International Journal of Pure and Applied Mathematics
Volume 89 No. 1 2013, 19-27
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v89i1.3

ijpam.eu

GRACEFUL LABELING OF HANGING THETA GRAPHS

T.N. Janakiraman[1], G. Sathiamoorthy[2 §]

[1]Department of Mathematics
National Institute of Technology
Tiruchirappalli, Tamilnadu, INDIA
[2]School of Humanities & Sciences
SASTRA University
Thanjavur, Tamilnadu, INDIA

Abstract: In this paper, we introduce a new class of hanging theta graphs and prove that those graphs admit graceful labeling.

AMS Subject Classification: 05C78
Key Words: graceful labeling, theta graph

1. Introduction

In this paper, G is a simple undirected graph with vertex set V (G) and edge set E (G) contains p vertices and q edges respectively. A function ϕ is called graceful labeling of graph G with q edges if ϕ is an injunction mapping from ϕ: V (G) $\rightarrow \{0, 1, 2, 3,..., q\}$and ϕ_1: E(G) $\rightarrow \{1, 2, \ldots, q\}$ such that ϕ_1(uv) is the difference between cardinality of ϕ(u) and ϕ(v). The resulting edge labels are distinct. A graph that admits graceful labeling is called graceful graph.

The concept of graceful labeling was introduced Rosa [5] with the name β valuation Golomb [3] given the name 'graceful label'. Gallian [2] given the

Received: April 9, 2013

§Correspondence author

extensive survey of contributions to graceful labeling of variety of graphs. The notation and terminology used in this paper are taken from [2]. There are many works relating to graceful labeling of trees, which are given in [2]. Besides tree graphs there are many other graphs, which are not trees that admit graceful labeling, are also given in [2]. The above contributions motivated us to give a graceful labeling of new graphs called "Hanging of Theta Graphs".

In[1], Barrientos has given graceful arbitrary super subdivisions of graphs and in our paper[4], we have given a graceful labeling for a special class of generalized fan graph. In different context the theta graphs were given by Shee and Ho [6, 7], Shiu and Kwong [8], Kwong, and Lee [9], Swaminathan and Jeyanthi [10].

2. Main Result

Definition 1. The theta graph Θ(1, 2,..., m+1) is the graph consisting of (m+1) internally disjoint paths of length 2 which have two common end points..

Select degree two vertex one from each of the "c" identical copies of theta graph and merge them to a point "t". Join "t" with another point "b" by an edge and call the resulting structure as Θ^* graph.

Here after we term the point "t" as supporting point and "b" as base point.

Now, consider 'n' copies of Θ^* graph say Θ^*(1), Θ^*(2),..., Θ^*(n) with base points b_1, b_2,..., b_n forming a path P_n(a path on n vertices). The resulting graph is called a hanging theta graph and is denoted by hΘ(n, c, m, 2).

The two common end vertices of j^{th} theta graph in Θ^*(i) are denoted by u(i, j, 1) and u(i, j, 2). The m two degree vertices in j^{th} theta graph in Θ^*(i) are denoted by v(i, j, 1), v(i, j, 2),..., v(i, j, m).

Let "e" denotes the number of edges in any Θ^* graph, and e = $c(2m+2)+1$. Clearly q = ne+ n – 1 or n(e+1) -1 denotes the total number of edges in hΘ(n, c, m, 2).

Theorem 2. *hΘ(n, c, m, 2) is graceful.*

Proof. **Section 1: c is odd.**

First, we label the vertices of $\Theta^{\square}\mathbf{(1)}$ as follows.

Step 1: $\phi(b_1) = 0$; $\phi(t_1) = q$.

Step 2: ϕ(u(1,i,k)) $=(2m+2)(i-1)+k$, $1 \leq i \leq c$, $1 \leq k \leq 2$.

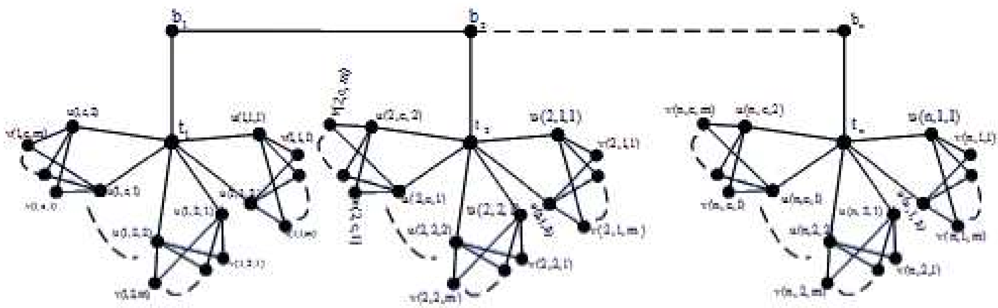

Figure 1: The general form of hΘ(n, c, m, 2)

Step 3:

$$\phi(v(1,i,k)) = \left\{ \begin{array}{c} q-2+(2m+2)(1-i)-2(j-1), \\ where\ 1 \leq\ i < \frac{c+3}{2}, 1 \leq\ j \leq\ m. \\ q-2-(2m+2)\frac{c-1}{2}-(2m+1)-2(j-1), \\ for i = \frac{c+3}{2}, 1 \leq\ j \leq\ m. \\ q-4-(2m-1)-(2m+2)(i-2)-2(j-1), \\ where \frac{c+5}{2} \leq\ i \leq\ c, 1 \leq\ j \leq\ m. \end{array} \right\}$$

Now, we label the vertices of $\Theta^{\square}$ (2) as follows.

Step 1: $\phi(t_2)$ = e, $\phi(b_2) = q - \phi(t_2)$.

Step 2: ϕ(u(2, i, k)) = $\phi(b_2) + (2m+2)(i-1)+k$, 1 ≤ i ≤ c, 1 ≤ k ≤ 2.

Step 3:

$$\phi(v(2,i,k)) = \left\{ \begin{array}{c} \phi(t_2)-2+(2m+2)(1-i)-2(j-1), \\ where\ 1 \leq\ i < \frac{c+3}{2}, 1 \leq\ j \leq\ m. \\ \phi(t_2)-2-(2m+2)(\frac{c-1}{2})-(2m+1)-2(j-1), \\ for i = \frac{c+3}{2}, 1 \leq\ j \leq\ m. \\ \phi(t_2)-4-(2m-1)-(2m+2)(i-2)-2(j-1), \\ where \frac{c+5}{2} \leq\ i \leq\ c, 1 \leq\ j \leq\ m. \end{array} \right\}$$

Now, we label the odd branches of $\Theta^{\square}$**(2d+1)** as follows

Step 1: $\phi(b_{2d+1}) = \phi(b_{2d-1})$ + e +1, $\phi(t_{2d+1})$ = q − $\phi(b_{2d+1})$, 1 ≤ d ≤ $\frac{n-1}{2}$.

Step 2: ϕ(u(2d+1, i, k)) = $\phi(b_{2d+1}) + (2m+2)(i-1)+k$, 1 ≤ i ≤ c, 1 ≤ k ≤ 2, 1 ≤ d ≤ $\frac{n-1}{2}$.

Step 3:

$$\phi(v(2d+1,i,j)) = \left\{ \begin{array}{c} \phi(t_{2d+1}) - 2 + (2m+2)(1-i) - 2(j-1), \\ where\ 1 \leq\ i < \frac{c+3}{2}, 1 \leq\ j \leq\ m, 1 \leq\ p \leq \frac{n-1}{2}. \\ \phi(t_{2d+1}) - 2 - (2m+2)(\frac{c-1}{2}) - (2m+1) - 2(j-1), \\ for\ i = \frac{c+3}{2}, 1 \leq\ j \leq\ m. \\ \phi(t_{2d+1}) - 4 - (2m-1) - (2m+2)(i-2) - 2(j-1), \\ where \frac{c+5}{2} \leq\ i \leq\ c, 1 \leq\ j \leq\ m, 1 \leq\ d \leq \frac{n-1}{2}. \end{array} \right\}$$

Now, label even branches of $\Theta^{\square}$**(2d+2)** as follows.

Step 1: $\phi(\mathrm{t}_{2d+2}) = \phi(\mathrm{t}_{2d}) + e + 1$, $\phi(\mathrm{b}_{2d+2}) = q - \phi(t_{2d+2})$, $1 \leq \mathrm{d} \leq \frac{n-2}{2}$.

Step 2: $\phi(\mathrm{u}(2\mathrm{d}+2, \mathrm{i}, \mathrm{k})) = \phi(\mathrm{b}_{2d+2}) + (2m+2)(i-1) + k$, $1 \leq \mathrm{i} \leq \mathrm{c}$, $1 \leq \mathrm{k} \leq 2$, $1 \leq \mathrm{d} \leq \frac{n-2}{2}$.

Step 3:

$$\phi(v(2d+2,i,j)) = \left\{ \begin{array}{c} (2m+2)(1-i) + \phi(t_{2d+2}) - 2 - 2(j-1), \\ where\ 1 \leq\ i\ < \frac{c+3}{2}, 1 \leq\ j \leq\ m, 2 \leq\ d \leq \frac{n-2}{2}. \\ \phi(t_{2d+2}) - 2 - (2m+2)(\frac{c-1}{2}) - (2m+1) - 2(j-1), \\ fori = \frac{c+3}{2}, 1 \leq\ j \leq\ m. \\ \phi(t_{2d+2}) - 4 - (2m-1) - (2m+2)(i-2) - 2(j-1), \\ where \frac{c+5}{2} \leq\ i \leq\ c, 1 \leq\ j \leq\ m, 2 \leq\ d \leq \frac{n-2}{2}. \end{array} \right\}$$

Now, we obtain edge labeling by assigned vertex values earlier, as follows.

Step 1: $\phi_1(b_1 t_1) = \mathrm{q}$.

Step 2: $\phi_1(b_{2d+1} t_{2d+1}) = \phi_1(b_{2d-1} t_{2d-1}) - 2(e+1)$, for $1 \leq \mathrm{d} \leq \frac{n-1}{2}$.

Step 3: $\phi_1(b_2 t_2) = \mathrm{q} - 2e$.

Step 4: $\phi_1(b_{2d+2} t_{2d+2}) = \phi_1(b_{2d} t_{2d}) - 2(e+1)$, $1 \leq \mathrm{d} \leq \frac{n-2}{2}$.

Step 5: $\phi_1(t_1 u(1,i,k) = (q-1) + (2m+2)(1-i) - (k-1)$, $1 \leq \mathrm{k} \leq 2$, $1 \leq \mathrm{i} \leq \mathrm{c}$.

Step 6: $\phi_1(t_{2d+1} u(2d+1,i,k) = \phi_1(t_{2d-1} u(2d-1,1,1)) - 2(e+1) + (2m+2)(1-i) - (k-1)$, $1 \leq \mathrm{k} \leq 2$, $1 \leq \mathrm{i} \leq \mathrm{c}$, $1 \leq \mathrm{d} \leq \frac{n-1}{2}$.

Step 7: $\phi_1(t_2 u(2,i,k)) = \mathrm{q} - 2e + 1 + (2m+2)(i-1) + (k-1)$, $1 \leq \mathrm{k} \leq 2$, $1 \leq \mathrm{i} \leq \mathrm{c}$.

Step 8: $\phi_1(t_{2d+2} u(2d+2,i,k)) = \phi_1(t_{2d} u(2d,1,1)) - 2(e+1) + (2m+2)(i-1) + (k-1)$, $1 \leq \mathrm{k} \leq 2$, $1 \leq \mathrm{i} \leq \mathrm{c}$, $1 \leq \mathrm{d} \leq \frac{n-2}{2}$.

Let E_s be the edge set of the sub graph $(\Theta^{\square}(s) - b_s)$ and $\phi_1(E_s)$ is common for odd and even.

c is odd.

Step 9:

$$\phi_1(E_1) = \left\{ \begin{array}{c} q-3-4(m+1)(i-1)-(k-1)-2(j-1), \\ where\ 1 \le\ i \le \frac{c+1}{2}, 1 \le\ j \le\ m, 1 \le\ k \le\ 2 \\ q-2-4(m+1)(i-1)-(2(j-1)+(k-1)), \\ for\ i = \frac{c+3}{2},\ 1 \le\ j \le\ m, 1 \le\ k \le\ 2 \\ q-2-4(m+1)(i-(\frac{c-3}{2}))-(k-1)-2(j-1), \\ where\ \frac{c+5}{2} \le\ i \le\ c, 1 \le\ j \le\ m, 1 \le\ k \le\ 2 \end{array} \right\}$$

Step 10:

$$\phi_1(E_2) = \left\{ \begin{array}{c} q-2e+3+4(m+1)(i-1)+(k-1)+2(j-1) \\ where\ 1 \le\ i \le\ \frac{c+1}{2}, 1 \le\ j \le\ m, 1 \le\ k \le\ 2 \\ q+(2m+2)-(e-1)+(2(j-1)+(k-1)), \\ for\ i = \frac{c+3}{2}, 1 \le\ j \le\ m, 1 \le\ k \le\ 2 \\ q-4(m+1)(i-(\frac{c+3}{2}))+2(j-1)+(k-1), \\ where \frac{c+5}{2} \le\ i \le\ c, 1 \le\ j \le\ m, 1 \le\ k \le\ 2 \end{array} \right\}$$

edge values in odd branches.

Step 11:

$$\phi_1(E_{2d+1}) = \left\{ \begin{array}{c} q-3-2d(e+1)-4(m+1)(i-1)-(k-1)-2(j-1), \\ 1 \le\ i \le\ \frac{c+1}{2}, 1 \le\ j \le\ m, 1 \le\ k \le\ 2, 1 \le\ d \le\ \frac{n-1}{2} \\ q-2-2d(e+1)-4(m+1)(i-1)-(k-1)-2(j-1), \\ fori = \frac{c+3}{2}, 1 \le\ j \le\ m, 1 \le\ k \le\ 2, 1 \le\ d \le\ \frac{n-1}{2}. \\ q-2-2d(e+1)-4(m+1)(i-(\frac{c-3}{2}))-(k-1)-2(j-1), \\ \frac{c+5}{2} \le\ i \le\ c, 1 \le\ j \le\ m, 1 \le\ k \le\ 2, 1 \le\ d \le\ \frac{n-1}{2}. \end{array} \right\}$$

edge values of even branches.

Step 12:

$$\phi_1(E_{2d+2}) = \left\{ \begin{array}{c} q+5-((2d+2)(e+1))+4(m+1)(i-1)+(k-1)+ \\ 2(j-1), for 1 \le\ i \le\ \frac{c+1}{2}, 1 \le\ j \le\ m, 1 \le\ k \le\ 2, \\ 1 \le\ p \le\ \frac{n-2}{2}. \\ q+2-((2d+1)(e+1))+(2m+2)+(k-1)+2(j-1), \\ for \imath = \frac{c+3}{2}, 1 \le\ j \le\ m, 1 \le\ k \le\ 2, \\ 1 \le\ d \le\ \frac{n-2}{2}. \\ q+2-((2d+1)(e+1))+6(m+1)(i-(\frac{c+3}{2}))+(k-1) \\ +2(j-1), for\ \frac{c+5}{2}\ \le\ i \le\ c, 1 \le\ j \le\ m, 1 \le\ k \le\ 2, \\ 1 \le\ d \le\ \frac{n-2}{2}. \end{array} \right\}$$

c is even.

First we label $\Theta^{\square}(1)$ as follows.
Step 1: $\phi(\mathrm{b}_1) = 0$; $\phi(\mathrm{t}_1) = \mathrm{q}$.
Step 2: $\phi(\mathrm{u}(1, \mathrm{i}, \mathrm{k})) = (2m+2)(i-1)+k$, $1 \le \mathrm{i} \le \mathrm{c}$, $1 \le \mathrm{k} \le 2$.
Step 3:

$$\phi(v(1,i,k)) = \left\{ \begin{array}{c} q-2+(2m+2)(1-i)-2(j-1), \\ where\ 1 \le\ i <\ \frac{c}{2}+1, 1 \le\ j \le\ m. \\ q-2+(2m+2)(1-\frac{c}{2})-(2m+1)2(j-1), \\ for\ i = \frac{c}{2}+1, 1 \le\ j \le\ m. \\ q-4-(2m+2)(i-2)-(2m-1)-2(j-1), \\ where\ \frac{c}{2}+2 \le\ i \le\ c, 1 \le\ j \le\ m. \end{array} \right\}$$

Now, we label the vertices of $\Theta^{\square}\mathbf{(2)}$ as follows.
Step 1: $\phi(\mathrm{t}_2) = \mathrm{e}$, $\phi(\mathrm{b}_2) = q - \phi(t_2)$.
Step 2: $\phi(\mathrm{u}(2, \mathrm{i}, \mathrm{k})) = \phi(b_2) + (2m+2)(i-1)+k$, $1 \le \mathrm{i} \le \mathrm{c}$, $1 \le \mathrm{k} \le 2$.
Step 3:

$$\phi(v(2,i,k)) = \left\{ \begin{array}{c} \phi(t_2)-2+(2m+2)(1-i)-2(j-1), \\ where\ 1 \le\ i\ <\ \frac{c}{2}+1, 1 \le\ j\ \le\ m. \\ \phi(t_2)-2+(2m+2)(1-\frac{c}{2})-(2m+1)-2(j-1), \\ for\ i = \frac{c}{2}+1, 1 \le\ j \le\ m. \\ \phi(t_2)-4-(2m-1)-2(j-1)-(2m+2)(i-2), \\ where\ \frac{c}{2}+2 \le\ i \le\ c, 1 \le\ j \le\ m. \end{array} \right\}$$

Now, we label the odd branches of $\Theta^{\square}\mathbf{(2d+1)}$ as follows.
Step 1: $\phi(\mathrm{b}_{2d+1}) = \phi(\mathrm{b}_{2d-1}) + \mathrm{e} + 1$, $\phi(\mathrm{t}_{2d+1}) = \mathrm{q} - \phi(\mathrm{b}_{2d+1})$, $1 \le \mathrm{d} \le \frac{n-1}{2}$.
Step 2: $\phi(\mathrm{u}(2\mathrm{d}+1, \mathrm{i}, \mathrm{k})) = \phi(\mathrm{b}_{2d+1}) + (2m+2)(i-1)+k$, $1 \le \mathrm{i} \le \mathrm{c}$, $1 \le \mathrm{k} \le 2$, $1 \le \mathrm{d} \le \frac{n-1}{2}$.
Step 3:

$$\phi(v(2d{+}1,i,j)) = \left\{ \begin{array}{c} (2m+2)(1-i)+\phi(t_{2d+1})-2-2(j-1), \\ where\ 1 \le\ i\ <\ \frac{c}{2}+1, 1 \le\ j \le\ m, 1 \le\ d \le\ \frac{n-1}{2}. \\ \phi(t_{2d+1})-(2m+2)(\frac{c}{2}-1)-2-2(j-1)-(2m+1), \\ for\ i = \frac{c}{2}+1, 1 \le\ j \le\ m. \\ \phi(t_{2d+1})-4-2(j-1)-(2m-1)-(2m+2)(i-2), \\ where\ \frac{c}{2}+2 \le\ i \le\ c, 1\ \le\ j\ \le\ m, 1 \le\ d \le\ \frac{n-1}{2}. \end{array} \right\}$$

Now, label even branches of $\Theta^{\square}\mathbf{(2d+2)}$ as follows.
Step 1: $\phi(\mathrm{t}_{2d+2}) = \phi(\mathrm{t}_{2d}) + e + 1$, $\phi(\mathrm{b}_{2d+2}) = q - \phi(t_{2d+2})$, $1 \le \mathrm{d} \le \frac{n-2}{2}$.
Step 2: $\phi(\mathrm{u}(2\mathrm{d}+2, \mathrm{i}, \mathrm{k})) = \phi(b_{2d+2}) + (2m+2)(i-1)+k$, $1 \le \mathrm{i} \le \mathrm{c}$, $1 \le \mathrm{k} \le 2$, $1 \le \mathrm{d} \le \frac{n-2}{2}$.

Step 3:

$$\phi(v(2d{+}2,i,j)) = \begin{cases} (2m+2)(1-i)+\phi(t_{2d+2})-2-2(j-1), \\ where\ 1 \le i < \frac{c}{2}+1, 1 \le j \le m, 2 \le d \le \frac{n-2}{2}. \\ \phi(t_{2d+2})-(2m+2)(\frac{c}{2}-1)-2-2(j-1)-(2m+1), \\ for\ i=\frac{c}{2}+1,\ 1 \le j \le m. \\ \phi(t_{2d+2})-4-2(j-1)-(2m-1)-(2m+2)(i-2), \\ where \frac{c}{2}+2 \le i \le ,1 \le j \le m, 2 \le d \le \frac{n-2}{2}. \end{cases}$$

The resultant edge label of **c is even** as follows.
(Step 1 to Step 8 is common for c is odd or even, which is already given.)
Step 9:

$$\phi_1(E_1) = \begin{cases} q-3-4(m+1)(i-1)-(k-1)-2(j-1), \\ where\ 1 \le i < \frac{c}{2}+1, 1 \le j \le m, 1 \le k \le 2 \\ q-2-4(m+1)(i-1)-(2(j-1)+(k-1)), \\ for\ i=\frac{c}{2}+1,\ 1 \le j \le m, 1 \le k \le 2 \\ q-2-4(m+1)(i-(\frac{c}{2}-1))-(k-1)-2(j-1), \\ where\ \frac{c}{2}+2 \le i \le c, 1 \le j \le m, 1 \le k \le 2 \end{cases}$$

Step 10:

$$\phi_1(E_2) = \begin{cases} q-2e+3+4(m+1)(i-1)+(k-1)+2(j-1) \\ where\ 1 \le i \le \frac{c}{2}+1, 1 \le j \le m, 1 \le k \le 2 \\ q-8(m+1)+(2(j-1)+(k-1)), \\ for\ i=\frac{c}{2}+1,\ 1 \le j \le m, 1 \le k \le 2 \\ q-4(m+1)(i-(\frac{c}{2}+1))+2(j-1)+(k-1), \\ where\ \frac{c}{2}+2 \le i \le c, 1 \le j \le m, 1 \le k \le 2 \end{cases}$$

edge values in odd branches.

Step 11:

$$\phi_1(E_{2d+1}) = \begin{cases} q-3-2d(e+1)-4(m+1)(i-1)-(k-1)-2(j-1), \\ for 1 \le i \le \frac{c}{2}+1,\ 1 \le j \le m, 1 \le k \le 2, 1 \le d \le \frac{n-1}{2}. \\ q-2-2d(e+1)-8(m+1)-(k-1)-2(j-1), \\ for\ i=\frac{c}{2}+1, 1 \le j \le m, 1 \le k \le 2, \\ 1 \le d \le \frac{n-1}{2}. \\ q-2-2d(e+1)-4(m+1)(i-(\frac{c}{2}-1))-(k-1)- \\ 2(j-1), for \frac{c}{2}+2 \le i \le c, 1 \le j \le m, \\ 1 \le k \le 2, 1 \le d \le \frac{n-1}{2}. \end{cases}$$

edge values of even branches.

Step 12:

$$\phi_1(E_{2d+2}) = \left\{ \begin{array}{c} q+1-(2d+2)e+4(m+1)(i-1)+(k-1)+ \\ 2(j-1), for 1 \leq\ i \leq\ \frac{c}{2}+1, 1 \leq\ j \leq\ m, \\ 1 \leq\ k \leq\ 2, 1 \leq\ d \leq\ \frac{n-2}{2}. \\ q+2-((2d+1)(e+1))+(k-1)+2(j-1), for \\ i=\frac{c}{2}+1, 1 \leq\ j \leq\ m, 1 \leq\ k \leq\ 2, \\ 1 \leq\ d \leq\ \frac{n-2}{2}. \\ q+2-((2d+1)(e+1))+4(m+1)(i-(\frac{c}{2}+1))+(k-1)+ \\ 2(j-1), for fracc2+2 \leq\ i \leq\ c, 1 \leq\ j \leq\ m, \\ 1 \leq\ k \leq\ 2, 1 \leq\ d \leq\ \frac{n-2}{2}. \end{array} \right\}$$

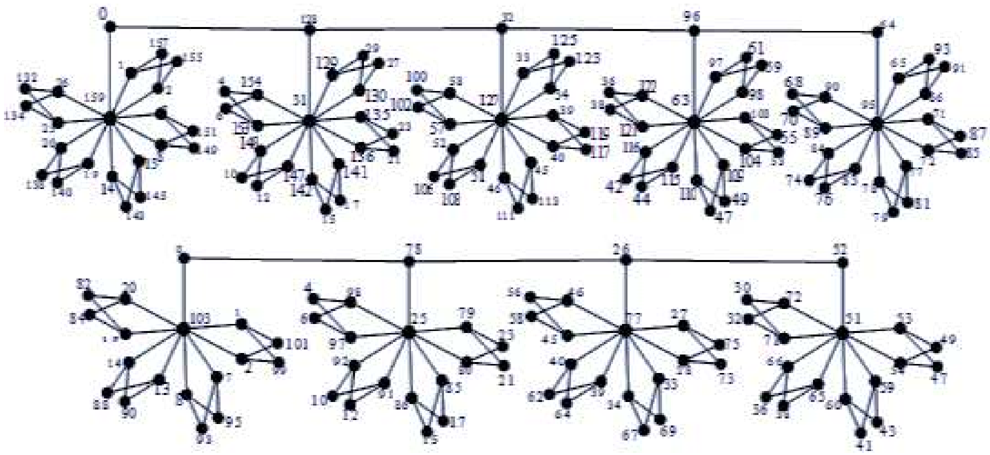

Figure 2: The example for h Θ(5, 5, 2, 2) and hΘ (4, 4, 2, 2) are given above

Some useful relations between the vertices of different branches for easy understanding are given below.

For *odd branches*:

1. $u((2d+1), i, k) = u(2d-1, i, k) + e + 1$, 1 ≤ d ≤ $\frac{n-1}{2}$, 1 ≤ i ≤ c, 1 ≤ k ≤ 2.

2. $v(2d+1, i, j) = v(2d-1, i, j) - (e+1)$, 1 ≤ d ≤ $\frac{n-1}{2}$, 1 ≤ i ≤ c, 1 ≤ k ≤ 2.

For *even branches*:

1. $u((2d+2), i, k) = u(2d, i, k) - (e+1)$, 1 ≤ d ≤ $\frac{n-2}{2}$, 1 ≤ i ≤ c, 1 ≤ k ≤ 2.

2. $v(2d+2, i, j) = v(2d, i, j) + (e+1)$, 1 ≤ d ≤ $\frac{n-2}{2}$, 1 ≤ i ≤ c, 1 ≤ k ≤ 2.

Hence we conclude the graph h Θ(n, c, m, 2) is graceful.

3. Conclusion

In this paper, we have proved the graceful labeling of new class of graphs called Hanging Theta graphs.

□

References

[1] C. Barrientos, Graceful arbitrary super subdivisions of graphs, *Indian J. pure Appl. Math.*, **38** (2007), 445-450.

[2] J.A. Gallian, A dynamic survey of graceful labeling, *The Electronic Journal of Combinatories*, Fifteenth Edition, **26** (2012).

[3] S.W. Golomb, How to number a graph, In: *Graph Theory and Computing* (Ed. R.C. Read), *Academic Press*, New York (1972), 23-37.

[4] T.N. Janakiraman, G. Sathiamoorthy, Graceful labeling of a special class of fan graph, *IJESACBT*, **2**, No. 2 (Apr-June 2011), 104-109.

[5] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs*, Internat. Symposium, Rome 1966), Gordon and Breach, N.Y. and Dunod Paris (1967), 349-355.

[6] S.C. Shee, Y.S. Ho, The cordiality of one-point union of n-copies of a graph, *Discrete Math.*, **117** (1993), 225-243.

[7] S.C. Shee, Y.S. Ho, The cordiality of the path-union of n copies of a graph, *Discrete Math.*, **151** (1996), 221-229.

[8] W.C. Shiu, H. Kwong, An algebraic approach for finding balance index sets, *Australasian J. Combin.*, **45** (2009), 139-155.

[9] H. Kwong, S.M. Lee, On edge-balance index sets of generalized theta graphs, *Cong.Numer.*, **198** (2009), 157-168.

[10] V. Swaminathan, P. Jeyanthi, Super edge-magic strength of generalized theta graph, *Inter. J. Management and Systems*, **22** (2006), 203-220.