ON THE DIOPHANTINE EQUATION $3^x + 17^y = z^2$

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Abstract: In this paper, we prove that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 17^y = z^2$ where x, y and z are non-negative integers.

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1. Introduction

In 2012, we [7] proved that $(1, 0, 2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. In this paper, we prove that $(1, 0, 2)$ is also a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 17^y = z^2$ where x, y and z are non-negative integers.

In 2007, Acu [1] proved that the Diophantine equation $2^x + 5^y = z^2$ has only two non-negative integer solutions where x, y and z are non-negative integers. The solutions (x, y, z) are $(3, 0, 3)$ and $(2, 1, 3)$.

In 2013, Chotchaisthit [2] proved that the Diophantine equation $2^x + 11^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(3, 0, 3)$.

Many related questions was solved as see in [3, 5, 6, 8, 9, 10, 11, 12, 13, 14].

2. Preliminaries

Proposition 2.1. [4] $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [7] $(1, 2)$ is a unique solution (x, z) for the Diophantine equation $3^x + 1 = z^2$ where x and z are non-negative integers.

Lemma 2.3. The Diophantine equation $1 + 17^y = z^2$ has no non-negative integer solution.

Proof. Suppose that there are non-negative integers y and z such that $1 + 17^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. It follows that $z^2 = 1 + 17^y \geq 1 + 17^1 = 18$. Hence, $z \geq 5$. Now, we consider on the equation $z^2 - 17^y = 1$. By Proposition 2.1, we have $y = 1$. Hence, $z^2 = 18$. This is a contradiction. $\square$

3. Results

Theorem 3.1. $(1, 0, 2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 17^y = z^2$ where x, y and z are non-negative integers.

Proof. Let x, y and z be non-negative integers such that $3^x + 17^y = z^2$. By Lemma 2.3, we have $x \geq 1$. Now, we will divide the number y into two cases.

Case $y = 0$. By Lemma 2.2, we have $x = 1$ and $z = 2$.

Case $y \geq 1$. Note that z is a positive even number. Then $z^2 \equiv 0 \pmod{4}$. It follows that $3^x \equiv 3 \pmod{4}$ since $17^y \equiv 1 \pmod{4}$. Hence, x is odd. This implies that $3^x \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$. Then $z^2 \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$. This is a contradiction since $z^2 \equiv 1, 2, 4, 8, 9, 13, 16 \pmod{17}$. $\square$

Corollary 3.2. The Diophantine equation $3^x + 17^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers.

Proof. Suppose that there are non-negative integers x, y and w such that $3^x + 17^y = w^4$. Let $z = w^2$. We obtain that $3^x + 17^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 2)$. Hence, $w^2 = z = 2$. This is a contradiction. $\square$

Corollary 3.3. *The Diophantine equation $9^u + 17^y = z^2$ has no non-negative integer solution where u, y and z are non-negative integers.*

Proof. Suppose that there are non-negative integers u, y and z such that $9^u + 17^y = z^2$. Let $x = 2u$. We obtain that $3^x + 17^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 2)$. Hence, $2u = x = 1$. This is a contradiction. $\square$

Corollary 3.4. *The Diophantine equation $9^u + 17^y = w^4$ has no non-negative integer solution where u, y and w are non-negative integers.*

Proof. This follows from Corollary 3.3 using $z = w^2$. $\square$

4. Open Problem

Let q be a positive odd prime number. Now, we questions that what is the set of all solutions (x, y, z) for the Diophantine equation $3^x + q^y = z^2$ where x, y and z are non-negative integers.

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