INTEGRAL MEAN ESTIMATES FOR POLYNOMIALS WITH RESTRICTED ZEROS

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Abstract: Let $P(z)$ be a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, it was proved by Aziz and Ahemad [2]:

$$n\left\{\frac{2\pi}{r}\int_{0}^{2\pi}|P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\frac{2\pi}{q^r}\int_{0}^{2\pi}|1 + ke^{i\theta}|^{qr} d\theta\right\}^{\frac{1}{qr}} \left\{\frac{2\pi}{p^r}\int_{0}^{2\pi}|P'(e^{i\theta})|^{pr} d\theta\right\}^{\frac{1}{pr}}.$$

In this paper, we establish some refinements and generalizations of above and some known polynomial inequalities concerning the polar derivative of a polynomial with restricted zeros.

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1. Introduction and Statement of the Results

Let $P(z)$ be a polynomial of degree $n$. It was shown by Turán [12] that if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$n \max_{|z|=1} |P(z)| \leq 2 \max_{|z|=1} |P'(z)| . \quad (1.1)$$

Inequality (1.1) is best possible with equality holds for $P(z) = \alpha z^n + \beta$, where

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$|\alpha| = |\beta| \neq 0$.

As an extension of (1.1), Malik [9] proved that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then

$$n \max_{|z|=1} |P(z)| \leq (1 + k) \max_{|z|=1} |P'(z)|. \tag{1.2}$$

Equality in (1.2) holds for $P(z) = (z + k)^n$, where $k \leq 1$.

On the other hand, for the class of polynomials $P(z) = a_n z^n + \sum_{j=1}^{n-1} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, of degree $n$ having all their zeros in $|z| \leq k$, $k \leq 1$, Aziz and Shah [6] proved that

$$n \max_{|z|=1} |P(z)| \leq (1 + k^\mu) \max_{|z|=1} |P'(z)| - \frac{n}{k^{n-\mu}} \min_{|z|=k} |P(z)|. \tag{1.3}$$

Malik [10] obtained a generalization of (1.1) in the sense that the left-hand side of (1.1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, he proved that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq 1$, then for each $q > 0$,

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \tag{1.4}$$

The corresponding extension of (1.2), which is a generalization of (1.4), was obtained by Aziz [1] who proved that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for each $q \geq 0$

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left| 1 + ke^{i\theta} \right|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \tag{1.5}$$

Inequality (1.5) reduces to the inequality (1.2) by letting $q \to \infty$.

As a generalization of (1.5), Aziz and Ahemad [2] proved that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{1/r} \leq \left\{ \int_0^{2\pi} \left| 1 + ke^{i\theta} \right|^q d\theta \right\}^{1/q} \left\{ \int_0^{2\pi} \left| P'(e^{i\theta}) \right|^p d\theta \right\}^{1/p}. \tag{1.6}$$

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P(z)$ of degree $n$ with respect to a point $\alpha \in \mathbb{C}$, then (see [11])

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$
The polynomial $D_\alpha P(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that
\[ \lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z) \]
uniformly with respect to $z$ for $|z| \leq R, R > 0$.

As an extension of (1.2) to the polar derivative, Aziz and Rather [3] proved that if all the zeros of $P(z)$ lie in $|z| \leq k$, where $k \leq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq k$,
\[ n(|\alpha| - k) \max_{|z|=1} |P(z)| \leq (1 + k) \max_{|z|=1} |D_\alpha P(z)|. \quad (1.7) \]

For the class of Lacunary polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}, \ 1 \leq \mu \leq n$, of degree $n$ having all their zeros in $|z| \leq k$, where $k \leq 1$, Aziz and Rather [4] also proved that if $\alpha$ is any real or complex number with $|\alpha| \geq S_{\mu}$ then
\[ n(|\alpha| - S_{\mu}) \max_{|z|=1} |P(z)| \leq (1 + S_{\mu}) \max_{|z|=1} |D_\alpha P(z)|, \quad (1.8) \]
where
\[ S_{\mu} = \frac{n|a_n| k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n|a_n| k^{\mu-1} + \mu |a_{n-\mu}|}. \quad (1.9) \]

In view of (1.9), one can easily verify that the inequality (1.8) is a generalization as well as a refinement of inequality (1.7).

In this paper, we consider the class of polynomials
\[ P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}, \ 1 \leq \mu \leq n, \]
having all its zeros in $|z| \leq k$, where $k \leq 1$ and establish some generalizations of inequalities (1.1),(1.2),(1.7),(1.5) and (1.8).

In this direction, we present the following interesting results which yields (1.8) as a special case.

**Theorem 1.1.** If $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}, \ 1 \leq \mu \leq n$, is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for all real or complex numbers $\alpha, \lambda$ with $|\alpha| \geq A_{\mu}, |\lambda| \leq 1$ and for each $r > 0, p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$,
\[ n(|\alpha| - A_{\mu}) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda me^{i\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} \left| 1 + A_{\mu} e^{i\theta} \right|^p r d\theta \right\}^{\frac{1}{pr}}. \]
\[ \times \left\{ \int_0^{2\pi} \left| D_{\alpha} P(e^{i\theta}) + \frac{\lambda m n e^{i(n-1)\theta}}{k^n} \right|^q d\theta \right\}^{\frac{1}{q^r}}, \tag{1.10} \]

where \( m = \min_{|z|=k} |P(z)| \) and

\[
A_\mu = \frac{n \left( |a_n| - \frac{m}{k^n} \right) k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n \left( |a_n| - \frac{m}{k^n} \right) k^{\mu-1} + \mu|a_{n-\mu}|}. \tag{1.11} \]

If we divide two sides of inequality (1.10) by \(|\alpha|\) and then letting \(|\alpha| \to \infty\), we obtain the following generalization of inequality (1.3).

**Corollary 1.2.** If \( P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu \leq n, \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq k\), where \( k \leq 1\), then for every real or complex number \( \lambda \) with \(|\lambda| \leq 1\) and for each \( r > 0, p > 1, q > 1 \) with \( p^{-1} + q^{-1} = 1\),

\[
n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m n e^{i(n-1)\theta}}{k^n} \right|^p d\theta \right\}^{\frac{1}{p^r}} \leq \left\{ \int_0^{2\pi} \left| 1 + A_\mu e^{i\theta} \right|^{pr} d\theta \right\}^{\frac{1}{pr}} \times \left\{ \int_0^{2\pi} \left| P'(e^{i\theta}) + \frac{\lambda m n e^{i(n-1)\theta}}{k^n} \right|^q d\theta \right\}^{\frac{1}{q^r}}, \tag{1.12} \]

where \( m = \min_{|z|=k} |P(z)| \) and \( A_\mu \) is given by (1.11).

**Remark 1.3.** If we take \( \lambda = 0 \) in (1.12), then we get a refinement of inequality (1.6).

Letting \( p \to \infty \), so that \( q \to 1 \) in (1.10), we obtain the following result.

**Corollary 1.4.** If \( P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu \leq n, \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq k\), where \( k \leq 1\), then for all real or complex numbers \( \alpha, \lambda \) with \(|\alpha| \geq A_\mu, |\beta| \leq 1\) and for each \( r > 0, \)

\[
n \left( |\alpha|-A_\mu \right) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m n e^{i(n-1)\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{p^r}} \leq (1 + A_\mu) \left\{ \int_0^{2\pi} \left| D_{\alpha} P(e^{i\theta}) + \frac{\lambda m n e^{i(n-1)\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{q^r}}, \tag{1.13} \]

where \( m = \min_{|z|=k} |P(z)| \) and \( A_\mu \) is given by (1.11).
By Lemma 2.4, we have $A_\mu \leq k^\mu$ and therefore, it can be verified that for $p > 1$ and $r > 0$,

$$\int_0^{2\pi} |1 + A_\mu e^{i\theta}|^{pr} d\theta \leq \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta. \tag{1.14}$$

In view of inequality (1.14), we get the following result.

**Corollary 1.5.** If $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for all real or complex numbers $\alpha, \lambda$ with $|\alpha| \geq A_\mu$, $|\lambda| \leq 1$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n(|\alpha| - A_\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{i\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{pr}} \leq \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}}, \tag{1.15}$$

where $m = \text{Min}_{|z|=1} |P(z)|$ and $A_\mu$ is given by (1.11).

### 2. Some Lemmas

For the proof of theorem 1.1, we need the following lemmas:

**Lemma 2.1.** If $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$ and $Q(z) = z^n P(1/z)$, then for $|z| = 1$

$$|Q'(z)| \leq S_\mu |P'(z)|, \tag{2.1}$$

where $S_\mu$ is given by (1.9).

The above lemma is due to Aziz and Rather [5].

**Lemma 2.2.** If $P(z) = \sum_{j=1}^n a_j z^j$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$ and $m = \text{min}_{|z|=k} |P(z)|$, then

$$|a_n| > \frac{m}{k^n}. \tag{2.2}$$
Proof. By hypothesis all the zeros of $P(z)$ lie in $|z| \leq k$. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result holds trivially. So, we assume $P(z)$ has all its zeros in $|z| < k$, $k \leq 1$, so that $m > 0$. If $Q(z) = z^n P(1/z)$, then $Q(z)$ has no zero in $|z| \leq 1/k$. By Minimum Modulus Principle

$$|Q(z)| \geq \min_{|z| = 1/k} |Q(z)|$$

for $|z| \leq 1/k$, where $1/k \geq 1$.

which in particular gives,

$$|a_n| = |Q(0)| > \frac{1}{k^n \min_{|z| = k} |P(z)|}.$$

This completes the proof of Lemma 2.2.

\[ \Box \]

**Lemma 2.3.** The function

$$S_\mu(x) = \frac{n x^k \mu + |a_n - \mu| k^{\mu - 1}}{n x k^{\mu - 1} + |a_n - \mu|},$$

where $k \leq 1$ and $\mu \geq 1$, is a non-increasing function of $x$.

**Proof.** The proof follows by considering the first derivative test for $S_\mu(x)$.

We also need following Lemma.

**Lemma 2.4 ([7]).** If $P(z) = a_n z^n + \sum_{j=1}^n a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$, then

$$A_\mu \leq k^\mu,$$

where $A_\mu$ is defined in (1.11).

**3. Proof of Theorem**

**Proof of Theorem 1.1.** Let $m = \min_{|z| = k} |P(z)|$. By hypothesis $P(z) = a_n z^n + \sum_{j=1}^n a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, has all its zeros in $|z| \leq k$, where $k \leq 1$. We show $F(z) = P(z) + \frac{\lambda m z^n}{k^n}$ has all its zeros in $|z| \leq k$ for every $\lambda$ with $|\lambda| \leq 1$. This is obvious if $m = 0$ that is, if $P(z)$ has a zero on $|z| = k$. Hence, we suppose that all the zeros of $P(z)$ lie in $|z| < k$, $k \leq 1$, so that $m > 0$. Now $m \leq |P(z)|$ for $|z| = k$, therefore if $\lambda$ is any complex number such that $|\lambda| < 1$, then

$$\left| \frac{m \lambda z^n}{k^n} \right| < |P(z)|$$

for $|z| = k$. 


Since all the zeros of $P(z)$ lie in $|z| < k$, it follows by Rouche’s theorem that all the zeros of

$$F(z) = P(z) + \frac{m\lambda z^n}{k^n} = \left(a_n + \frac{\lambda m}{k^n}\right) z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$$

also lie in $|z| < k$, $k \leq 1$.

Let $G(z) = z^n F(1/z) = z^n P(1/z) + \frac{m\lambda}{k^n}$ then it can be easily verified for $|z| = 1$,

$$|F'(z)| = |nG(z) - zG'(z)| \quad \text{and} \quad |G'(z)| = |nF(z) - zF'(z)|. \quad (3.1)$$

Applying Lemma 2.1 to the polynomial $F(z)$, we get for $|z| = 1$,

$$S'_\mu |F'(z)| \geq |G'(z)|, \quad (3.2)$$

where

$$S'_\mu = n \frac{|a_n + \frac{m\lambda}{k^n}| k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n |a_n + \frac{m\lambda}{k^n}| k^{\mu-1} + \mu |a_{n-\mu}|}. \quad (3.3)$$

Since for every $\lambda$ with $|\lambda| < 1$, we have by Lemma 2.2,

$$|a_n + \frac{m\lambda}{k^n}| \geq |a_n| - \frac{m|\lambda|}{k^n} \geq |a_n| - \frac{m}{k^n}, \quad (3.4)$$

therefore it follows from Lemma 2.3, (3.3) and (3.4) that for every $\lambda$ with $|\lambda| < 1$,

$$S'_\mu = n \frac{|a_n + \frac{m\lambda}{k^n}| k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n |a_n + \frac{m\lambda}{k^n}| k^{\mu-1} + \mu |a_{n-\mu}|} \leq n \frac{(|a_n| - \frac{m}{k^n}) k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n (|a_n| - \frac{m}{k^n}) k^{\mu-1} + \mu |a_{n-\mu}|} = A_\mu \quad (3.5)$$

Using (3.1) and (3.5) in (3.2), we obtain

$$A_\mu |F'(z)| \geq |nF(z) - zF'(z)| \quad \text{for} \quad |z| = 1. \quad (3.6)$$

This gives for every real or complex number $\alpha$ with $|\alpha| \geq A_\mu$, we have

$$|D_\alpha F(z)| = |nF(z) + (\alpha - z)F'(z)| \geq |\alpha||F'(z)| - |nF(z) - zF'(z)| \geq (|\alpha| - A_\mu)|F'(z)| \quad \text{for} \quad |z| = 1. \quad (3.7)$$
Replacing $F(z)$ by $P(z) + \frac{\lambda mnz^{n-1}}{k^n}$, we obtain for $|z| = 1$,

$$\left| D_\alpha P(z) + \frac{\lambda mnz^{n-1}}{k^n} \right| \geq (|\alpha| - A_\mu) \left| P'(z) + \frac{\lambda mnz^{n-1}}{k^n} \right|. \quad (3.8)$$

Since $F(z)$ has all its zeros in $|z| \leq k \leq 1$, it follows by Gauss-Lucas theorem that all the zeros of $F'(z)$ also lie in $|z| \leq k \leq 1$. This shows that the polynomial

$$z^{n-1} F'(1/z) \equiv nG(z) - zG'(z)$$

has all its zeros in $|z| \geq (1/k) \geq 1$. Therefore, it follows from (3.6) that the function

$$w(z) = \frac{zG'(z)}{A_\mu(nG(z) - zG'(z))}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + A_\mu w(z)$ is subordinate to the function $1 + A_\mu z$ for $|z| \leq 1$. Hence by a well known property of subordination \[8\], we have for each $r > 0$,

$$\int_0^{2\pi} \left| 1 + A_\mu w(e^{i\theta}) \right|^r d\theta \leq \int_0^{2\pi} \left| 1 + A_\mu e^{i\theta} \right|^r d\theta. \quad (3.9)$$

Now

$$1 + A_\mu w(z) = \frac{nG(z)}{nG(z) - zG'(z)},$$

and

$$|F'(z)| = |z^{n-1} F'(1/z)| = |nG(z) - zG'(z)| \quad \text{for} \quad |z| = 1,$$

therefore for $|z| = 1$,

$$n |G(z)| = |1 + A_\mu w(z)||nG(z) - zG'(z)| = |1 + A_\mu w(z)||F'(z)|.$$

Equivalently for $|z| = 1$:

$$n \left| z^n P(1/z) + \frac{\bar{\lambda} m}{k^n} \right| = \left| 1 + A_\mu w(z) \right| \left| P'(z) + \frac{\lambda mnz^{n-1}}{k^n} \right|,$$

which implies for $|z| = 1$,

$$n \left| P(z) + \frac{\lambda m z^n}{k^n} \right| = \left| 1 + A_\mu w(z) \right| \left| P'(z) + \frac{\lambda mnz^{n-1}}{k^n} \right|. \quad (3.10)$$
From (3.8) and (3.10), we deduce that for each $r > 0$,

$$n^r(|\alpha| - A_\mu)^r \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{i\theta} n}{k^n} \right|^r d\theta \leq \int_0^{2\pi} \left| 1 + A_\mu w(e^{i\theta}) \right|^r \left| D_\alpha P(e^{i\theta}) + \frac{\lambda m n e^{i(n-1)\theta}}{k^n} \right|^r d\theta.$$ 

This gives with the help of (3.9) and using Hölder’s inequality for $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n^r(|\alpha| - A_\mu)^r \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{i\theta} n}{k^n} \right|^r d\theta \leq \left( \int_0^{2\pi} |1 + A_\mu e^{i\theta}|^{pr} d\theta \right)^{1/p} \times \left( \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + \frac{\lambda m n e^{i(n-1)\theta}}{k^n} \right|^{qr} d\theta \right)^{1/q},$$

equivalently

$$n(|\alpha| - A_\mu)^r \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\lambda m e^{i\theta} n}{k^n} \right|^r d\theta \right\}^{r/p} \leq \left\{ \int_0^{2\pi} |1 + A_\mu e^{i\theta}|^{pr} d\theta \right\}^{1/p} \times \left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + \frac{\lambda m n e^{i(n-1)\theta}}{k^n} \right|^{qr} d\theta \right\}^{1/q}.$$ 

This completes the proof of theorem 1.1. \hfill \Box

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References


