**ON IMPLICIT MANN TYPE ITERATION PROCESS FOR
STRICTLY HEMICONTRACTIVE MAPPINGS IN
REAL SMOOTH BANACH SPACES**

So Bong Jeong¹, Arif Rafiq², Shin Min Kang^{3 §}

¹Department of Physical Education
Gyeongsang National University
Jinju, 660-701, KOREA

²Department of Mathematics
Lahore Leads University
Lahore 54810, PAKISTAN

³Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, KOREA

Abstract: In this paper, we proved that the implicit Mann type iteration process can be applied to approximate the fixed point of strictly hemicontractive mappings in real smooth Banach spaces.

AMS Subject Classification: 47J25

Key Words: implicit Mann type iteration process, strictly hemicontractive mapping, real smooth Banach space

1. Introduction

Let K be a nonempty subset of an arbitrary Banach space X and X^* be its dual space. The symbols $D(T)$ and $F(T)$ stand for the domain and the set of fixed points of T (for a single-valued mapping $T : X \rightarrow X$, $x \in X$ is called a

Received: August 30, 2013

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[§]Correspondence author

fixed point of T iff $Tx = x$). We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. In a smooth Banach space, J is single-valued (we denoted by j).

Let $T : D(T) \subset X \rightarrow X$ be a mapping.

Definition 1.1. The mapping T is called *Lipshitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\|$$

for all $x, y \in K$. If $L = 1$, then T is called *nonexpansive* and if $0 \leq L < 1$, then T is called *contractive*.

Definition 1.2. (see [1], [2]) (1) The mapping T is said to be *pseudocontractive* if

$$\|x - y\| \leq \|x - y + r((I - T)x - (I - T)y)\|$$

for all $x, y \in D(T)$ and $r > 0$.

(2) The mapping T is said to be *strongly pseudocontractive* if there exists a constant $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

for all $x, y \in D(T)$ and $r > 0$.

(3) The mapping T is said to be *local strongly pseudocontractive* if for each $x \in D(T)$ there exists a constant $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

for all $y \in D(T)$ and $r > 0$.

(4) The mapping T is said to be *strictly hemicontractive* if $F(T) \neq \emptyset$ and there exists a constant $t > 1$ such that

$$\|x - q\| \leq \|(1 + r)(x - q) - rt(Tx - q)\|$$

for all $x \in D(T)$, $q \in F(T)$ and $r > 0$.

Clearly, every strongly pseudocontractive mapping with a nonempty fixed point set is strictly hemicontractive, but the converse does not hold in general (see [2]).

Chidume [1] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudocontractive mapping from a bounded closed convex subset of L_p (or l_p) into itself. Schu [9] generalized the result in [1] to both uniformly continuous strongly pseudocontractive mappings and real smooth Banach spaces. Park [7] extended the result in [1] to both strongly pseudocontractive mappings and certain smooth Banach spaces. Rhoades [8] proved that the Mann and Ishikawa iteration methods may exhibit different behaviors for different classes of nonlinear mappings. Afterwards, several generalizations have been made in various directions (see for example [2], [4]-[6], [10]).

In 2001, Xu and Ori [11] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \dots, N\}$) with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in K$,

$$\begin{aligned} x_1 &= (1 - \alpha_1)x_0 + \alpha_1 T_1 x_1, \\ x_2 &= (1 - \alpha_2)x_1 + \alpha_2 T_2 x_2, \\ &\vdots \\ x_N &= (1 - \alpha_N)x_{N-1} + \alpha_N T_N x_N, \\ x_{N+1} &= (1 - \alpha_{N+1})x_N + \alpha_{N+1} T_{N+1} x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = (1 - \alpha_n)x_{n-1} + \alpha_n T_n x_n, \quad n \geq 1, \quad (XO)$$

where $T_n = T_{n \pmod N}$ (here the $\pmod N$ function takes values in I). Xu and Ori [11] proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [6], Oslilike proved the following results.

Theorem 1.3. *Let E be a real Banach space and K be a nonempty closed convex subset of E . Let $\{T_i : i \in I\}$ be N strictly pseudocontractive mappings from K to K with $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying:*

- (i) $0 < \alpha_n < 1$,
- (ii) $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$,

(iii) $\sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (XO) . Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$.

Remark 1.4. One can easily see that for $\alpha_n = 1 - \frac{1}{n^{1/2}}$, $\sum_{n=1}^{\infty} (1 - \alpha_n)^2 = \infty$. Hence the results of Osilike [6] are needed to be improved.

Let K be a nonempty closed bounded convex subset of a real smooth Banach space X and $T : K \rightarrow K$ be a continuous strictly hemiccontractive mapping. Under some conditions we obtain that the implicit Mann type iteration process mainly due to Xu and Ori [11] converges strongly to a unique fixed point of T .

In this paper, we improve the corresponding results in [6].

2. Preliminaries

We need the following results.

Lemma 2.1. (see [7]) *Let X be a smooth Banach space. Suppose one of the following holds:*

- (a) J is uniformly continuous on any bounded subsets of X ,
- (b) $\langle x - y, j(x) - j(y) \rangle \leq \|x - y\|^2$ for all $x, y \in X$,
- (c) for any bounded subset D of X , there exists a function $c : [0, \infty) \rightarrow [0, \infty)$ such that

$$\operatorname{Re} \langle x - y, j(x) - j(y) \rangle \leq c(\|x - y\|)$$

for all $x, y \in D$, where c satisfies $\lim_{t \rightarrow 0^+} \frac{c(t)}{t} = 0$.

Then for any $\epsilon > 0$ and any bounded subset K , there exists $\delta > 0$ such that

$$\|sx + (1 - s)y\|^2 \leq (1 - 2s) \|y\|^2 + 2s \operatorname{Re} \langle x, j(y) \rangle + 2s\epsilon \quad (2.1)$$

for all $x, y \in K$ and $s \in [0, \delta]$.

Remark 2.2. 1. If X is uniformly smooth, then (a) in Lemma 2.1 holds.
2. If X is a Hilbert space, then (b) in Lemma 2.1 holds.

Lemma 2.3. (see [2]) *Let $T : D(T) \subset X \rightarrow X$ be a mapping with $F(T) \neq \emptyset$. Then T is strictly hemiccontractive if and only if there exists a constant $t > 1$ such that for all $x \in D(T)$ and $q \in F(T)$, there exists $j(x - q) \in J(x - q)$ satisfying*

$$\operatorname{Re} \langle x - Tx, j(x - q) \rangle \geq (1 - t^{-1}) \|x - q\|^2. \quad (2.2)$$

Lemma 2.4. (see [5]) *Let X be an arbitrary normed linear space and $T : D(T) \subset X \rightarrow X$ be a mapping. If T is strictly hemiccontractive, then $F(T)$ is a singleton.*

3. Main Results

We now prove our main results.

Lemma 3.1. *Let $\{\theta_n\}$ and $\{\beta_n\}$ be nonnegative real sequences and $\epsilon' > 0$ be a constant satisfying*

$$\beta_{n+1} \leq (1 - \theta_n^l)\beta_n + \epsilon'\theta_n, \quad l \geq 1, n \geq 1,$$

where $\sum_{n=1}^{\infty} \theta_n^l = \infty$ and $\theta_n \leq 1$ for all $n \geq 1$. Then $\limsup_{n \rightarrow \infty} \beta_n \leq \epsilon'$.

Proof. By a straightforward argument, for $n \geq k \geq 1$,

$$\beta_{n+1} \leq \beta_k \prod_{j=k}^n (1 - \theta_j^l) + \epsilon' \sum_{j=k}^n \theta_j \prod_{i=j+1}^n (1 - \theta_i^l). \quad (3.1)$$

Note that $\sum_{j=k}^n \theta_j \prod_{i=j+1}^n (1 - \theta_i^l) \leq 1$. It follows from (3.1) that

$$\beta_{n+1} \leq \exp \left(- \sum_{j=k}^n \theta_j^l \right) \beta_k + \epsilon'. \quad (3.2)$$

Thus (3.2) ensures that

$$\limsup_{n \rightarrow \infty} \beta_n \leq \epsilon'.$$

This completes the proof. □

Remark 3.2. If $l = 1$, then Lemma 3.1 reduces to Lemma 1 of Park [7].

Theorem 3.3. *Let X be a real smooth Banach space satisfying any one of the Axioms (a)-(c) of Lemma 2.1. Let K be a nonempty closed bounded convex subset of X and $T : K \rightarrow K$ be a continuous strictly hemiccontractive mapping. Suppose that $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying:*

- (iv) $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (v) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (XO) with $T_i = T$ ($i = 1, \dots, N$). Then the sequence $\{x_n\}$ converges strongly to a unique fixed point q of T .

Proof. By [3, Corollary 1], T has a unique fixed point q in K . It follows from Lemma 2.4 that $F(T)$ is a singleton. That is, $F(T) = \{q\}$ for some $q \in K$.

Now for $k = \frac{1}{t}$, where t satisfies (2.2), we will prove that $\{x_n\}$ is bounded.

Indeed, from (XO) we have

$$\begin{aligned}
 & \|x_n - q\|^2 \\
 &= \langle x_n - q, j(x_n - q) \rangle \\
 &= \langle (1 - \alpha_n)x_{n-1} + \alpha_n Tx_n - q, j(x_n - q) \rangle \\
 &= \langle (1 - \alpha_n)(x_{n-1} - q) + \alpha_n(Tx_n - q), j(x_n - q) \rangle \\
 &= (1 - \alpha_n) \langle x_{n-1} - q, j(x_n - q) \rangle + \alpha_n \langle Tx_n - q, j(x_n - q) \rangle \\
 &\leq (1 - \alpha_n) \|x_{n-1} - q\| \|x_n - q\| + k\alpha_n \|x_n - q\|^2,
 \end{aligned}$$

which implies that

$$\|x_n - q\| \leq (1 - \alpha_n) \|x_{n-1} - q\| + k\alpha_n \|x_n - q\|,$$

and consequently, we obtain

$$\|x_n - q\| \leq \frac{1 - \alpha_n}{1 - k\alpha_n} \|x_{n-1} - q\| \leq \|x_{n-1} - q\|.$$

Now induction yields

$$\|x_n - q\| \leq \|x_0 - q\|, \quad n \geq 1.$$

So, from the above discussion, we can conclude that the sequence $\{x_n\}$ is bounded. Also since T is continuous, so set

$$M := \sup_{n \geq 1} \|x_n - q\| + \sup_{n \geq 1} \|Tx_n - q\|. \quad (3.3)$$

Using (3.3) and Lemma 2.1, we infer that

$$\begin{aligned}
 & \|x_n - q\|^2 \\
 &= \|(1 - \alpha_n)x_{n-1} + \alpha_n Tx_n - q\|^2 \\
 &= \|(1 - \alpha_n)(x_{n-1} - q) + \alpha_n(Tx_n - q)\|^2 \\
 &\leq (1 - 2\alpha_n) \|x_{n-1} - q\|^2 + 2\alpha_n \langle Tx_n - q, j(x_{n-1} - q) \rangle + 2\epsilon\alpha_n \\
 &= (1 - 2\alpha_n) \|x_{n-1} - q\|^2 + 2\alpha_n \langle Tx_n - q, j(x_n - q) \rangle \\
 &\quad + 2\alpha_n \langle Tx_n - q, j(x_{n-1} - q) - j(x_n - q) \rangle + 2\epsilon\alpha_n \\
 &\leq (1 - 2\alpha_n) \|x_{n-1} - q\|^2 + 2k\alpha_n \|x_n - q\|^2 \\
 &\quad + 2\alpha_n \|Tx_n - q\| \|j(x_{n-1} - q) - j(x_n - q)\| + 2\epsilon\alpha_n \\
 &\leq (1 - 2\alpha_n) \|x_{n-1} - q\|^2 + 2k\alpha_n \|x_n - q\|^2 + 2M\alpha_n \delta_n + 2\epsilon\alpha_n,
 \end{aligned} \quad (3.4)$$

where

$$\delta_n = \|j(x_{n-1} - q) - j(x_n - q)\|.$$

Since J is uniformly continuous on any bounded subsets of X and

$$\begin{aligned} \|x_{n-1} - x_n\| &= \|x_{n-1} - (1 - \alpha_n)x_{n-1} - \alpha_n Tx_n\| \\ &= \alpha_n \|x_{n-1} - Tx_n\| \\ &\leq 2M\alpha_n \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which implies that

$$\delta_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.5)$$

Consider

$$\begin{aligned} \|x_n - p\|^2 &= \|(1 - \alpha_n)x_{n-1} + \alpha_n Tx_n - p\|^2 \\ &= \|(1 - \alpha_n)(x_{n-1} - q) + \alpha_n(Tx_n - q)\|^2 \\ &\leq (1 - \alpha_n) \|x_{n-1} - q\|^2 + \alpha_n \|Tx_n - q\|^2 \\ &\leq \|x_{n-1} - q\|^2 + M^2\alpha_n, \end{aligned} \quad (3.6)$$

where the first inequality holds by the convexity of $\|\cdot\|^2$.

For given any $\epsilon > 0$ and the bounded subset K , there exists a $\delta > 0$ satisfying (2.1). Note that (3.5) and (iv) ensure that there exists an N such that

$$\alpha_n < \min \left\{ \delta, \frac{1}{2(1-k)}, \frac{\epsilon}{4M^2k} \right\} \quad \text{and} \quad \delta_n \leq \frac{\epsilon}{4M}, \quad n \geq N.$$

Substituting (3.6) in (3.4) to obtain

$$\begin{aligned} \|x_n - q\|^2 &\leq (1 - 2(1-k)\alpha_n) \|x_{n-1} - q\|^2 + 2M\alpha_n\delta_n \\ &\quad + 2\epsilon\alpha_n + 2M^2k\alpha_n^2 \\ &\leq (1 - 2(1-k)\alpha_n) \|x_{n-1} - q\|^2 + 3\epsilon\alpha_n \end{aligned} \quad (3.7)$$

for all $n \geq N$.

Putting

$$\beta_n = \|x_{n-1} - q\|, \quad \theta_n = 2(1-k)\alpha_n \quad \text{and} \quad \epsilon' = \frac{3\epsilon}{2(1-k)},$$

we have from (3.7)

$$\beta_{n+1} \leq (1 - \theta_n)\beta_n + \epsilon'\theta_n, \quad n \geq 1.$$

Set $\delta = \frac{1}{2(1-k)}$ for $k \leq \frac{1}{2}$. Because $\alpha_n \leq \delta$ we imply $2(1-k)\alpha_n \leq 1$. Observe that $\sum_{n=1}^{\infty} \theta_n = \infty$ and $\theta_n < 1$ for all $n \geq 1$. It follows from Lemma 3.1 that

$$\limsup_{n \rightarrow \infty} \|x_n - q\|^2 \leq \epsilon'.$$

Letting $\epsilon' \rightarrow 0^+$, we obtain that $\limsup_{n \rightarrow \infty} \|x_n - q\|^2 = 0$, which implies that $x_n \rightarrow q$ as $n \rightarrow \infty$. This completes the proof. $\square$

Corollary 3.4. *Let X be a real smooth Banach space satisfying any one of the Axioms (a)-(c) of Lemma 2.1. Let K be a nonempty closed bounded convex subset of X and $T : K \rightarrow K$ be a Lipschitz strictly hemicontractive mapping. Suppose that $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the conditions (iv) and (v).*

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (XO) with $T_i = T$ ($i = 1, \dots, N$). Then the sequence $\{x_n\}$ converges strongly to a unique fixed point q of T .

Remark 3.5. Similar results can be found for the iteration processes involved error term, we omit the details.

Remark 3.6. Theorem 3.3 and Corollary 3.4 extend and improve the Theorem 1.3 in the following directions.

We do not need the assumption $\liminf_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$ as in Theorem 1.3.

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