END BEHAVIOR ANALYSIS FOR SOLUTIONS OF LIMIT AT INFINITY

Annop Kaewkhao¹, Somkid Intep^{2 §}

^{1,2}Department of Mathematics

Faculty of Science

Burapha University

Chonburi, 20131, THAILAND

Abstract: In this paper, we study properties of function leaders and establish useful theorems to determine limits of functions that have indeterminate form, $\frac{\infty}{\infty}$ and $\infty - \infty$, as x tend to infinity.

AMS Subject Classification: 26A06, 26A03

Key Words: end behavior, function leader, indeterminate form

1. Introduction

A limit determination of a function is not only one of the important things in Calculus, but also essential tool in applied mathematics. Many times we have to confront whether a function, say $f(x)$, converges to a real constant, L , when x is extremely large. If it does, we can write that $\lim_{x \rightarrow \infty} f(x) = L$, and call that f has the limit L as x tend to infinity. However, if $f(x)$ and $g(x)$ are increased without bound when x is very large, no one can guarantee that $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ will exist or not. Precisely, we call the limit in such this case that an indeterminate form.

Received: August 1, 2013

© 2013 Academic Publications, Ltd.
url: www.acadpubl.eu

[§]Correspondence author

Apart from L'Hospital's rule, there is no formal prove in investigating the end behavior of f and g to determine $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$, even in the good reference books in calculus such [1] and [2].

In this article, we will explain what is meant by the word "end behavior", and state useful theorems for finding the limit in the form $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ in the next section.

2. End Behavior Analysis

If a function $g(x)$ has the same behavior as $f(x)$ when x tend to infinity, we will call that f and g has the same end behavior. To make analysis compact, we set up a definition of a function leader, which has the same end behavior as the function. Then, we study useful properties of leader, and give some example to convince.

Definition 1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a, r \in \mathbb{R}$. We call ax^r , denote by $\lceil f(x) \rceil$, to be a leader of f if $\lim_{x \rightarrow \infty} \frac{f(x)}{ax^r} = 1$, i.e., $\lim_{x \rightarrow \infty} \frac{f(x)}{\lceil f(x) \rceil} = 1$. We also call r to be a degree of $\lceil f(x) \rceil$ which denoted by $\circ \lceil f(x) \rceil$.

By the definition above, it is easy to see that $\lim_{x \rightarrow \infty} \frac{\lceil f(x) \rceil}{f(x)} = 1$.

Remark 2. Any real-valued function is not necessary to have a leader, however, if one has, there is unique.

Example 3. Let $f(x) = \sin \frac{1}{x}$. Since

$$\lim_{x \rightarrow \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1,$$

the leader of $f(x)$ is x^{-1} .

Let $g(x) = e^x$. There are no real constants, a and r , such that

$$\lim_{x \rightarrow \infty} \frac{e^x}{ax^r} = 1.$$

Hence, $g(x)$ does not have a function leader.

Since a function could have a unique leader, we find following nice properties:

Proposition 4. Let f and g be real-valued functions which have leaders, then

$$\text{a) } \left\lceil \frac{1}{f} \right\rceil = \frac{1}{\lceil f \rceil},$$

$$\text{b) } \lceil f \cdot g \rceil = \lceil f \rceil \cdot \lceil g \rceil, \quad \text{and}$$

$$\text{c) } \left\lceil \frac{f}{g} \right\rceil = \frac{\lceil f \rceil}{\lceil g \rceil}.$$

Theorem 5. Let f and g be real-valued functions which have leaders such that $\lceil f \rceil + \lceil g \rceil \neq 0$. Then

$$\lceil f + g \rceil = \begin{cases} \lceil f \rceil & \text{when } \circ \lceil f \rceil > \circ \lceil g \rceil, \\ \lceil g \rceil & \text{when } \circ \lceil f \rceil < \circ \lceil g \rceil, \\ \lceil f \rceil + \lceil g \rceil & \text{when } \circ \lceil f \rceil = \circ \lceil g \rceil. \end{cases}$$

Proof. Suppose $\lceil f \rceil = ax^r$ and $\lceil g \rceil = bx^s$ such that $ax^r + bx^s \neq 0$. Obviously, the leader of $f + g$ is $\lceil f \rceil$ or $\lceil g \rceil$ when $\circ \lceil f \rceil \neq \circ \lceil g \rceil$. Now, consider the case $\circ \lceil f \rceil = \circ \lceil g \rceil$, i.e., $r = s$. Since $\lceil f \rceil + \lceil g \rceil \neq 0$, it follows that $a + b \neq 0$. So, we have

$$\frac{f + g}{\lceil f \rceil + \lceil g \rceil} = \frac{f + g}{(a + b)x^r} = \frac{f}{(a + b)x^r} + \frac{g}{(a + b)x^r}.$$

Hence,

$$\lim_{x \rightarrow \infty} \frac{f + g}{\lceil f \rceil + \lceil g \rceil} = \frac{a}{a + b} + \frac{b}{a + b} = 1,$$

and therefore $\lceil f + g \rceil = \lceil f \rceil + \lceil g \rceil$. □

Lemma 6. Let $\lim_{x \rightarrow \infty} f(x) = L > 0$. Then

$$\lim_{x \rightarrow \infty} g(x) = \infty \quad \text{if and only if} \quad \lim_{x \rightarrow \infty} f(x)g(x) = \infty.$$

Proof. Suppose $\lim_{x \rightarrow \infty} g(x) = \infty$. Let $N \in \mathbb{N}$, by the hypotheses, there are N_0 and $N_1 \in \mathbb{N}$ such that

$$\begin{aligned} |f(x) - L| &< \frac{L}{2} \quad \text{for all } x \geq N_0, \\ \text{i.e.,} \quad \frac{L}{2} &< f(x) < \frac{3L}{2}, \end{aligned} \tag{1}$$

and $g(x) > \frac{2N}{L}$ for all $x \geq N_1$.

Now, let $N_\varepsilon = \max\{N_0, N_1\}$. We have that

$$f(x)g(x) > \frac{L}{2} \cdot \frac{2N}{L} = N \text{ for all } x \geq N_\varepsilon.$$

Therefore, $\lim_{x \rightarrow \infty} f(x)g(x) = \infty$.

For conversion, suppose $\lim_{x \rightarrow \infty} f(x)g(x) = \infty$ so that there is $N_3 \in \mathbb{N}$ such that

$$f(x)g(x) > \frac{3LN}{2} \text{ for all } x \geq N_3.$$

Let $N_4 = \max\{N_0, N_3\}$. Then, by (1), $g(x) > N$ for all $x \geq N_4$.

It follows that $\lim_{x \rightarrow \infty} g(x) = \infty$. □

Theorem 7. Let f be a real-valued function which has leader. Then

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \lceil f(x) \rceil.$$

Proof. Consider,

$$\lim_{x \rightarrow \infty} f = \lim_{x \rightarrow \infty} \frac{f}{\lceil f \rceil} \cdot \lceil f \rceil.$$

If $\lim_{x \rightarrow \infty} \lceil f \rceil$ exists,

$$\lim_{x \rightarrow \infty} f = \lim_{x \rightarrow \infty} \lceil f \rceil.$$

On the other hand, if $\lim_{x \rightarrow \infty} \lceil f \rceil = \infty$, then, by Lemma 6,

$$\lim_{x \rightarrow \infty} f = \lim_{x \rightarrow \infty} \lceil f \rceil.$$

Therefore, the proof completed. □

Now we are ready to state the useful theorem to find the limit of the form

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}.$$

Theorem 8. Let f and g be real-valued functions which have leaders. If $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow \infty} g(x) = \infty$, then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{\lceil f(x) \rceil}{\lceil g(x) \rceil}.$$

Proof. It is straightforward from Theorem 7 and Proposition 4 that

$$\lim_{x \rightarrow \infty} \frac{f}{g} = \lim_{x \rightarrow \infty} \left\lceil \frac{f}{g} \right\rceil = \lim_{x \rightarrow \infty} \frac{\lceil f \rceil}{\lceil g \rceil}.$$

□

Example 9. It is easy to check that $2x^3$ is the leader of

$$f(x) = \sqrt{4x^6 + \sqrt{5x^7 + \sqrt{6x^8}}},$$

and x^3 is the leader of $g(x) = (x^{3/2} + 2x + 1)^2$. Therefore, by Theorem 8,

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{2x^3}{x^3} = 2.$$

From this example, we can see that in finding the solution our method is more convenient than the L'Hospital's rule as the quotient, $\frac{f'(x)}{g'(x)}$, is complicated.

Remark 10. If $\lim_{x \rightarrow \infty} (f(x) - g(x)) = \infty - \infty$, then we multiply both the numerator and the denominator by its conjugation, $f(x) + g(x)$. The limit, then, transforms to $\frac{\infty}{\infty}$. We can now apply Theorem 8 to get a limit solution. For example,

$$\begin{aligned} & \lim_{x \rightarrow \infty} \sqrt{x^4 + \sqrt{x^6 + x}} - (x^2 - x) \\ &= \lim_{x \rightarrow \infty} \left[\sqrt{x^4 + \sqrt{x^6 + x}} - (x^2 - x) \right] \cdot \frac{\left[\sqrt{x^4 + \sqrt{x^6 + x}} + (x^2 - x) \right]}{\sqrt{x^4 + \sqrt{x^6 + x}} + (x^2 - x)} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^6 + x} + 2x^3 - x^2}{\sqrt{x^4 + \sqrt{x^6 + x}} + (x^2 - x)} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^6 + 2x^3}}{\sqrt{x^4 + x^2}} = \lim_{x \rightarrow \infty} \frac{3x}{2} = \infty. \end{aligned}$$

3. Conclusion

We established the useful theorem to find limits as x tend to infinity for any functions that have their leaders, especially a function which has indeterminate form $\frac{\infty}{\infty}$ or $\infty - \infty$. We also emphasize that our method is easier and more convenient than other methods.

Interestingly, what are necessary and sufficient conditions to ensure that a function can has a leader?

Acknowledgements

The authors would like to thank the Faculty of science, Burapha university, Thailand for financial support.

References

- [1] H. Anton, I. Bivens, S. Davis, *Calculus*, 8-th Edition, John Wiles & Sons, USA (2005).
- [2] G.B. Tomas, R.L. Finney, *Calculus and Analytic Geometry*, 8-th Edition, Addison-Wesley, USA (1993).