IMPROVED BOUNDS ON POISSON APPROXIMATION FOR INDEPENDENT BERNOULLI RANDOM SUMMANDS

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Abstract: Bounds for the distance between the distribution function of random sums of independent Bernoulli random variables and an appropriate Poisson distribution function are improved. The bounds in this study are more appropriate for measuring the accuracy of Poisson approximation.

AMS Subject Classification: 62E17, 60F05, 60G05

Key Words: Bernoulli random variable, distribution function, Poisson approximation, random sums

1. Introduction

Let $X_1, X_2, \dots$ be a sequence of independent Bernoulli random variables, each with probability of success $p_i = P(X_i = 1) = 1 - P(X_i = 0)$, and let N be a non-negative integer-valued random variable and independent of the X_i 's. Let us consider the sum $S_N = \sum_{i=1}^N X_i$, which is called *random sums*. For the special case, $N = n \in \mathbb{N}$ is fixed, there has been some research on topics related to the approximation of the distribution of the sum S_n by an appropriate Poisson distribution, which can be found in [1], [3], [4]–[7], etc. Let $\lambda_N = \sum_{i=1}^N p_i$, $\lambda = E(\lambda_N)$ and U_λ a Poisson random variable with mean λ . Yannaros [8] gave

a uniform bound for the distance between the distribution functions of S_N and U_λ as follows:

$$\sup_{x \in \mathbb{N}} |P(S_N \leq x) - P(U_\lambda \leq x)| \leq E|\lambda_N - \lambda| + E \left(\frac{1 - e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^N p_i^2 \right), \quad (1.1)$$

and Kongudomthrap and Chaidee [2] gave a non-uniform bound in the form of

$$|P(S_N \leq x) - P(U_\lambda \leq x)| \leq \frac{3\lambda}{x} + E \left\{ \frac{1 - e^{-\lambda_N}}{\lambda_N} \min \left\{ 1, \frac{e^{\lambda_N}}{x+1} \right\} \sum_{i=1}^N p_i^2 \right\}, \quad (1.2)$$

where $x \in \mathbb{N}$ and $P(S_N = 0) = E \left[\prod_{i=1}^N (1 - p_i) \right]$.

In this study, we focus on improving the bounds in (1.1) and (1.2) to be more appropriate, which are in Section 2. Concluding remarks are presented in the last section.

2. Results

The following theorem presents non-uniform and uniform bounds for the distance between the distribution functions of S_N and U_λ , which are the desired results.

Theorem 2.1. *For $x \in \mathbb{N}$, then we have*

$$\begin{aligned} |P(S_N \leq x) - P(U_\lambda \leq x)| &\leq \min \left\{ \frac{2\lambda}{x+1}, \min \left\{ 1, \sqrt{\frac{2}{e\lambda}} \right\} E|\lambda_N - \lambda| \right\} \\ &+ \min \left\{ E \left(\frac{1 - e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^N p_i^2 \right), E \left(\frac{2(e^{\lambda_N} - \lambda_N - 1)}{\lambda_N^2(x+1)} \sum_{i=1}^N p_i^2 \right), \frac{1}{x} E \left(\sum_{i=1}^N p_i^2 \right) \right\} \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} \sup_{x \in \mathbb{N}} |P(S_N \leq x) - P(U_\lambda \leq x)| &\leq \min \left\{ \lambda, \min \left\{ 1, \sqrt{\frac{2}{e\lambda}} \right\} E|\lambda_N - \lambda| \right\} \\ &+ \min \left\{ E \left(\frac{1 - e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^N p_i^2 \right), E \left(\frac{e^{\lambda_N} - \lambda_N - 1}{\lambda_N^2} \sum_{i=1}^N p_i^2 \right) \right\}. \end{aligned} \quad (2.2)$$

Proof. Because (2.2) is directly obtained from (2.1), it suffices to show that (2.1) holds. It follows the fact that

$$\begin{aligned}
 |P(S_N \leq x) - P(U_\lambda \leq x)| &\leq \sum_{n=0}^{\infty} P(N = n) |P(S_n \leq x) - P(U_\lambda \leq x)| \\
 &\leq \sum_{n=0}^{\infty} P(N = n) |P(S_n \leq x) - P(U_{\lambda_n} \leq x)| \\
 &\quad + |P(U_{\lambda_N} \leq x) - P(U_\lambda \leq x)|. \tag{2.3}
 \end{aligned}$$

Applying the corresponding results in [5] and [1], we obtain

$$|P(S_n \leq x) - P(U_{\lambda_n} \leq x)| \leq \min \left(\frac{1 - e^{-\lambda_n}}{\lambda_n}, \frac{2(e^{\lambda_n} - \lambda_n - 1)}{\lambda_n^2(x+1)}, \frac{1}{x} \right) \sum_{i=1}^n p_i^2$$

and

$$|P(U_{\lambda_N} \leq x) - P(U_\lambda \leq x)| \leq \min \left\{ 1, \sqrt{\frac{2}{e\lambda}} \right\} E|\lambda_N - \lambda|,$$

respectively. In addition, we have that

$$\begin{aligned}
 |P(U_{\lambda_N} \leq x) - P(U_\lambda \leq x)| &\leq P(U_\lambda \geq x+1) + \sum_{n=0}^{\infty} P(N = n) P(U_{\lambda_n} \geq x+1) \\
 &\leq \frac{E(U_\lambda)}{x+1} + \sum_{n=0}^{\infty} P(N = n) \frac{E(U_{\lambda_n})}{x+1} \\
 &= \frac{\lambda + \lambda}{x+1} = \frac{2\lambda}{x+1}.
 \end{aligned}$$

Taking these bounds into (2.3), hence (2.1) holds. $\square$

If X_i 's are identically distributed, then the following corollary is an immediate consequence of the Theorem 2.1.

Corollary 2.1. *Let $x \in \mathbb{N}$, if $p_1 = p_2 = \dots = p$, then we have the following:*

$$\begin{aligned}
 |P(S_N \leq x) - P(U_\lambda \leq x)| &\leq \min \left\{ E(1 - e^{-pN})p, E \left[\frac{2(e^{pN} - pN - 1)}{N(x+1)} \right], \right. \\
 &\quad \left. \frac{p^2 E(N)}{x} \right\} + \min \left\{ \frac{2pE(N)}{x+1}, \min \left\{ 1, \sqrt{\frac{2}{eE(N)p}} \right\} pE|N - E(N)| \right\} \tag{2.4}
 \end{aligned}$$

and

$$\begin{aligned} \sup_{x \in \mathbb{N}} |P(S_N \leq x) - P(U_\lambda \leq x)| &\leq \min \left\{ E(1 - e^{-pN})p, E \left(\frac{e^{pN} - pN - 1}{N} \right) \right\} \\ &+ \min \left\{ pE(N), \min \left\{ 1, \sqrt{\frac{2}{eE(N)p}} \right\} pE|N - E(N)| \right\}. \end{aligned} \quad (2.5)$$

Remark. Let us consider:

1. $\frac{2\lambda}{x+1} < \frac{3\lambda}{x}$,
2. $\min \left(\frac{1-e^{-\lambda N}}{\lambda_N}, \frac{2(e^{\lambda N} - \lambda_N - 1)}{\lambda_N^2(x+1)}, \frac{1}{x} \right) \leq \frac{1-e^{-\lambda N}}{\lambda_N} \min \left\{ 1, \frac{e^{\lambda N}}{x+1} \right\}$,
3. $\min \left\{ \lambda, \min \left\{ 1, \sqrt{\frac{2}{e\lambda}} \right\} E|\lambda_N - \lambda| \right\} \leq E|\lambda_N - \lambda|$ and
4. $\min \left\{ E \left(\frac{1-e^{-\lambda N}}{\lambda_N} \right), E \left(\frac{e^{\lambda N} - \lambda_N - 1}{\lambda_N^2} \right) \right\} \leq E \left(\frac{1-e^{-\lambda N}}{\lambda_N} \right)$.

Thus, the bounds in (2.2) and (2.1) are sharper than the bounds in (1.1) and (1.2) respectively.

3. Conclusion

In this study, non-uniform and uniform bounds for the distance between the distribution function of random sums of independent Bernoulli random variables and an appropriate Poisson distribution function were improved. They are sharper than those reported in [8] and [2]. Thus, the bounds obtained in this study are more appropriate for measuring the accuracy of the approximation.

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