ON THE DIOPHANTINE EQUATION $5^x + 23^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $5^x + 23^y = z^2$ has no non-negative integer solution where x , y and z are non-negative integers.

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1. Introduction

Recently, Sroysang [8, 11] showed that both the Diophantine equations $31^x + 32^y = z^2$ and $23^x + 32^y = z^2$ have no non-negative integer solution where x , y and z are non-negative integers. Moreover, he [6] showed that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for $3^x + 5^y = z^2$ where x , y and z are non-negative integers. In this paper, we will show that the Diophantine equation $5^x + 23^y = z^2$ has no non-negative integer solution where x , y and z are non-negative integers. For related equations of type $a^x + b^y = z^2$, we list the results as follows.

In 2007, Acu [1] showed that $(3, 0, 3)$ and $(2, 1, 3)$ are only two non-negative integer solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x , y and z are non-negative integers

In 2011, Suvarnamani, Singta and Chotchaisthit [12] showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers.

In 2012, Chotchaisthit [4] solved the Diophantine equation $4^x + p^y = z^2$, where x, y and z are non-negative integers, for all positive prime number p . In the same year, Sroysang [7] showed that $(1, 0, 3)$ is a unique non-negative integer solution (x, y, z) for $8^x + 19^y = z^2$ where x, y and z are non-negative integers.

In 2013, Sroysang [9, 10] showed that (i) $(0, 1, 3)$ is a unique non-negative integer solution (x, y, z) for $7^x + 8^y = z^2$ where x, y and z are non-negative integers, and (ii) $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three non-negative integer solutions (x, y, z) for $2^x + 3^y = z^2$ where x, y and z are non-negative integers. In the same year, Chotchaisthit [3] showed that the Diophantine equation $2^x + 11^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(3, 0, 3)$.

2. Preliminaries

In 1844, Catalan [2] conjectured that $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$. And then the conjecture was proved by Mihailescu [5] in 2004.

Proposition 2.1. [5] *(the Catalan's conjecture) $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.*

Lemma 2.2. [6] *The Diophantine equation $5^x + 1 = z^2$ has no non-negative integer solution where x and z are non-negative integers.*

Lemma 2.3. [11] *The Diophantine equation $1 + 23^y = z^2$ has no non-negative integer solution where y and z are non-negative integers.*

3. Results

Theorem 3.1. *The Diophantine equation $5^x + 23^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $5^x + 23^y = z^2$. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. Note that z is

even. Then $z^2 \equiv 0 \pmod{4}$. Moreover, $5^x \equiv 1 \pmod{4}$. It follows that $23^y \equiv 3 \pmod{4}$. Thus, y is odd. Hence, $23^y \equiv 2 \pmod{5}$ or $23^y \equiv 3 \pmod{5}$. This implies that $z^2 \equiv 2 \pmod{5}$ or $z^2 \equiv 3 \pmod{5}$. This is impossible. $\square$

Corollary 3.2. *The Diophantine equation $5^x + 23^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $5^x + 23^y = w^4$. Let $z = w^2$. Then $5^x + 23^y = z^2$. By Theorem 3.1, the equation $5^x + 23^y = z^2$ has no non-negative integer solution. This is a contradiction. $\square$

Corollary 3.3. *The Diophantine equation $23^u + 25^v = z^2$ has no non-negative integer solution where u, v and z are non-negative integers.*

Proof. Suppose that there are non-negative integers u, v and z such that $23^u + 25^v = w^4$. Let $x = 2v$ and $y = u$. Then $5^x + 23^y = z^2$. By Theorem 3.1, the equation $5^x + 7^y = z^2$ has no non-negative integer solution. This is a contradiction. $\square$

Corollary 3.4. *The Diophantine equation $23^u + 25^v = w^4$ has no non-negative integer solution where u, v and w are non-negative integers.*

Proof. Suppose that there are non-negative integers u, v and w such that $23^u + 25^v = w^4$. Let $z = w^2$. Then $23^u + 25^v = z^2$. By Corollary 3.3, the equation $23^u + 25^v = z^2$ has no non-negative integer solution. This is a contradiction. $\square$

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