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POISSON APPROXIMATION FOR RANDOM SUMS OF GEOMETRIC RANDOM VARIABLES

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Abstract: In this paper, we determine bounds with different Poisson mean for the total variation distance between the distribution of random sums of independent geometric random variables and an appropriate Poisson distribution. Two examples have been given to illustrate the results obtained.

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1. Introduction

Let $X_1, ..., X_n$ be n ($n \in \mathbb{N}$) independently distributed geometric random variables, each with $P(X_i = k) = (1 - p_i)^k p_i$, $k = 0, 1, ...$, and let $S_n = \sum_{i=1}^{n} X_i$ and U_λ a Poisson random variable with mean λ. It is well-known that if all $q_i = 1 - p_i$ are small, then the distribution of S_n, denoted by $\mathcal{L}(S_n)$, can be approximated by an appropriate Poisson distribution with mean λ, denoted by $\mathcal{L}(U_\lambda)$. In this case, Teerapabolarn and Wongkasem [2] gave a bound for the total variation distance between the distributions of S_n and U_{λ_n} as follows:

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$$d(\mathcal{L}(S_n), \mathcal{L}(U_{\lambda_n})) = \sup_{A \subseteq \mathbb{N}\cup\{0\}} |P(S_n \in A) - P(U_{\lambda_n} \in A)|$$
$$\leq \sum_{i=1}^{n} \min\left\{\frac{1-e^{-\lambda_n}}{\lambda_n p_i}, 1\right\} \frac{q_i^2}{p_i} \tag{1.1}$$

for $\lambda_n = \sum_{i=1}^{n} \frac{q_i}{p_i}$. For $\lambda_n = \sum_{i=1}^{n} q_i$, Vellaisamy and Upadhye [3] gave a bound in the form of

$$d(\mathcal{L}(S_n), \mathcal{L}(U_{\lambda_n})) \leq \sum_{i=1}^{n} \min\left\{\frac{0.42888}{\sqrt{\lambda_n}}, 1\right\} \frac{q_i^2}{p_i}. \tag{1.2}$$

Let us consider the sum $S_N = \sum_{i=1}^{N} X_i$, where N is a non-negative integer valued random variable and independent of the X_i's. The sum is called *random sums* of independent geometric random random variables. In this study, we are interest to approximate $\mathcal{L}(S_N)$ by $\mathcal{L}(U_\lambda)$ when $\lambda = E(\lambda_N)$, which are in Section 2. We give some examples to illustrate the results of this study in the last section.

2. Result

The following theorem presents bounds with different Poisson mean for the total variation distance between $\mathcal{L}(S_N)$ and $\mathcal{L}(U_\lambda)$.

Theorem 2.1. *With the above definitions:*

1. For $\lambda_N = \sum_{i=1}^{N} \frac{q_i}{p_i}$,

$$d(\mathcal{L}(S_N), \mathcal{L}(U_\lambda)) \leq \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda|$$
$$+ \min\left\{E\left(\frac{1-e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^{N} \frac{q_i^2}{p_i^2}\right), E\left(\sum_{i=1}^{N} \frac{q_i^2}{p_i}\right)\right\}. \tag{2.1}$$

2. For $\lambda_N = \sum_{i=1}^{N} q_i$,

$$d(\mathcal{L}(S_N), \mathcal{L}(U_\lambda)) \leq \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda|$$
$$+ \min\left\{E\left(\frac{0.42888}{\sqrt{\lambda_N}} \sum_{i=1}^{N} \frac{q_i^2}{p_i}\right), E\left(\sum_{i=1}^{N} \frac{q_i^2}{p_i}\right)\right\}. \tag{2.2}$$

Proof. 1. We have

$$
\begin{aligned}
d(\mathcal{L}(S_N), \mathcal{L}(U_\lambda)) &\leq d(\mathcal{L}(S_N), \mathcal{L}(U_{\lambda_N})) + d(\mathcal{L}(U_{\lambda_N}), \mathcal{L}(U_\lambda)) \\
&= \sum_{n=0}^{\infty} P(N=n) d(\mathcal{L}(S_n), \mathcal{L}(U_{\lambda_n})) + d(\mathcal{L}(U_{\lambda_N}), \mathcal{L}(U_\lambda)) \\
&\leq \sum_{n=0}^{\infty} P(N=n) \sum_{i=1}^{n} \min\left\{\frac{1-e^{-\lambda}}{\lambda p_i}, 1\right\} \frac{q_i^2}{p_i} \\
&\qquad + \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda| \qquad (2.3) \\
&= \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda| \\
&\qquad + \min\left\{E\left(\frac{1-e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^{N} \frac{q_i^2}{p_i^2}\right), E\left(\sum_{i=1}^{N} \frac{q_i^2}{p_i}\right)\right\},
\end{aligned}
$$

where the first and second terms of the right hand side of (2.3) follow (1.1) and [1] on pp. 12, respectively.

2. Substituting the bound in the first term of the right hand side of (2.3) by the bound in (1.2), we have

$$
\begin{aligned}
d(\mathcal{L}(S_N), \mathcal{L}(U_\lambda)) &\leq \sum_{n=0}^{\infty} P(N=n) \sum_{i=1}^{n} \min\left\{\frac{0.42888}{\sqrt{\lambda_n}}, 1\right\} \frac{q_i^2}{p_i} \\
&\qquad + \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda| \\
&= \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda| \\
&\qquad + \min\left\{E\left(\frac{0.42888}{\sqrt{\lambda_N}} \sum_{i=1}^{N} \frac{q_i^2}{p_i}\right), E\left(\sum_{i=1}^{N} \frac{q_i^2}{p_i}\right)\right\}.
\end{aligned}
$$

Hence, the proof is complete. □

If X_i's are identically distributed, then the following corollary is an immediately consequence of the Theorem 2.1,

Corollary 2.1. *If $p_1 = p_2 = \cdots = p$, then we have the following:*

1. For $\lambda_N = \frac{Nq}{p}$,

$$d(\mathcal{L}(S_N), \mathcal{L}(U_\lambda)) \leq \min\left\{1, \sqrt{\frac{2p}{eE(N)q}}\right\} \frac{q}{p} E|N - E(N)| + \min\left\{E(1 - e^{-\frac{Nq}{p}})\frac{q}{p}, E(N)\frac{q^2}{p}\right\}. \tag{2.4}$$

2. For $\lambda_N = Nq$,

$$d(\mathcal{L}(S_N), \mathcal{L}(U_\lambda)) \leq \min\left\{1, \sqrt{\frac{2}{eE(N)q}}\right\} qE|N - E(N)| + \min\left\{\frac{0.42888q^{\frac{3}{2}}E(\sqrt{N})}{p}, E(N)\frac{q^2}{p}\right\}. \tag{2.5}$$

3. Examples

This section, we give two examples to illustrate the results in the case of X_i's are identically distributed, which are in the Corollary 2.1.

Example 3.1. For n ($n \in \mathbb{N}$) is fixed, let N be a positive integer-valued random variable with probability function

$$P(N = k) = \begin{cases} \frac{1}{2}, & k = n, \\ \frac{1}{2}, & k = 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $E(N) = \frac{3n}{2}$ and $E|N - E(N)| = \frac{n}{2}$. Let $p_1 = p_2 = \cdots = p$, then we have

$$d(\mathcal{L}(S_N), \mathcal{L}(U_{\frac{3nq}{2p}})) \leq \min\left\{1, \sqrt{\frac{4p}{3enq}}\right\} \frac{nq}{2p} + \min\left\{\frac{q}{p}, \frac{3nq^2}{2p}\right\}$$

and

$$d(\mathcal{L}(S_N), \mathcal{L}(U_{\frac{3nq}{2}})) \leq \min\left\{1, \sqrt{\frac{4}{3enq}}\right\} \frac{nq}{2} + \min\left\{0.42888\sqrt{\frac{3nq^3}{2p^2}}, \frac{3nq^2}{2p}\right\}.$$

Example 3.2. Let N be a positive integer-valued random variable with probability function

$$P(N = n) = \frac{1}{2^n},\ n = 1, 2, ...,$$

then we have $E(N) = 2$ and $E|N - E(N)| = 1$. If $p_1 = p_2 = \cdots = p$, then we obtain

$$d(\mathcal{L}(S_N), \mathcal{L}(U_{\frac{2q}{p}})) \leq \min\left\{1, \sqrt{\frac{p}{eq}}\right\}\frac{q}{p} + \min\left\{\frac{q}{p}, \frac{2q^2}{p}\right\}$$

and

$$d(\mathcal{L}(S_N), \mathcal{L}(U_{2q})) \leq \min\left\{1, \sqrt{\frac{1}{eq}}\right\} q + \min\left\{0.42888\sqrt{\frac{2q^3}{p^2}}, \frac{2q^2}{p}\right\}.$$

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