INTEGRAL MEAN ESTIMATES FOR POLYNOMIALS WITH RESTRICTED ZEROS

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Abstract: Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, it was proved by Aziz and Ahemad [2]:

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^{qr} d\theta \right\}^{\frac{1}{qr}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{pr} d\theta \right\}^{\frac{1}{pr}}.$$

In this paper, we establish some refinements and generalizations of above and some known polynomial inequalities concerning the polar derivative of a polynomial with restricted zeros.

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1. Introduction and Statement of the Results

Let $P(z)$ be a polynomial of degree n . It was shown by Turán [12] that if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$n \max_{|z|=1} |P(z)| \leq 2 \max_{|z|=1} |P'(z)|. \quad (1.1)$$

Inequality (1.1) is best possible with equality holds for $P(z) = \alpha z^n + \beta$, where

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$|\alpha| = |\beta| \neq 0$.

As an extension of (1.1), Malik [9] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then

$$n \max_{|z|=1} |P(z)| \leq (1+k) \max_{|z|=1} |P'(z)|. \quad (1.2)$$

Equality in (1.2) holds for $P(z) = (z+k)^n$, where $k \leq 1$.

On the other hand, for the class of polynomials $P(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, of degree n having all their zeros in $|z| \leq k$, $k \leq 1$, Aziz and Shah [6] proved that

$$n \max_{|z|=1} |P(z)| \leq (1+k^\mu) \max_{|z|=1} |P'(z)| - \frac{n}{k^{n-\mu}} \min_{|z|=k} |P(z)|. \quad (1.3)$$

Malik [10] obtained a generalization of (1.1) in the sense that the left-hand side of (1.1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, he proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for each $q > 0$,

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \leq \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta \right\}^{\frac{1}{q}} \max_{|z|=1} |P'(z)|. \quad (1.4)$$

The corresponding extension of (1.2), which is a generalization of (1.4), was obtained by Aziz [1] who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for each $q \geq 0$

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^q d\theta \right\}^{\frac{1}{q}} \max_{|z|=1} |P'(z)|. \quad (1.5)$$

Inequality (1.5) reduces to the inequality (1.2) by letting $q \rightarrow \infty$.

As a generalization of (1.5), Aziz and Ahemad [2] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^{qr} d\theta \right\}^{\frac{1}{qr}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{pr} d\theta \right\}^{\frac{1}{pr}} \quad (1.6)$$

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P(z)$ of degree n with respect to a point $\alpha \in \mathbb{C}$, then (see [11])

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \rightarrow \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

As an extension of (1.2) to the polar derivative, Aziz and Rather [3] proved that if all the zeros of $P(z)$ lie in $|z| \leq k$, where $k \leq 1$, then for every real or complex number α with $|\alpha| \geq k$,

$$n(|\alpha| - k) \max_{|z|=1} |P(z)| \leq (1 + k) \max_{|z|=1} |D_\alpha P(z)|. \quad (1.7)$$

For the class of Lacunary polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, of degree n having all their zeros in $|z| \leq k$, where $k \leq 1$, Aziz and Rather [4] also proved that if α is any real or complex number with $|\alpha| \geq S_\mu$ then

$$n(|\alpha| - S_\mu) \max_{|z|=1} |P(z)| \leq (1 + S_\mu) \max_{|z|=1} |D_\alpha P(z)|, \quad (1.8)$$

where

$$S_\mu = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}. \quad (1.9)$$

In view of (1.9), one can easily verify that the inequality (1.8) is a generalization as well as a refinement of inequality (1.7).

In this paper, we consider the class of polynomials

$$P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}, \quad 1 \leq \mu \leq n,$$

having all its zeros in $|z| \leq k$, where $k \leq 1$ and establish some generalizations of inequalities (1.1), (1.2), (1.7), (1.5) and (1.8).

In this direction, we present the following interesting results which yields (1.8) as a special case.

Theorem 1.1. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for all real or complex numbers α, λ with $|\alpha| \geq A_\mu$, $|\lambda| \leq 1$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,*

$$n(|\alpha| - A_\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{i n \theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + A_\mu e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}}$$

$$\times \left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + \frac{\lambda \alpha m n e^{i(n-1)\theta}}{k^n} \right|^{qr} d\theta \right\}^{\frac{1}{qr}}, \quad (1.10)$$

where $m = \min_{|z|=k} |P(z)|$ and

$$A_\mu = \frac{n \left(|a_n| - \frac{m}{k^n} \right) k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n \left(|a_n| - \frac{m}{k^n} \right) k^{\mu-1} + \mu |a_{n-\mu}|}. \quad (1.11)$$

If we divide two sides of inequality (1.10) by $|\alpha|$ and then letting $|\alpha| \rightarrow \infty$, we obtain the following generalization of inequality (1.3).

Corollary 1.2. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex number λ with $|\lambda| \leq 1$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,*

$$\begin{aligned} n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{in\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{r}} &\leq \left\{ \int_0^{2\pi} |1 + A_\mu e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \\ &\times \left\{ \int_0^{2\pi} \left| P'(e^{i\theta}) + \frac{\lambda m n e^{i(n-1)\theta}}{k^n} \right|^{qr} d\theta \right\}^{\frac{1}{qr}}, \end{aligned} \quad (1.12)$$

where $m = \min_{|z|=k} |P(z)|$ and A_μ is given by (1.11).

Remark 1.3. If we take $\lambda = 0$ in (1.12), then we get a refinement of inequality (1.6).

Letting $p \rightarrow \infty$, so that $q \rightarrow 1$ in (1.10), we obtain the following result.

Corollary 1.4. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for all real or complex numbers α, λ with $|\alpha| \geq A_\mu$, $|\beta| \leq 1$ and for each $r > 0$,*

$$\begin{aligned} n(|\alpha| - A_\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{in\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{r}} \\ \leq (1 + A_\mu) \left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + \frac{\lambda m n \alpha e^{i(n-1)\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{r}}, \end{aligned} \quad (1.13)$$

where $m = \min_{|z|=k} |P(z)|$ and A_μ is given by (1.11).

By Lemma 2.4, we have $A_\mu \leq k^\mu$ and therefore, it can be verified that for $p > 1$ and $r > 0$,

$$\int_0^{2\pi} |1 + A_\mu e^{i\theta}|^{pr} d\theta \leq \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta. \quad (1.14)$$

In view of inequality (1.14), we get the following result.

Corollary 1.5. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for all real or complex numbers α, λ with $|\alpha| \geq A_\mu$, $|\lambda| \leq 1$ and for each $r > 0$, $p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$,*

$$\begin{aligned} n(|\alpha| - A_\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{in\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{r}} &\leq \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \\ &\times \left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + \frac{\lambda \alpha m n e^{i(n-1)\theta}}{k^n} \right|^{qr} d\theta \right\}^{\frac{1}{qr}}, \end{aligned} \quad (1.15)$$

where $m = \min_{|z|=k} |P(z)|$ and A_μ is given by (1.11).

2. Some Lemmas

For the proof of theorem 1.1, we need the following lemmas:

Lemma 2.1. *If $P(z) = a_n + \sum_{j=\mu}^n a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, then for $|z| = 1$*

$$|Q'(z)| \leq S_\mu |P'(z)|, \quad (2.1)$$

where S_μ is given by (1.9).

The above lemma is due to Aziz and Rather [5].

Lemma 2.2. *If $P(z) = \sum_{j=1}^n a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then*

$$|a_n| > \frac{m}{k^n}. \quad (2.2)$$

Proof. By hypothesis all the zeros of $P(z)$ lie in $|z| \leq k$. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result holds trivially. So, we assume $P(z)$ has all its zeros in $|z| < k$, $k \leq 1$, so that $m > 0$. If $Q(z) = z^n \overline{P(1/\bar{z})}$, then $Q(z)$ has no zero in $|z| \leq 1/k$. By Minimum Modulus Principle

$$|Q(z)| \geq \min_{|z|=1/k} |Q(z)| \quad \text{for } |z| \leq 1/k, \quad \text{where } 1/k \geq 1,$$

which in particular gives,

$$|a_n| = |Q(0)| > \frac{1}{k^n} \min_{|z|=k} |P(z)|.$$

This completes the proof of Lemma 2.2.. □

Lemma 2.3. *The function*

$$S_\mu(x) = \frac{nxk^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{nxk^{\mu-1} + \mu|a_{n-\mu}|}, \quad (2.3)$$

where $k \leq 1$ and $\mu \geq 1$, is a non-increasing function of x .

Proof. The proof follows by considering the first derivative test for $S_\mu(x)$. □

We also need following Lemma.

Lemma 2.4 ([7]). *If $P(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then*

$$A_\mu \leq k^\mu, \quad (2.4)$$

where A_μ is defined in (1.11).

3. Proof of Theorem

Proof of Theorem 1.1. Let $m = \min_{|z|=k} |P(z)|$. By hypothesis $P(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, has all its zeros in $|z| \leq k$, where $k \leq 1$. We show $F(z) = P(z) + \frac{\lambda m z^n}{k^n}$ has all its zeros in $|z| \leq k$ for every λ with $|\lambda| \leq 1$. This is obvious if $m = 0$ that is, if $P(z)$ has a zero on $|z| = k$. Hence, we suppose that all the zeros of $P(z)$ lie in $|z| < k$, $k \leq 1$, so that $m > 0$. Now $m \leq |P(z)|$ for $|z| = k$, therefore if λ is any complex number such that $|\lambda| < 1$, then

$$\left| \frac{m \lambda z^n}{k^n} \right| < |P(z)| \quad \text{for } |z| = k.$$

Since all the zeros of $P(z)$ lie in $|z| < k$, it follows by Rouché's theorem that all the zeros of

$$F(z) = P(z) + \frac{m\lambda z^n}{k^n} = \left(a_n + \frac{\lambda m}{k^n}\right) z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}$$

also lie in $|z| < k, k \leq 1$.

Let $G(z) = z^n \overline{F(1/\bar{z})} = z^n \overline{P(1/\bar{z})} + \frac{m\bar{\lambda}}{k^n}$ then it can be easily verified for $|z| = 1$,

$$|F'(z)| = |nG(z) - zG'(z)| \quad \text{and} \quad |G'(z)| = |nF(z) - zF'(z)|. \quad (3.1)$$

Applying Lemma 2.1 to the polynomial $F(z)$, we get for $|z| = 1$,

$$S'_\mu |F'(z)| \geq |G'(z)|, \quad (3.2)$$

where

$$S'_\mu = \frac{n \left| a_n + \frac{m\lambda}{k^n} \right| k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n \left| a_n + \frac{m\lambda}{k^n} \right| k^{\mu-1} + \mu |a_{n-\mu}|}. \quad (3.3)$$

Since for every λ with $|\lambda| < 1$, we have by Lemma 2.2,

$$\left| a_n + \frac{m\lambda}{k^n} \right| \geq |a_n| - \frac{m|\lambda|}{k^n} \geq |a_n| - \frac{m}{k^n}, \quad (3.4)$$

therefore it follows from Lemma 2.3, (3.3) and (3.4) that for every λ with $|\lambda| < 1$,

$$\begin{aligned} S'_\mu &= \frac{n \left| a_n + \frac{m\lambda}{k^n} \right| k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n \left| a_n + \frac{m\lambda}{k^n} \right| k^{\mu-1} + \mu |a_{n-\mu}|} \\ &\leq \frac{n \left(|a_n| - \frac{m}{k^n} \right) k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n \left(|a_n| - \frac{m}{k^n} \right) k^{\mu-1} + \mu |a_{n-\mu}|} = A_\mu \end{aligned} \quad (3.5)$$

Using (3.1) and (3.5) in (3.2), we obtain

$$A_\mu |F'(z)| \geq |nF(z) - zF'(z)| \quad \text{for} \quad |z| = 1. \quad (3.6)$$

This gives for every real or complex number α with $|\alpha| \geq A_\mu$, we have

$$\begin{aligned} |D_\alpha F(z)| &= |nF(z) + (\alpha - z)F'(z)| \\ &\geq |\alpha| |F'(z)| - |nF(z) - zF'(z)| \\ &\geq (|\alpha| - A_\mu) |F'(z)| \quad \text{for} \quad |z| = 1. \end{aligned} \quad (3.7)$$

Replacing $F(z)$ by $P(z) + \frac{\lambda m z^n}{k^n}$, we obtain for $|z| = 1$,

$$\left| D_\alpha P(z) + \frac{\lambda \alpha m n z^{n-1}}{k^n} \right| \geq (|\alpha| - A_\mu) \left| P'(z) + \frac{\lambda \alpha m n z^{n-1}}{k^n} \right|. \quad (3.8)$$

Since $F(z)$ has all its zeros in $|z| \leq k \leq 1$, it follows by Gauss-Lucas theorem that all the zeros of $F'(z)$ also lie in $|z| \leq k \leq 1$. This shows that the polynomial

$$z^{n-1} \overline{F'(1/\bar{z})} \equiv nG(z) - zG'(z)$$

has all its zeros in $|z| \geq (1/k) \geq 1$. Therefore, it follows from (3.6) that the function

$$w(z) = \frac{zG'(z)}{A_\mu(nG(z) - zG'(z))}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + A_\mu w(z)$ is subordinate to the function $1 + A_\mu z$ for $|z| \leq 1$. Hence by a well known property of subordination [8], we have for each $r > 0$,

$$\int_0^{2\pi} \left| 1 + A_\mu w(e^{i\theta}) \right|^r d\theta \leq \int_0^{2\pi} \left| 1 + A_\mu e^{i\theta} \right|^r d\theta. \quad (3.9)$$

Now

$$1 + A_\mu w(z) = \frac{nG(z)}{nG(z) - zG'(z)},$$

and

$$|F'(z)| = |z^{n-1} \overline{F'(1/\bar{z})}| = |nG(z) - zG'(z)| \quad \text{for } |z| = 1,$$

therefore for $|z| = 1$,

$$n|G(z)| = |1 + A_\mu w(z)| |nG(z) - zG'(z)| = |1 + A_\mu w(z)| |F'(z)|.$$

Equivalently for $|z| = 1$:

$$n \left| z^n \overline{P(1/\bar{z})} + \frac{\bar{\lambda} m}{k^n} \right| = |1 + A_\mu w(z)| \left| P'(z) + \frac{\lambda \alpha m n z^{n-1}}{k^n} \right|,$$

which implies for $|z| = 1$,

$$n \left| P(z) + \frac{\lambda m z^n}{k^n} \right| = |1 + A_\mu w(z)| \left| P'(z) + \frac{\lambda \alpha m n z^{n-1}}{k^n} \right|. \quad (3.10)$$

From (3.8) and (3.10), we deduce that for each $r > 0$,

$$\begin{aligned} n^r(|\alpha| - A_\mu)^r \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{in\theta}}{k^n} \right|^r d\theta \\ \leq \int_0^{2\pi} |1 + A_\mu w(e^{i\theta})|^r \left| D_\alpha P(e^{i\theta}) + \frac{\lambda \alpha m n e^{i(n-1)\theta}}{k^n} \right|^r d\theta. \end{aligned}$$

This gives with the help of (3.9) and using Hölder's inequality for $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,

$$\begin{aligned} n^r(|\alpha| - A_\mu)^r \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{in\theta}}{k^n} \right|^r d\theta &\leq \left(\int_0^{2\pi} |1 + A_\mu e^{i\theta}|^{pr} d\theta \right)^{1/p} \\ &\times \left(\int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + \frac{\lambda \alpha m n e^{i(n-1)\theta}}{k^n} \right|^{qr} d\theta \right)^{1/q}, \end{aligned}$$

equivalently

$$\begin{aligned} n(|\alpha| - A_\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{in\theta}}{k^n} \right|^r d\theta \right\}^r &\leq \left\{ \int_0^{2\pi} |1 + A_\mu e^{i\theta}|^{pr} d\theta \right\}^{1/pr} \\ &\times \left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + \frac{\lambda \alpha m n e^{i(n-1)\theta}}{k^n} \right|^{qr} d\theta \right\}^{1/qr}. \end{aligned}$$

This completes the proof of theorem 1.1. □

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