}$$

Note we have two time parameters and recall

$$B(t,T) = \exp\left\{-\int_t^T f(t,u)du\right\}.$$

With d denoting a differential in the t variable:

$$d\left\{-\int_t^T f(t,u)du\right\} \quad = \quad f(t,t)dt - \int_t^T (df(t,u))du$$

$$\begin{aligned} &= r(t)dt - \int_t^T [\alpha(t,u)dt + \sigma(t,u)dW(t)]\,du \\ &= (r(t) - \alpha^*(t,T))dt + \sigma^*(t,T)dW(t), \end{aligned}$$

where

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)du \tag{9}$$

$$\sigma^*(t,T) = \int_t^T \sigma(t,u)du. \tag{10}$$

Recall, by definition, $f(t,u)$ is an $\mathcal{F}_t$-adapted process. Therefore,

$$X(t) := -\int_t^T f(t,u)du$$

is an $\mathcal{F}_t$-adapted process. In fact it is an It process with,

$$dX(t) = (r(t) - \alpha^*(t,T))dt - \sigma^*(t,T)dW(t).$$

Also, $B(t,T) = e^{X(t)}$, so

$$\begin{aligned} dB(t,T) &= e^{X(t)}\left[r(t) - \alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\right]dt - e^{X(t)}\sigma^*(t,T)dW(t) \\ &= B(t,T)\left[\left(r(t) - \alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\right)dt - \sigma^*(t,T)dW(t)\right]. \end{aligned}$$

Now, the discounted $B(t,T)$ will be a martingale under P (so P is a risk-neutral measure), if for $0 \le t \le T \le T^*$,

$$\alpha^*(t,T) = \frac{1}{2}\left(\sigma^*(t,T)\right)^2.$$

from the definitions of α^* and σ^* this means

$$\int_t^T \alpha(t,u)du = \frac{1}{2}\left(\int_t^T \sigma(t,u)du\right)^2.$$

This is equivalent to

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du.$$

If P itself is not a risk-neutral measure there may be a probability P^λ under which $\left(\exp\int_0^t -r(u)du\right)B(t,T)$ is a martingale. This is the content of the following result due to Heath, Jarrow and Morton [2].

Theorem 4. *For each $T \in]0, T^*]$ suppose $\alpha(u,T)$ and $\sigma(u,T)$ are adapted processes. We assume $\sigma(u,T) > 0$ for all u, T, and $f(0,T)$ is a deterministic function of T. The instantaneous forward rate $f(t,T)$ is defined by*

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T)dW(u).$$

Then the term structure model determined by the processes $f(t,T)$ does not allow arbitrage if and only if there is an adapted process $\lambda(t)$ such that

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du + \sigma(t,T)\lambda(t), \text{ for all } 0 \le t \le T \le T^*, \tag{11}$$

and the process

$$\Lambda^\lambda(t) := \exp\left(-\int_0^t \lambda(u)dW(u) - \frac{1}{2}\int_0^t \lambda(u)^2 du\right) \tag{12}$$

is an $(\mathcal{F}_t, P)$ martingale.

4. Application of HJM Model when σ is a Constant

In the HJM model, we assume that the function σ is a positive constant. We assume that there is an underlying probability space $(\Omega, \mathcal{F}, P)$, equipped with a standard filtration $\mathcal{F}_t$ Under the risk-neutral measure P, we would like to price a call with maturity θ and strike price K, on a zero-coupon bond with maturity $T > \theta$.

Under the risk-neutral measure P, the instantaneous forward rate dynamics is given by

$$f(t,u) = f(0,u) + \sigma^2 t(u - t/2) + \sigma W_t. \tag{13}$$

The price of zero-coupon bond with maturity T at time θ is

$$\begin{aligned} B(\theta,T) &= \exp\left\{-\int_\theta^T f(\theta,u)du\right\} \\ &= \frac{\exp\left\{-\int_0^T f(\theta,u)du\right\}}{\exp\left\{-\int_0^\theta f(\theta,u)du\right\}} \\ &= \frac{\exp\left\{-\int_0^T f(0,u)du\right\}\exp\left\{-\int_0^T \left[\sigma^2\theta(u-\theta/2) + \sigma W_\theta\right]du\right\}}{\exp\left\{-\int_0^\theta f(0,u)du\right\}\exp\left\{-\int_0^\theta \left[\sigma^2\theta(u-\theta/2) + \sigma W_\theta\right]du\right\}} \end{aligned}$$

$$= \frac{B(0,T)}{B(0,\theta)} \exp\left(-\sigma(T-\theta)W_\theta - \frac{\sigma^2\theta T(T-\theta)}{2}\right).$$

We have, for $\lambda \in \mathbf{R}$

$$\begin{aligned} E\left[\exp\left\{-\sigma\int_0^\theta W_s ds\right\}\exp\{\lambda W_\theta\}\right] &= E\left[\exp\left\{-\sigma\int_0^\theta W_s ds + \lambda W_\theta\right\}\right] \\ &= \exp\left\{\frac{1}{2}\begin{bmatrix}-\sigma & \lambda\end{bmatrix}\begin{bmatrix}\frac{\theta^3}{3} & \frac{\theta^2}{2} \\ \frac{\theta^2}{2} & \theta\end{bmatrix}\right\} \\ &= \exp\left\{\frac{1}{2}\left(\frac{\sigma^2\theta^3}{3} - \lambda\sigma\theta^2 + \lambda^2\theta\right)\right\}. \end{aligned}$$

We define a new probability measures P_1 and P_2 with densities with respect to the risk-neutral measure P respectively by setting

$$\frac{dP_1}{dP} = \frac{\exp\left\{-\int_0^\theta r(s)ds\right\}B(\theta,T)}{B(0,T)}, \tag{14}$$

$$\frac{dP_2}{dP} = \frac{\exp\left\{-\int_0^\theta r(s)ds\right\}B(\theta,T)}{B(0,\theta)}. \tag{15}$$

Using the above result we have, $\forall\lambda \in$
$textbfR$

$$\begin{aligned} E^{P_1}\left[\exp\{\lambda W_\theta\}\right] &= E\left(\frac{e^{-\int_0^\theta r(s)ds}B(\theta,T)}{B(0,T)}\exp\{\lambda W_\theta\}\right) \\ &= \exp\left(\frac{-\sigma^2\theta T(T-\theta)}{2} - \frac{1}{6}\sigma^2\theta^3\right)\times \\ &\quad \times\exp\left(\frac{1}{6}\sigma^2\theta^3 - \frac{1}{2}(\lambda-\sigma(T-\theta)\sigma\theta^2 + \frac{1}{2}(\lambda-\sigma(T-\theta))^2\right)\theta \\ &= \exp\left[\lambda(\sigma\theta(T-\theta) - \frac{\sigma\theta^2}{2}) + \frac{1}{2}\lambda^2\theta\right]. \end{aligned}$$

It follows that, under the probability P_1, W_θ is a normal random with mean

$$E^{P_1}(W_\theta) = \sigma\theta(T-\theta) - \frac{\sigma\theta^2}{2} \tag{16}$$

and variance

$$Var^{P_1}(W_\theta) = \theta. \tag{17}$$

Similarly, $\forall\lambda \in \mathbf{R}$

$$\begin{aligned} E^{P_2}\left[\exp\{\lambda W_\theta\}\right] &= E\left(\frac{e^{-\int_0^\theta r(s)ds}}{B(0,\theta)}\exp\{\lambda W_\theta\}\right) \\ &= \exp\left[\lambda(-\frac{\sigma\theta^2}{2}) + \frac{1}{2}\lambda^2\theta\right]. \end{aligned}$$

Then, under the probability P_2, W_θ is a normal random with mean

$$E^{P_2}(W_\theta) = -\frac{\sigma\theta^2}{2} \tag{18}$$

and variance

$$Var^{P_2}(W_\theta) = \theta. \tag{19}$$

We know that, the price of call with maturity θ and strike price K at time 0, on a zero-coupon bond with maturity $T > \theta$ is define as

$$C_0 = E\left[e^{-\int_0^\theta r(s)ds}(B(\theta,T) - K)^+\right]. \tag{20}$$

We have

$$\begin{aligned} B(\theta,T) - K &\geq 0 \\ &\Longrightarrow \frac{B(0,T)}{B(0,\theta)}\exp\left(-\sigma(T-\theta)W_\theta - \frac{\sigma^2\theta T(T-\theta)}{2}\right) - K \geq 0 \\ &\Longrightarrow W_\theta \leq d_1, \end{aligned}$$

with

$$d_1 = -\frac{\sigma\theta T}{2} - \frac{\log\left(K\frac{B(0,\theta)}{B(0,T)}\right)}{\sigma(T-\theta)}. \tag{21}$$

Consequently,

$$\begin{aligned} C_0 &= \int_\Omega \chi_{\{x\leq d_1\}}\exp\left\{-\int_0^\theta r(s)ds\right\}(B(\theta,T)dP \\ &\quad - \int_\Omega \chi_{\{x\leq d_1\}}\exp\left\{-\int_0^\theta r(s)ds\right\}KdP \\ &= \int_\Omega \chi_{\{x\leq d_1\}}B(0,T)dP_1 - \int_\Omega \chi_{\{x\leq d_1\}}B(0,\theta)dP_2 \\ &= B(0,T)\int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi\theta}}\exp\left\{-\frac{1}{2}\left(x - \sigma\theta(T-\theta) + \frac{\sigma\theta^2}{2}\right)^2\right\}dx \end{aligned}$$

$$-KB(0,\theta)\int_{-\infty}^{d_1}\exp\left\{-\frac{1}{2\theta}\left(x+\frac{\sigma\theta^2}{2}\right)\right\}dx.$$

Therefore,

$$C_0 = B(0,T)\mathcal{N}\left(\frac{d_1-\sigma\theta(T-\theta)+\frac{\sigma\theta^2}{2}}{\sqrt{\theta}}\right) - KB(0,\theta)\mathcal{N}\left(\frac{d_1+\sigma\theta^2}{\sqrt{\theta}}\right).$$

Let use set

$$d = \frac{\sigma\sqrt{\theta}(T-\theta)}{2} - \frac{\log\left(K\frac{B(0,\theta)}{B(0,T)}\right)}{\sigma\sqrt{\theta}(T-\theta)}. \tag{22}$$

Using these notation, we have

$$C_0 = B(0,T)\mathcal{N}(d) - KB(0,\theta)\mathcal{N}\left(d-\sigma\sqrt{\theta}(T-\theta)\right), \tag{23}$$

where

$$\mathcal{N}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\zeta}\exp\left(-\frac{x^2}{2}\right)dx.$$

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