ON THE DIOPHANTINE EQUATION $3^x + 17^y = z^2$

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Abstract: In this paper, we prove that $(1, 0, 2)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $3^x + 17^y = z^2$ where $x, y$ and $z$ are non-negative integers.

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1. Introduction

In 2012, we [7] proved that $(1, 0, 2)$ is a unique solution $(x, y, z)$ for the Diophantine equation $3^x + 5^y = z^2$ where $x, y$ and $z$ are non-negative integers. In this paper, we prove that $(1, 0, 2)$ is also a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $3^x + 17^y = z^2$ where $x, y$ and $z$ are non-negative integers.

In 2007, Acu [1] proved that the Diophantine equation $2^x + 5^y = z^2$ has only two non-negative integer solutions where $x, y$ and $z$ are non-negative integers. The solutions $(x, y, z)$ are $(3, 0, 3)$ and $(2, 1, 3)$.

In 2013, Chotchaisthit [2] proved that the Diophantine equation $2^x + 11^y = z^2$ has a unique non-negative integer solution where $x, y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(3, 0, 3)$.
Many related questions was solved as see in [3, 5, 6, 8, 9, 10, 11, 12, 13, 14].

2. Preliminaries

**Proposition 2.1.** \[4\] (3, 2, 2, 3) is a unique solution \((a, b, x, y)\) for the Diophantine equation \(a^x - b^y = 1\) where \(a, b, x\) and \(y\) are integers with \(\min\{a, b, x, y\} > 1\).

**Lemma 2.2.** \[7\] (1, 2) is a unique solution \((x, z)\) for the Diophantine equation \(3^x + 1 = z^2\) where \(x\) and \(z\) are non-negative integers.

**Lemma 2.3.** The Diophantine equation \(1 + 17^y = z^2\) has no non-negative integer solution.

Proof. Suppose that there are non-negative integers \(y\) and \(z\) such that \(1 + 17^y = z^2\). If \(y = 0\), then \(z^2 = 2\) which is impossible. Then \(y \geq 1\). It follows that \(z^2 = 1 + 17^y \geq 1 + 17^1 = 18\). Hence, \(z \geq 5\). Now, we consider on the equation \(z^2 - 17^y = 1\). By Proposition 2.1, we have \(y = 1\). Hence, \(z^2 = 18\). This is a contradiction. 

3. Results

**Theorem 3.1.** (1, 0, 2) is a unique solution \((x, y, z)\) for the Diophantine equation \(3^x + 17^y = z^2\) where \(x, y\) and \(z\) are non-negative integers.

Proof. Let \(x, y\) and \(z\) be non-negative integers such that \(3^x + 17^y = z^2\). By Lemma 2.3, we have \(x \geq 1\). Now, we will divide the number \(y\) into two cases.

Case \(y = 0\). By Lemma 2.2, we have \(x = 1\) and \(z = 2\).

Case \(y \geq 1\). Note that \(z\) is a positive even number. Then \(z^2 \equiv 0 \pmod{4}\). It follows that \(3^x \equiv 3 \pmod{4}\) since \(17^y \equiv 1 \pmod{4}\). Hence, \(x\) is odd. This implies that \(3^x \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}\). Then \(z^2 \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}\). This is a contradiction since \(z^2 \equiv 1, 2, 4, 8, 9, 13, 16 \pmod{17}\).

**Corollary 3.2.** The Diophantine equation \(3^x + 17^y = w^4\) has no non-negative integer solution where \(x, y\) and \(w\) are non-negative integers.

Proof. Suppose that there are non-negative integers \(x, y\) and \(w\) such that \(3^x + 17^y = w^4\). Let \(z = w^2\). We obtain that \(3^x + 17^y = z^2\). By Theorem 3.1, we have \((x, y, z) = (1, 0, 2)\). Hence, \(w^2 = z = 2\). This is a contradiction.
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**Corollary 3.3.** The Diophantine equation $9^u + 17^y = z^2$ has no non-negative integer solution where $u, y$ and $z$ are non-negative integers.

**Proof.** Suppose that there are non-negative integers $u, y$ and $z$ such that $9^u + 17^y = z^2$. Let $x = 2u$. We obtain that $3^x + 17^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 2)$. Hence, $2u = x = 1$. This is a contradiction. □

**Corollary 3.4.** The Diophantine equation $9^u + 17^y = w^4$ has no non-negative integer solution where $u, y$ and $w$ are non-negative integers.

**Proof.** This follows from Corollary 3.3 using $z = w^2$. □

4. Open Problem

Let $q$ be a positive odd prime number. Now, we questions that what is the set of all solutions $(x, y, z)$ for the Diophantine equation $3^x + q^y = z^2$ where $x, y$ and $z$ are non-negative integers.

References


