END BEHAVIOR ANALYSIS FOR SOLUTIONS
OF LIMIT AT INFINITY

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Abstract: In this paper, we study properties of function leaders and establish useful theorems to determine limits of functions that have indeterminate form, \( \infty \) and \( \infty - \infty \), as \( x \) tend to infinity.

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1. Introduction

A limit determination of a function is not only one of the important things in Calculus, but also essential tool in applied mathematics. Many times we have to confront whether a function, say \( f(x) \), converges to a real constant, \( L \), when \( x \) is extremely large. If it does, we can write that \( \lim_{x \to \infty} f(x) = L \), and call that \( f \) has the limit \( L \) as \( x \) tend to infinity. However, if \( f(x) \) and \( g(x) \) are increased without bound when \( x \) is very large, no one can guarantee that \( \lim_{x \to \infty} \frac{f(x)}{g(x)} \) will exist or not. Precisely, we call the limit in such this case that an indeterminate form.
Apart from L’Hospital’s rule, there is no formal prove in investigating the end behavior of $f$ and $g$ to determine $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$, even in the good reference books in calculus such [1] and [2].

In this article, we will explain what is meant by the word “end behavior”, and state useful theorems for finding the limit in the form $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ in the next section.

2. End Behavior Analysis

If a function $g(x)$ has the same behavior as $f(x)$ when $x$ tend to infinity, we will call that $f$ and $g$ has the same end behavior. To make analysis compact, we set up a definition of a function leader, which has the same end behavior as the function. Then, we study useful properties of leader, and give some example to convince.

**Definition 1.** Let $f : \mathbb{R} \to \mathbb{R}$ and $a, r \in \mathbb{R}$. We call $ax^r$, denote by $\lceil f(x) \rceil$, to be a leader of $f$ if $\lim_{x \to \infty} \frac{f(x)}{ax^r} = 1$, i.e., $\lim_{x \to \infty} \frac{f(x)}{\lceil f(x) \rceil} = 1$. We also call $r$ to be a degree of $\lceil f(x) \rceil$ which denoted by $d \lceil f(x) \rceil$.

By the definition above, it is easy to see that $\lim_{x \to \infty} \frac{\lceil f(x) \rceil}{f(x)} = 1$.

**Remark 2.** Any real-valued function is not necessary to have a leader, however, if one has, there is unique.

**Example 3.** Let $f(x) = \sin \frac{1}{x}$. Since

$$\lim_{x \to \infty} \sin \frac{1}{x} = 1,$$

the leader of $f(x)$ is $x^{-1}$.

Let $g(x) = e^x$. There are no real constants, $a$ and $r$, such that

$$\lim_{x \to \infty} \frac{e^x}{ax^r} = 1.$$

Hence, $g(x)$ does not have a function leader.

Since a function could have a unique leader, we find following nice properties:
Proposition 4. Let $f$ and $g$ be real-valued functions which have leaders, then

\begin{itemize}
  \item[a)] $\left\lceil \frac{1}{f} \right\rceil = \frac{1}{\lceil f \rceil}$,
  \item[b)] $\lceil f \cdot g \rceil = \lceil f \rceil \cdot \lceil g \rceil$, and
  \item[c)] $\left\lceil \frac{f}{g} \right\rceil = \frac{\lceil f \rceil}{\lceil g \rceil}$.
\end{itemize}

Theorem 5. Let $f$ and $g$ be real-valued functions which have leaders such that $\lceil f \rceil + \lceil g \rceil \neq 0$. Then

$$\lceil f + g \rceil = \begin{cases} 
\lceil f \rceil & \text{when } \circ \lceil f \rceil > \circ \lceil g \rceil, \\
\lceil g \rceil & \text{when } \circ \lceil f \rceil < \circ \lceil g \rceil, \\
\lceil f \rceil + \lceil g \rceil & \text{when } \circ \lceil f \rceil = \circ \lceil g \rceil.
\end{cases}$$

Proof. Suppose $\lceil f \rceil = ax^r$ and $\lceil g \rceil = bx^s$ such that $ax^r + bx^s \neq 0$. Obviously, the leader of $f + g$ is $\lceil f \rceil$ or $\lceil g \rceil$ when $\circ \lceil f \rceil \neq \circ \lceil g \rceil$. Now, consider the case $\circ \lceil f \rceil = \circ \lceil g \rceil$, i.e., $r = s$. Since $\lceil f \rceil + \lceil g \rceil \neq 0$, it follows that $a + b \neq 0$. So, we have

$$\frac{f + g}{\lceil f \rceil + \lceil g \rceil} = \frac{f + g}{(a + b)x^r} = \frac{f}{(a + b)x^r} + \frac{g}{(a + b)x^r}.$$ 

Hence,

$$\lim_{x \to \infty} \frac{f + g}{\lceil f \rceil + \lceil g \rceil} = \frac{a}{a + b} + \frac{b}{a + b} = 1,$$

and therefore $\lceil f + g \rceil = \lceil f \rceil + \lceil g \rceil$. \hfill \Box

Lemma 6. Let $\lim_{x \to \infty} f(x) = L > 0$. Then

$$\lim_{x \to \infty} g(x) = \infty \text{ if and only if } \lim_{x \to \infty} f(x)g(x) = \infty.$$ 

Proof. Suppose $\lim_{x \to \infty} g(x) = \infty$. Let $N \in \mathbb{N}$, by the hypotheses, there are $N_0$ and $N_1 \in \mathbb{N}$ such that

$$|f(x) - L| < \frac{L}{2} \text{ for all } x \geq N_0,$$

i.e.,

$$\frac{L}{2} < f(x) < \frac{3L}{2}. \quad (1)$$
and \( g(x) > \frac{2N}{L} \) for all \( x \geq N_1 \).

Now, let \( N_{\varepsilon} = \max\{N_0, N_1\} \). We have that
\[
f(x)g(x) > \frac{L}{2} \cdot \frac{2N}{L} = N \text{ for all } x \geq N_{\varepsilon}.
\]

Therefore, \( \lim_{x \to \infty} f(x)g(x) = \infty \).

For conversion, suppose \( \lim_{x \to \infty} f(x)g(x) = \infty \) so that there is \( N_3 \in \mathbb{N} \) such that
\[
f(x)g(x) > \frac{3LN}{2} \text{ for all } x \geq N_3.
\]

Let \( N_4 = \max\{N_0, N_3\} \). Then, by (1), \( g(x) > N \) for all \( x \geq N_4 \).

It follows that \( \lim_{x \to \infty} g(x) = \infty \).

\[\text{Theorem 7.} \quad \text{Let } f \text{ be a real-valued function which has leader. Then}
\]
\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \lceil f(x) \rceil.
\]

\[\text{Proof.} \quad \text{Consider,}
\]
\[
\lim_{x \to \infty} f = \lim_{x \to \infty} \frac{f}{\lceil f \rceil} \cdot \lceil f \rceil.
\]

If \( \lim_{x \to \infty} \lceil f \rceil \) exists,
\[
\lim_{x \to \infty} f = \lim_{x \to \infty} \lceil f \rceil.
\]

On the other hand, if \( \lim_{x \to \infty} \lceil f \rceil = \infty \), then, by Lemma 6,
\[
\lim_{x \to \infty} f = \lim_{x \to \infty} \lceil f \rceil.
\]

Therefore, the proof completed.

\[\text{Theorem 8.} \quad \text{Let } f \text{ and } g \text{ be real-valued functions which have leaders. If}
\]
\[
\lim_{x \to \infty} f(x) = \infty \text{ and } \lim_{x \to \infty} g(x) = \infty, \text{ then}
\]
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\lceil f(x) \rceil}{\lceil g(x) \rceil}.
\]
Proof. It is straightforward from Theorem 7 and Proposition 4 that
\[
\lim_{x \to \infty} \frac{f}{g} = \lim_{x \to \infty} \left\lfloor \frac{f}{g} \right\rfloor = \lim_{x \to \infty} \left\lfloor \frac{f}{g} \right\rfloor.
\]

Example 9. It is easy to check that \(2x^3\) is the leader of
\[
f(x) = \sqrt{4x^6 + \sqrt{5x^7}} + \sqrt{6x^8},
\]
and \(x^3\) is the leader of \(g(x) = (x^{3/2} + 2x + 1)^2\). Therefore, by Theorem 8,
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{2x^3}{x^3} = 2.
\]

From this example, we can see that in finding the solution our method is more convenient than the L’Hospital’s rule as the quotient, \(\frac{f'(x)}{g'(x)}\), is complicated.

Remark 10. If \(\lim_{x \to \infty} (f(x) - g(x)) = \infty - \infty\), then we multiply both the numerator and the denominator by its conjugation, \(f(x) + g(x)\). The limit, then, transforms to \(\infty\). We can now apply Theorem 8 to get a limit solution. For example,
\[
\lim_{x \to \infty} \sqrt{x^4 + \sqrt{x^6 + x} - (x^2 - x)} \\
= \lim_{x \to \infty} \left[ \sqrt{x^4 + \sqrt{x^6 + x} - (x^2 - x)} \right] \cdot \left[ \frac{\sqrt{x^4 + \sqrt{x^6 + x} + (x^2 - x)}}{\sqrt{x^4 + \sqrt{x^6 + x} + (x^2 - x)}} \right] \\
= \lim_{x \to \infty} \frac{\sqrt{x^6 + x} + 2x^3 - x^2}{\sqrt{x^4 + \sqrt{x^6 + x} + (x^2 - x)}} \\
= \lim_{x \to \infty} \frac{\sqrt{x^6 + 2x^3}}{\sqrt{x^4 + x^2}} = \lim_{x \to \infty} \frac{3x}{2} = \infty.
\]
3. Conclusion

We established the useful theorem to find limits as $x$ tend to infinity for any functions that have their leaders, especially a function which has indeterminate form $\frac{\infty}{\infty}$ or $\infty - \infty$. We also emphasize that our method is easier and more convenient than other methods.

Interestingly, what are necessary and sufficient conditions to ensure that a function can has a leader?

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References
