RIGHT RESIDUATED LATTICES AND RIGHT TRIANGLE ALGEBRAS

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Abstract: In this paper, we introduce the notions of right (left) triangle algebras in right (left) residuated lattices. Every right (left) residuated lattices induce right (left) triangle algebras. Moreover, every right (left) triangle algebras are isomorphic their right (left) approximation spaces.

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1. Introduction

Pawlak [10,11] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Moreover, Georgescu and Popescue [6,7] introduced a generalized residuated lattice which is induced by two implications. By using these concepts, information systems and decision rules are investigated [2,9,14]. Deschrijver, et.al. [3-5,12,13] introduced triangle algebras and interval-valued residuated lattices.

In this paper, we introduce the notions of right (left) triangle algebras in right (left) residuated lattices. Every right (left) residuated lattices induce

right (left) triangle algebras. Moreover, every right (left) triangle algebras are isomorphic their right (left) approximation spaces.

Definition 1. (see [6], [7]) A structure $(L, \vee, \wedge, \odot, \Rightarrow, \perp, \top)$ is called a *right residuated lattice* if it satisfies the following conditions:

(R1) $(L, \vee, \wedge, \top, \perp)$ is a bounded where $\perp$ is the bottom element and $\top$ is the top element;

(R2) $(L, \odot, \top)$ is a monoid;

(RR) it satisfies a right residuation , i.e.

$$a \odot b \leq c \text{ iff } b \leq a \Rightarrow c.$$

A structure $(L, \vee, \wedge, \odot, \rightarrow, \perp, \top)$ is called a *left residuated lattice* if it satisfies (R1), (R2) and

(LR) it satisfies a left residuation , i.e.

$$a \odot b \leq c \text{ iff } a \leq b \rightarrow c.$$

A structure $(L, \vee, \wedge, \odot, \Rightarrow, \rightarrow, \perp, \top)$ is called a *generalized residuated lattice* if it is a right and left residuated lattice.

We call that a generalized residuated lattice has the law of double negation if $a = (a^*)^0 = (a^0)^*$ where $a^0 = a \rightarrow \perp$ and $a^* = a \Rightarrow \perp$.

Remark 2. (see [6], [7], [12]) (1) A generalized residuated lattice is a residuated lattice $(\rightarrow, \Rightarrow)$ iff $\odot$ is commutative.

(2) A left-continuous t-norm $([0, 1], \leq, \odot)$ defined by $a \rightarrow b = \bigvee \{c \mid a \odot c \leq b\}$ is a residuated lattice

(3) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

Lemma 3. (see [4], [5]) Let $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, \perp, \top)$ be a generalized residuated lattice. For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(2) $x \odot y \leq x \wedge y$.

(3) $x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \rightarrow y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigwedge_{i \in \Gamma} (x_i \rightarrow y)$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

- (4) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.
- (5) $x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z)$ and $x \Rightarrow (y \rightarrow z) = y \rightarrow (x \Rightarrow z)$.
- (6) $x \odot (x \rightarrow y) \leq y$ and $(x \Rightarrow y) \odot x \leq y$.
- (7) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z$ and $(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z$.
- (8) $x \rightarrow y = \top$ iff $x \leq y$ iff $x \Rightarrow y = \top$.

Definition 4. A structure $\mathcal{A} = (A, \wedge, \vee, \odot, \Rightarrow, \nu_1, \mu_1, \perp, e_1, \top)$ is called a *right triangle algebra* if it satisfies the following conditions:

- (R) $(A, \wedge, \vee, \odot, \Rightarrow, \perp, \top)$ is a right residuated lattice.
- (T1) $\nu_1(x) \leq x$ and $\nu_1(x) \leq \nu_1(\nu_1(x))$.
- (T2) $\nu_1(x \wedge y) = \nu_1(x) \wedge \nu_1(y)$ and $\nu_1(x \vee y) = \nu_1(x) \vee \nu_1(y)$.
- (T3) $\nu_1(e_1) = \perp$.
- (T4) $\nu_1 \circ \mu_1 = \mu_1$.
- (S1) $x \leq \mu_1(x)$ and $\mu_1(x) \geq \mu_1(\mu_1(x))$.
- (S2) $\mu_1(x \wedge y) = \mu_1(x) \wedge \mu_1(y)$ and $\mu_1(x \vee y) = \mu_1(x) \vee \mu_1(y)$.
- (S3) $\mu_1(e_1) = \top$.
- (S4) $\mu_1 \circ \nu_1 = \nu_1$.
- (R5) $\nu_1(x \Rightarrow y) \leq \nu_1(x) \Rightarrow \nu_1(y)$.
- (R6) $(\nu_1(x) \Leftrightarrow \nu_1(y)) \odot (\mu_1(x) \Leftrightarrow \mu_1(y)) \leq (x \Leftrightarrow y)$.
- (R7) $\nu_1(x) \Rightarrow \nu_1(y) \leq \nu_1(\nu_1(x) \Rightarrow \nu_1(y))$.

The pair $[\nu_1(x), \mu_1(x)]$ is called a *right rough approximation* for $x \in A$. $R_r(A) = \{[\nu_1(x), \mu_1(x)] \mid x \in A\}$ is called a *right rough approximation space*.

A structure $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, \nu_2, \mu_2, \perp, e_2, \top)$ is called a *left triangle algebra* if ν_2, μ_2 satisfy (T1)-(T4), (S1)-(S4), respectively and following conditions:

- (L) $(A, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ is a left residuated lattice.
- (L5) $\nu_2(x \rightarrow y) \leq \nu_2(x) \rightarrow \nu_2(y)$.
- (L6) $(\mu_2(x) \Leftrightarrow \mu_2(y)) \odot (\nu_2(x) \Leftrightarrow \nu_2(y)) \leq (x \Leftrightarrow y)$.
- (L7) $\nu_2(x) \rightarrow \nu_2(y) \leq \nu_2(\nu_2(x) \rightarrow \nu_2(y))$.

The pair $[\nu_2(x), \mu_2(x)]$ is called a *left rough approximation* for $x \in A$. $R_l(A) = \{[\nu_2(x), \mu_2(x)] \mid x \in A\}$ is called a *left rough approximation space*.

Remark 5. (1) If $\odot$ is commutative (or $\Rightarrow = \rightarrow$), $\nu_1 = \nu_2$, $\mu_1 = \mu_2$ and $e_1 = e_2$ in Definition 4, then $(A, \wedge, \vee, \odot, \Rightarrow, \nu_1, \mu_1, \perp, e_1, \top)$ is a triangle algebra in [12].

(2) In Definition 4, $\nu_i(\top) = \top$ and $\mu_i(\perp) = \perp$ because $\nu_i(\top) = \nu_i(\mu_i(e_i)) = \mu_i(e_i) = \top$ and $\mu_i(\perp) = \mu_i(\nu_i(e_i)) = \nu_i(e_i) = \perp$.

2. Right Residuated Lattices and Right Triangle Algebras

Theorem 6. Let $(A, \wedge, \vee, \odot, \Rightarrow, \perp, \top)$ be a right residuated lattice and $[A]^{[2]} = \{[x, y] \mid x \leq y\}$. Define

$$\begin{aligned} [x_1, x_2] \wedge [y_1, y_2] &= [x_1 \wedge y_1, x_2 \wedge y_2] \\ [x_1, x_2] \vee [y_1, y_2] &= [x_1 \vee y_1, x_2 \vee y_2] \\ [x_1, x_2] \odot [y_1, y_2] &= [x_1 \odot y_1, x_2 \odot y_2] \\ [x_1, x_2] \Rightarrow [y_1, y_2] &= [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2]. \end{aligned}$$

Then the following properties hold.

- (1) $([A]^{[2]}, \wedge, \vee, \odot, \Rightarrow, [\perp, \perp], [\top, \top])$ is a right residuated lattice.
- (2) $(D(A), \wedge, \vee, \odot, \Rightarrow, [\perp, \perp], [\top, \top])$ is a right residuated lattice where $D(A) = \{[x, x] \mid x \in A\}$. Moreover, $\eta_1 : A \rightarrow D(A)$ with $\eta_1(x) = [x, x]$ is an isomorphism; i.e., η_1 is bijective, $\eta_1(x \vee y) = \eta_1(x) \vee \eta_1(y)$, $\eta_1(x \wedge y) = \eta_1(x) \wedge \eta_1(y)$, $\eta_1(x \odot y) = \eta_1(x) \odot \eta_1(y)$, $\eta_1(x \Rightarrow y) = \eta_1(x) \Rightarrow \eta_1(y)$.
- (3) If we define $\pi_1([x_1, x_2]) = [x_1, x_1]$ and $\pi_2([x_1, x_2]) = [x_2, x_2]$, then $([A]^{[2]}, \wedge, \vee, \odot, \Rightarrow, \pi_1, \pi_2, [\perp, \perp], [\perp, \top], [\top, \top])$ is a right triangle algebra.

Proof. (1)

$$\begin{aligned} [x_1, x_2] \odot [y_1, y_2] &= [x_1 \odot y_1, x_2 \odot y_2] \leq [z_1, z_2] \\ \text{iff } x_1 \odot y_1 &\leq z_1, x_2 \odot y_2 \leq z_2 \text{ iff } y_1 \leq x_1 \Rightarrow z_1, y_2 \leq x_2 \Rightarrow z_2 \\ \text{iff } [y_1, y_2] &\leq [(x_1 \Rightarrow z_1) \wedge (x_2 \Rightarrow z_2), x_2 \Rightarrow z_2] = [x_1, x_2] \Rightarrow [z_1, z_2] \end{aligned}$$

Other cases are easily proved.

(2) $\eta_1(x \Rightarrow y) = [x \Rightarrow y, x \Rightarrow y] = [x, x] \Rightarrow [y, y] = \eta_1(x) \Rightarrow \eta_1(y)$. Other cases are easily proved.

(3) (R5) $\pi_1([x_1, x_2] \Rightarrow [y_1, y_2]) = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), (x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2)] \leq [(x_1 \Rightarrow y_1), (x_1 \Rightarrow y_1)] = \pi_1([x_1, x_2]) \Rightarrow \pi_1([y_1, y_2])$.

(R6)

$$\begin{aligned}
& \pi_1([x_1, x_2]) \Leftrightarrow \pi_1([y_1, y_2]) \odot \pi_2([x_1, x_2]) \Leftrightarrow \pi_2([y_1, y_2]) \\
& = ([x_1, x_1] \Leftrightarrow [y_1, y_1]) \odot ([x_2, x_2] \Leftrightarrow [y_2, y_2]) \\
& = ([x_1 \Leftrightarrow y_1, x_1 \Leftrightarrow y_1]) \odot ([x_2 \Leftrightarrow y_2, x_2 \Leftrightarrow y_2]) \\
& = ((x_1 \Leftrightarrow y_1) \odot (x_2 \Leftrightarrow y_2), (x_1 \Leftrightarrow y_1) \odot (x_2 \Leftrightarrow y_2)) \\
& \leq ((x_1 \Leftrightarrow y_1) \wedge (x_2 \Leftrightarrow y_2), (x_2 \Leftrightarrow y_2)) \\
& = [x_1, x_2] \Leftrightarrow [y_1, y_2].
\end{aligned}$$

(R7)

$$\begin{aligned}
& \pi_1([x_1, x_2]) \Rightarrow \pi_1([y_1, y_2]) = [x_1, x_1] \Rightarrow [y_1, y_1] \\
& = [x_1 \Rightarrow y_1, x_1 \Rightarrow y_1] = \pi_1(\pi_1([x_1, x_2]) \Rightarrow \pi_1([y_1, y_2])).
\end{aligned}$$

Hence $([A]^{[2]}, \wedge, \vee, \odot, \Rightarrow, \pi_1, \pi_2, [\perp, \perp], [\perp, \top], [\top, \top])$ is a right triangle algebra. $\square$

The following corollary is similarly proved that of Theorem 6.

Corollary 7. *Let $(A, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ be a left residuated lattice and $[A]^{[2]} = \{[x, y] \mid x \leq y\}$. Define*

$$\begin{aligned}
[x_1, x_2] \wedge [y_1, y_2] &= [x_1 \wedge y_1, x_2 \wedge y_2] \\
[x_1, x_2] \vee [y_1, y_2] &= [x_1 \vee y_1, x_2 \vee y_2] \\
[x_1, x_2] \odot [y_1, y_2] &= [x_1 \odot y_1, x_2 \odot y_2] \\
[x_1, x_2] \rightarrow [x_1, x_2] &= [(x_1 \rightarrow y_1) \wedge (x_2 \rightarrow y_2), x_2 \rightarrow y_2].
\end{aligned}$$

Then the following properties hold.

- (1) $([A]^{[2]}, \wedge, \vee, \odot, \rightarrow, [\perp, \perp], [\top, \top])$ is a left residuated lattice.
- (2) $(D(A), \wedge, \vee, \odot, \rightarrow, [\perp, \perp], [\top, \top])$ is a left residuated lattice where $D(A) = \{[x, x] \mid x \in A\}$. Moreover, $\eta_2 : A \rightarrow D(A)$ with $\eta_2(x) = [x, x]$ is an isomorphism in a sense in Theorem 6(2).
- (3) If we define $\pi_1([x_1, x_2]) = [x_1, x_1]$ and $\pi_2([x_1, x_2]) = [x_2, x_2]$, then $([A]^{[2]}, \wedge, \vee, \odot, \rightarrow, \pi_1, \pi_2, [\perp, \perp], [\perp, \top], [\top, \top])$ is a left triangle algebra.

Theorem 8. *Let $(A, \wedge, \vee, \otimes, \Rightarrow, \nu_1, \mu_1, \perp, e_1, \top)$ be a right triangle lattice. Then the following properties hold:*

- (1) $(E_r(A), \wedge, \vee, \otimes, \Rightarrow, \perp, \top)$ is a right residuated lattice where $E_r(A) = \{x \in A \mid \nu_1(x) = x\} = \{x \in A \mid \nu_1(x) = x\}$.
- (2) $([E_r(A)]^{[2]}, \wedge, \vee, \odot, \Rightarrow, \pi_1, \pi_2, [\perp, \perp], [\perp, \top], [\top, \top])$ is a right triangle algebra.

Proof. (1) Let $\nu_1(x) = x$. Then $\nu_1(x) = \mu_1(\nu_1(x)) = \mu_1(x)$. Hence $\mu_1(x) = x$. Let $\mu_1(x) = x$. Then $\mu_1(x) = \nu_1(\mu_1(x)) = \nu_1(x)$. Hence $\nu_1(x) = x$. Thus $\{x \in A \mid \nu_1(x) = x\} = \{x \in A \mid \mu_1(x) = x\}$.

(R1) For $x, y \in E_r(A)$, $\nu_1(x \wedge y) = \nu_1(x) \wedge \nu_1(y) = x \wedge y$ and $\nu_1(x \vee y) = \nu_1(x) \vee \nu_1(y) = x \vee y$. Then $x \wedge y, x \vee y \in E_r(A)$. Since $\nu_1(\perp) = \perp$ and $\nu_1(\top) = \top$, then $\perp, \top \in E_r(A)$.

Since $x \Rightarrow y = \nu_1(x) \Rightarrow \nu_1(y) \leq \nu_1(\nu_1(x) \Rightarrow \nu_1(y)) \leq \nu_1(x) \Rightarrow \nu_1(y) = x \Rightarrow y$, then $x \Rightarrow y \in E_r(A)$. Since $y \leq x \Rightarrow x \otimes y$, we have $\nu_1(y) \leq \nu_1(x \Rightarrow x \otimes y) = \nu_1(x) \Rightarrow \nu_1(x \otimes y)$. Hence $\nu_1(x) \otimes \nu_1(y) \leq \nu_1(x \otimes y)$. So, $x \otimes y = \nu_1(x) \otimes \nu_1(y) \leq \nu_1(x \otimes y) \leq x \otimes y$, then $x \otimes y \in E_r(A)$.

(R2) For $x, y, z \in E_r(A)$, $(\nu_1(x) \otimes \nu_1(y)) \otimes \nu_1(z) = (x \otimes y) \otimes z = x \otimes (y \otimes z) = \nu_1(x) \otimes (\nu_1(y) \otimes \nu_1(z))$ and $x \otimes \top = \nu_1(x) \otimes \nu_1(\top) = \nu_1(x) \otimes \top = \nu_1(x) = x$.

(RR) $x \otimes y \leq z$ iff $\nu_1(x \otimes y) = \nu_1(x) \otimes \nu_1(y) \leq \nu_1(z)$ iff $\nu_1(y) \leq \nu_1(x) \Rightarrow \nu_1(z) = \nu_1(x \Rightarrow z)$ iff $y \leq x \Rightarrow z$. $\square$

The following corollary is similarly proved that of Theorem 8.

Corollary 9. *Let $(A, \wedge, \vee, \otimes, \rightarrow, \nu_2, \mu_2, \perp, e_2, \top)$ be a left triangle lattice. Then the following properties hold.*

(1) $(E_l(A), \wedge, \vee, \otimes, \rightarrow, \perp, \top)$ is a left residuated lattice where $E_l(A) = \{x \in A \mid \nu_2(x) = x\} = \{x \in A \mid \mu_2(x) = x\}$.

(2) $([E_l(A)]^{[2]}, \wedge, \vee, \odot, \rightarrow, \pi_1, \pi_2, [\perp, \perp], [\perp, \top], [\top, \top])$ is a left triangle algebra.

Example 10. Let $K = \{(x, y) \in R^2 \mid x > 0\}$ be a set and we define an operation $\otimes : K \times K \rightarrow K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1).$$

Then $(K, \otimes)$ is a group with $e = (1, 0)$, $(x, y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a, b) \in R^2 \mid a = 1, b \geq 0, \text{ or } a > 1\}$ because $P \cap P^{-1} = \{(1, 0)\}$, $P \odot P \subset P$, $(a, b)^{-1} \odot P \odot (a, b) = P$ and $P \cup P^{-1} = K$. For $(x_1, y_1), (x_2, y_2) \in K$, we define

$$\begin{aligned} (x_1, y_1) \leq (x_2, y_2) &\Leftrightarrow (x_1, y_1)^{-1} \odot (x_2, y_2) \in P, (x_2, y_2) \odot (x_1, y_1)^{-1} \in P \\ &\Leftrightarrow x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2. \end{aligned}$$

Then $(K, \leq, \otimes)$ is a lattice-group. The structure $(L, \odot, \Rightarrow, \rightarrow, (\frac{1}{2}, 1), (1, 0))$ is defined a lattice where $\perp = (\frac{1}{2}, 1)$ is the least element and $\top = (1, 0)$ is the

greatest element from the following statements, for $(x_1, y_1)(x_2, y_2) \in L$,

$$\begin{aligned} (x_1, y_1) \odot (x_2, y_2) &= (x_1, y_1) \otimes (x_2, y_2) \vee (\tfrac{1}{2}, 1) = (x_1 x_2, x_1 y_2 + y_1) \vee (\tfrac{1}{2}, 1), \\ (x_1, y_1) \Rightarrow (x_2, y_2) &= ((x_1, y_1)^{-1} \otimes (x_2, y_2)) \wedge (1, 0) = (\tfrac{x_2}{x_1}, \tfrac{y_2 - y_1}{x_1}) \wedge (1, 0), \\ (x_1, y_1) \rightarrow (x_2, y_2) &= ((x_2, y_2) \otimes (x_1, y_1)^{-1}) \wedge (1, 0) = (\tfrac{x_2}{x_1}, -\tfrac{x_2 y_1}{x_1} + y_2) \wedge (1, 0). \end{aligned}$$

(1) The structure $(L, \odot, \Rightarrow, (\tfrac{1}{2}, 1), (1, 0))$ is a right residuated lattice because L is a totally order and the following statements:

$$\begin{aligned} (x_1, y_1) \vee (x_2, y_2) &= (x_2, y_2) && \text{iff } (x_1, y_1) \leq (x_2, y_2) \\ (x_1, y_1) \wedge (x_2, y_2) &= (x_1, y_1) && \text{iff } (x_1, y_1) \leq (x_2, y_2) \\ (x_1, y_1) \odot (x_2, y_2) &\leq (x_3, y_3) && \text{iff } (x_2, y_2) \leq ((x_1, y_1)^{-1} \odot (x_3, y_3)) \wedge (1, 0) \\ &&& \text{iff } (x_2, y_2) \leq (x_1, y_1) \Rightarrow (x_3, y_3). \end{aligned}$$

We define

$$L^{[2]} = \{[(x_1, y_1), (x_2, y_2)] \mid ((x_1, y_1), (x_2, y_2)) \in L \times L, (x_1, y_1) \leq (x_2, y_2)\}$$

$$e_1 = [(\tfrac{1}{2}, 1), (1, 0)] = [\perp, \top],$$

$$[(x_1, y_1), (x_2, y_2)] \wedge [(x_3, y_3), (x_4, y_4)] = [(x_1, y_1) \wedge (x_3, y_3), (x_2, y_2) \wedge (x_4, y_4)]$$

$$[(x_1, y_1), (x_2, y_2)] \vee [(x_3, y_3), (x_4, y_4)] = [(x_1, y_1) \vee (x_3, y_3), (x_2, y_2) \vee (x_4, y_4)]$$

$$[(x_1, y_1), (x_2, y_2)] \odot [(x_3, y_3), (x_4, y_4)] = [(x_1, y_1) \odot (x_3, y_3), (x_2, y_2) \odot (x_4, y_4)]$$

$$\begin{aligned} [(x_1, y_1), (x_2, y_2)] &\Rightarrow [(x_3, y_3), (x_4, y_4)] \\ &= [((x_1, y_1) \Rightarrow (x_3, y_3)) \wedge ((x_2, y_2) \Rightarrow (x_4, y_4)), (x_2, y_2) \Rightarrow (x_4, y_4)] \end{aligned}$$

Moreover, we define $\pi_1, \pi_2 : L^{[2]} \rightarrow L^{[2]}$ as

$$\pi_1([(x_1, y_1), (x_2, y_2)]) = [(x_1, y_1), (x_1, y_1)],$$

$$\pi_2([(x_1, y_1), (x_2, y_2)]) = [(x_2, y_2), (x_2, y_2)].$$

Then $(L^{[2]}, \wedge, \vee, \odot, \Rightarrow, \pi_1, \pi_2, [\perp, \perp], e_1 = [\perp, \top], [\top, \top])$ is a right triangle algebra from Theorem 6(3). Moreover, from Theorem 8, $(E_r(L^{[2]}), \wedge, \vee, \otimes, \Rightarrow, \perp, \top)$ is a right residuated lattice where

$$E_r(L^{[2]}) = \{[(x_1, y_1), (x_1, y_1)] \mid \pi_1([(x_1, y_1), (x_1, y_1)]) = [(x_1, y_1), (x_1, y_1)]\}.$$

(2) The structure $(L, \odot, \rightarrow, (\frac{1}{2}, 1), (1, 0))$ is a left residuated lattice because L is a totally order and the following statements:

$$\begin{aligned} (x_1, y_1) \odot (x_2, y_2) \leq (x_3, y_3) & \quad \text{iff } (x_1, y_1) \leq ((x_3, y_3) \odot ((x_1, y_1)^{-1})) \wedge (1, 0) \\ & \quad \text{iff } (x_1, y_1) \leq (x_2, y_2) \rightarrow (x_3, y_3). \end{aligned}$$

We define

$$\begin{aligned} L^{[2]} &= \{[(x_1, y_1), (x_2, y_2)] \mid ((x_1, y_1), (x_2, y_2)) \in L \times L, (x_1, y_1) \leq (x_2, y_2)\} \\ e &= [(\frac{1}{2}, 1), (1, 0)] = [\perp, \top], \end{aligned}$$

$$\begin{aligned} [(x_1, y_1), (x_2, y_2)] &\rightarrow [(x_3, y_3), (x_4, y_4)] \\ &= [((x_1, y_1) \rightarrow (x_3, y_3)) \wedge ((x_2, y_2) \rightarrow (x_4, y_4)), (x_2, y_2) \rightarrow (x_4, y_4)]. \end{aligned}$$

Moreover, we define $\pi_1, \pi_2 : L^{[2]} \rightarrow L^{[2]}$ as

$$\pi_1([(x_1, y_1), (x_2, y_2)]) = [(x_1, y_1), (x_1, y_1)],$$

$$\pi_2([(x_1, y_1), (x_2, y_2)]) = [(x_2, y_2), (x_2, y_2)].$$

Then $(L^{[2]}, \wedge, \vee, \odot, \rightarrow, \pi_1, \pi_2, [\perp, \perp], e_2 = [\perp, \top], [\top, \top])$ is a left triangle algebra from Corollary 7. Moreover, from Corollary 9, $(E_l(L^{[2]}), \wedge, \vee, \otimes, \rightarrow, \perp, \top)$ is a left residuated lattice where

$$E_l(L^{[2]}) = \{[(x_1, y_1), (x_1, y_1)] \mid \pi_1([(x_1, y_1), (x_1, y_1)]) = [(x_1, y_1), (x_1, y_1)]\}.$$

Theorem 11. *Let $(A, \wedge, \vee, \otimes, \Rightarrow, \nu_1, \mu_1, \perp, e_1, \top)$ be a right triangle algebra and $R_r(A) = \{[\nu_1(x), \mu_1(x)] \mid x \in A\}$. Define*

$$\begin{aligned} [\nu_1(x), \mu_1(x)] \wedge [\nu_1(y), \mu_1(y)] &= [\nu_1(x) \wedge \nu_1(y), \mu_1(x) \wedge \mu_1(y)], \\ [\nu_1(x), \mu_1(x)] \vee [\nu_1(y), \mu_1(y)] &= [\nu_1(x) \vee \nu_1(y), \mu_1(x) \vee \mu_1(y)], \\ [\nu_1(x), \mu_1(x)] \odot [\nu_1(y), \mu_1(y)] &= [\nu_1(x \otimes y), \mu_1(x \otimes y)], \\ [\nu_1(x), \mu_1(x)] \Rightarrow [\nu_1(y), \mu_1(y)] &= [\nu_1(x \Rightarrow y), \mu_1(x \Rightarrow y)], \end{aligned}$$

$$\pi_1([\nu_1(x), \mu_1(x)]) = [\nu_1(x), \nu_1(x)], \quad \pi_2([\nu_1(x), \mu_1(x)]) = [\mu_1(x), \mu_1(x)].$$

Then we have the following properties:

$$(1) \quad x \leq y \text{ iff } \nu_1(x) \leq \nu_1(y) \text{ and } \mu_1(x) \leq \mu_1(y) \text{ iff } [\nu_1(x), \mu_1(x)] \leq [\nu_1(y), \mu_1(y)].$$

(2) $(R_r(A), \wedge, \vee, \odot, \Rightarrow, \pi_1, \pi_2, [\perp, \perp], [\perp, \top], [\top, \top])$ is a right triangle algebra with $\pi_1(\phi_1(x)) = \nu_1(x), \pi_2(\phi_1(x)) = \mu_1(x)$.

(3) Define $\phi_1 : A \rightarrow R_r(A)$ as $\phi_1(x) = [\nu_1(x), \mu_1(x)]$. Then ϕ_1 is an isomorphism; i.e, ϕ_1 is bijective,

$$\phi_1(x \wedge y) = \phi_1(x) \wedge \phi_1(y), \quad \phi_1(x \vee y) = \phi_1(x) \vee \phi_1(y)$$

$$\phi_1(x \odot y) = \phi_1(x) \odot \phi_1(y), \quad \phi_1(x \Rightarrow y) = \phi_1(x) \Rightarrow \phi_1(y).$$

Moreover, $\pi_1(\phi_1(x)) = \nu_1(x), \pi_2(\phi_1(x)) = \mu_1(x)$.

Proof. (1) First, we show that $x \leq y$ iff $\nu_1(x) \leq \nu_1(y)$ and $\mu_1(x) \leq \mu_1(y)$. Let $x \leq y$. Then $x \wedge y = x$. $\nu_1(x) \wedge \nu_1(y) = \nu_1(x)$ and $\mu_1(x) \wedge \mu_1(y) = \mu_1(x)$. Hence $\nu_1(x) \leq \nu_1(y)$ and $\mu_1(x) \leq \mu_1(y)$.

Conversely, since $\nu_1(x) \leq \nu_1(y)$ and $\mu_1(x) \leq \mu_1(y)$, $\nu_1(x \wedge y) = \nu_1(x)$ and $\mu_1(x \wedge y) = \mu_1(x)$. By (T5), $x \wedge y = x$. Hence $x \leq y$.

Second, we show that $\nu_1(x) \leq \nu_1(y)$ and $\mu_1(x) \leq \mu_1(y)$ iff $[\nu_1(x), \mu_1(x)] \leq [\nu_1(y), \mu_1(y)]$. Let $\nu_1(x) \leq \nu_1(y)$ and $\mu_1(x) \leq \mu_1(y)$. Then $[\nu_1(x), \mu_1(x)] \wedge [\nu_1(y), \mu_1(y)] = [\nu_1(x), \mu_1(x)]$. Hence $[\nu_1(x), \mu_1(x)] \leq [\nu_1(y), \mu_1(y)]$. Conversely, it is similarly proved.

(2) (R1) Since $[\nu_1(\perp), \mu_1(\perp)] = [\perp, \perp]$ and $[\nu_1(\top), \mu_1(\top)] = [\top, \top]$, then $[\perp, \perp]$ is the bottom element and $[\top, \top]$ is the top element.

(R2)

$$\begin{aligned} & ([\nu_1(x), \mu_1(x)] \odot [\nu_1(y), \mu_1(y)]) \odot [\nu_1(z), \mu_1(z)] \\ &= [\nu_1(x \otimes y), \mu_1(x \otimes y)] \odot [\nu_1(z), \mu_1(z)] \\ &= [\nu_1((x \otimes y) \otimes z), \mu_1((x \otimes y) \otimes z)] \\ &= [\nu_1(x \otimes (y \otimes z)), \mu_1(x \otimes (y \otimes z))] \\ &= [\nu_1(x), \mu_1(x)] \odot ([\nu_1(y), \mu_1(y)] \odot [\nu_1(z), \mu_1(z)]) \end{aligned}$$

$$\begin{aligned} [\nu_1(x), \mu_1(x)] \odot [\top, \top] &= [\nu_1(x), \mu_1(x)] \odot [\nu_1(\top), \mu_1(\top)] \\ &= [\nu_1(x \otimes \top), \mu_1(x \otimes \top)] = [\nu_1(x), \mu_1(x)] \end{aligned}$$

(RR)

$$\begin{aligned} & [\nu_1(x), \mu_1(x)] \odot [\nu_1(y), \mu_1(y)] \leq [\nu_1(z), \mu_1(z)] \\ & \text{iff } \nu_1(x \otimes y) \leq \nu_1(z), \mu_1(x \otimes y) \leq \mu_1(z) \\ & \text{iff } \nu_1((x \otimes y) \vee z) = \nu_1(z), \mu_1((x \otimes y) \vee z) = \mu_1(z) \\ & \text{iff } (x \otimes y) \vee z = z, \text{ iff } (x \otimes y) \leq z, \text{ iff } y \leq x \Rightarrow z, \\ & \text{iff } \nu_1(y) \leq \nu_1(x \Rightarrow z), \mu_1(y) \leq \mu_1(x \Rightarrow z) \\ & \text{iff } [\nu_1(y), \mu_1(y)] \leq [\nu_1(x \Rightarrow z), \mu_1(x \Rightarrow z)] \\ & \text{iff } [\nu_1(y), \mu_1(y)] \leq [\nu_1(x), \mu_1(x)] \Rightarrow [\nu_1(z), \mu_1(z)] \end{aligned}$$

$$\begin{aligned}
& \pi_1([\nu_1(x), \mu_1(x)] \Rightarrow [\nu_1(y), \mu_1(y)]) = [\nu_1(x \Rightarrow y), \nu_1(x \Rightarrow y)] \\
& \pi_1([\nu_1(x), \mu_1(x)] \Rightarrow \pi_1([\nu_1(y), \mu_1(y)])) = [\nu_1(x), \nu_1(x)] \Rightarrow [\nu_1(y), \nu_1(y)] \\
& = [\nu_1(\nu_1(x)), \mu_1(\nu_1(x))] \Rightarrow [\nu_1(\nu_1(y)), \mu_1(\nu_1(y))] \\
& = [\nu_1(\nu_1(x) \Rightarrow \nu_1(y)), \mu_1(\nu_1(x) \Rightarrow \nu_1(y))]
\end{aligned}$$

Since $\nu_1(x) \odot \nu_1(x \Rightarrow y) \leq \nu_1(x \odot (x \Rightarrow y)) \leq \nu_1(y)$, then $\nu_1(x \Rightarrow y) \leq \nu_1(x) \Rightarrow \nu_1(y) \leq \nu_1(\nu_1(x) \Rightarrow \nu_1(y))$ from (R7). Moreover,

$$\nu_1(x) \Rightarrow \nu_1(y) \leq \mu_1(\nu_1(x) \Rightarrow \nu_1(y)) \leq \mu_1(\nu_1(x)) \Rightarrow \mu_1(\nu_1(y)) = \nu_1(x) \Rightarrow \nu_1(y).$$

(R5), (R7)

$$\begin{aligned}
& \pi_1([\nu_1(x), \mu_1(x)] \Rightarrow [\nu_1(y), \mu_1(y)]) = [\nu_1(x \Rightarrow y), \nu_1(x \Rightarrow y)] \\
& \leq [\nu_1(\nu_1(x) \Rightarrow \nu_1(y)), \mu_1(\nu_1(x) \Rightarrow \nu_1(y))] \\
& = \pi_1([\nu_1(x), \mu_1(x)] \Rightarrow \pi_1([\nu_1(y), \mu_1(y)]))
\end{aligned}$$

$$\begin{aligned}
& \pi_1([\nu_1(x), \mu_1(x)] \Rightarrow \pi_1([\nu_1(y), \mu_1(y)])) \\
& = [\nu_1(\nu_1(x) \Rightarrow \nu_1(y)), \mu_1(\nu_1(x) \Rightarrow \nu_1(y))] \\
& = [\nu_1(x) \Rightarrow \nu_1(y), \nu_1(x) \Rightarrow \nu_1(y)] \\
& = \pi_1(\pi_1([\nu_1(x), \mu_1(x)] \Rightarrow \pi_1([\nu_1(y), \mu_1(y)])))
\end{aligned}$$

(R6) Since $\nu_1(\mu_1(x)) = \mu_1(x)$, $\mu_1(x), \mu_1(y) \in E_r(A)$. Then $\mu_1(x) \Rightarrow \mu_1(y) \in E_r(A)$. Hence $\nu_1(\mu_1(x) \Rightarrow \mu_1(y)) = \mu_1(x) \Rightarrow \mu_1(y)$. Moreover,

$$\nu_1(\mu_1(x) \Rightarrow \mu_1(y)) = \mu_1(\nu_1(\mu_1(x) \Rightarrow \mu_1(y))) = \mu_1(\mu_1(x) \Rightarrow \mu_1(y)).$$

$$\begin{aligned}
& \pi_2([\nu_1(x), \mu_1(x)] \Rightarrow \pi_2([\nu_1(y), \mu_1(y)])) = [\mu_1(x), \mu_1(x)] \Rightarrow [\mu_1(y), \mu_1(y)] \\
& = [\nu_1(\mu_1(x)), \mu_1(\mu_1(x))] \Rightarrow [\nu_1(\mu_1(y)), \mu_1(\mu_1(y))] \\
& = [\nu_1(\mu_1(x) \Rightarrow \mu_1(y)), \mu_1(\mu_1(x) \Rightarrow \mu_1(y))] \\
& = [\mu_1(x) \Rightarrow \mu_1(y), \mu_1(x) \Rightarrow \mu_1(y)].
\end{aligned}$$

$$\begin{aligned}
& [\nu_1(x), \mu_1(x)] \odot (\pi_1([\nu_1(x), \mu_1(x)] \Rightarrow \pi_1([\nu_1(y), \mu_1(y)])) \\
& \odot (\pi_2([\nu_1(x), \mu_1(x)] \Rightarrow \pi_2([\nu_1(y), \mu_1(y)]))) \\
& = [\nu_1(\nu_1(x)), \mu_1(\mu_1(x))] \odot ([\nu_1(\nu_1(x) \Rightarrow \nu_1(y)), \mu_1(\nu_1(x) \Rightarrow \nu_1(y))]) \\
& \odot ([\nu_1(\mu_1(x) \Rightarrow \mu_1(y)), \mu_1(\mu_1(x) \Rightarrow \mu_1(y))]) \\
& = [\nu_1(\nu_1(x) \otimes (\nu_1(x) \Rightarrow \nu_1(y)) \otimes (\mu_1(x) \Rightarrow \mu_1(y))), \\
& \quad \mu_1(\mu_1(x) \otimes (\nu_1(x) \Rightarrow \nu_1(y)) \otimes (\mu_1(x) \Rightarrow \mu_1(y)))] \\
& \leq [\nu_1(\nu_1(y)), \mu_1(\mu_1(y))] = [\nu_1(y), \mu_1(y)]
\end{aligned}$$

Hence

$$\begin{aligned}
(\pi_1([\nu_1(x), \mu_1(x)]) &\Rightarrow \pi_1([\nu_1(y), \mu_1(y)]) \odot (\pi_2([\nu_1(x), \mu_1(x)]) \\
&\Rightarrow \pi_2([\nu_1(y), \mu_1(y)])) \leq [\nu_1(x), \mu_1(x)] \Rightarrow [\nu_1(y), \mu_1(y)].
\end{aligned}$$

By a similar method, the condition (T6) holds. Other cases are easily proved.

(3) Define $\phi_1 : A \rightarrow R_r(A)$ as $\phi_1(x) = [\nu_1(x), \mu_1(x)]$. Since $\nu_1(x) \leq \mu_1(x)$ and $\nu_1(x), \mu_1(x) \in E_r(X)$, ϕ_1 is well-defined. Moreover, $\phi_1(\perp) = [\nu_1(\perp), \mu_1(\perp)] = [\perp, \perp]$, $\phi_1(e_1) = [\nu_1(e_1), \mu_1(e_1)] = [\perp, \top]$ and $\phi_1(\top) = [\nu_1(\top), \mu_1(\top)] = [\top, \top]$. If $\phi_1(x) = \phi_1(y)$, then $\nu_1(x) = \nu_1(y)$ and $\mu_1(x) = \mu_1(y)$. By (T5), $x = y$. Hence ϕ_1 is injective.

For $[\nu_1(x), \mu_1(y)] \in R_r(A)$,

$$\begin{aligned}
\nu_1((\nu_1(x) \vee e_1) \wedge \mu_1(x)) &= (\nu_1(\nu_1(x)) \vee \nu_1(e_1)) \wedge \nu_1(\mu_1(x)) \\
&= (\nu_1(x) \vee \nu_1(e_1)) \wedge \nu_1(\mu_1(x)) = \nu_1(x) \wedge \mu_1(x) = \nu_1(x) \\
\mu_1((\nu_1(x) \vee e_1) \wedge \mu_1(x)) &= (\mu_1(\nu_1(x)) \vee \mu_1(e_1)) \wedge \mu_1(\mu_1(x)) \\
&= (\mu_1(x) \vee \mu_1(e_1)) \wedge \mu_1(\mu_1(x)) = \top \wedge \mu_1(x) = \mu_1(x).
\end{aligned}$$

By (T7), $(\nu_1(x) \vee e_1) \wedge \mu_1(x) = x$. Similarly, $\nu_1(x) \vee (e_1 \wedge \mu_1(x)) = x$. Moreover,

$$\phi_1(x) = \phi_1((\nu_1(x) \vee e_1) \wedge \mu_1(x)) = \phi_1(\nu_1(x) \vee (e_1 \wedge \mu_1(x))) = [\nu_1(x), \mu_1(y)].$$

Hence ϕ_1 is surjective.

$$\begin{aligned}
\nu_1(x \otimes y) &= \nu_1((\nu_1(x) \vee (e_1 \wedge \mu_1(x))) \otimes (\nu_1(y) \vee (e_1 \wedge \mu_1(y)))) \\
\mu_1(x \otimes y) &= \mu_1((\nu_1(x) \vee (e_1 \wedge \mu_1(x))) \otimes (\nu_1(y) \vee (e_1 \wedge \mu_1(y)))) \\
\phi_1(x \otimes y) &= [\nu_1(x \otimes y), \mu_1(x \otimes y)] \\
&= [\nu_1((\nu_1(x) \vee (e_1 \wedge \mu_1(x))) \otimes (\nu_1(y) \vee (e_1 \wedge \mu_1(y))))], \\
&\quad \mu_1((\nu_1(x) \vee (e_1 \wedge \mu_1(x))) \otimes (\nu_1(y) \vee (e_1 \wedge \mu_1(y))))] \\
&= [\nu_1(x), \mu_1(x)] \odot [\nu_1(y), \mu_1(y)] \\
&= \phi_1(x) \odot \phi_1(y). \\
\phi_1(x \Rightarrow y) &= [\nu_1(x \Rightarrow y), \mu_1(x \Rightarrow y)] \\
&= [\nu_1((\nu_1(x) \vee (e_1 \wedge \mu_1(x))) \Rightarrow (\nu_1(y) \vee (e_1 \wedge \mu_1(y))))], \\
&\quad \mu_1((\nu_1(x) \vee (e_1 \wedge \mu_1(x))) \Rightarrow (\nu_1(y) \vee (e_1 \wedge \mu_1(y))))] \\
&= [\nu_1(x), \mu_1(x)] \Rightarrow [\nu_1(y), \mu_1(y)] \\
&= \phi_1(x) \Rightarrow \phi_1(y). \quad \square
\end{aligned}$$

From Theorem 11, we similarly prove the following corollary.

Corollary 12. *Let $(A, \wedge, \vee, \otimes, \rightarrow, \nu_2, \mu_2, \perp, e_2, \top)$ be a left triangle algebra and $R_l(A) = \{[\nu_2(x), \mu_2(x)] \mid x \in A\}$. Define*

$$\begin{aligned}
[\nu_2(x), \mu_2(x)] \wedge [\nu_2(y), \mu_2(y)] &= [\nu_2(x) \wedge \nu_2(y), \mu_2(x) \wedge \mu_2(y)], \\
[\nu_2(x), \mu_2(x)] \vee [\nu_2(y), \mu_2(y)] &= [\nu_2(x) \vee \nu_2(y), \mu_2(x) \vee \mu_2(y)], \\
[\nu_2(x), \mu_2(x)] \odot [\nu_2(y), \mu_2(y)] &= [\nu_2(x \otimes y), \mu_2(x \otimes y)], \\
[\nu_2(x), \mu_2(x)] \rightarrow [\nu_2(y), \mu_2(y)] &= [\nu_2(x \rightarrow y), \mu_2(x \rightarrow y)],
\end{aligned}$$

$$\pi_1([\nu_2(x), \mu_2(x)]) = [\nu_2(x), \nu_2(x)], \pi_2([\nu_2(x), \mu_2(x)]) = [\mu_2(x), \mu_2(x)].$$

Then we have the following properties.

$$(1) x \leq y \text{ iff } \nu_2(x) \leq \nu_2(y) \text{ and } \mu_2(x) \leq \mu_2(y) \text{ iff } [\nu_2(x), \mu_2(x)] \leq [\nu_2(y), \mu_2(y)].$$

$$(2) (R_l(A), \wedge, \vee, \odot, \Rightarrow, \pi_1, \pi_2, [\perp, \perp], [\perp, \top], [\top, \top]) \text{ is a left triangle algebra.}$$

(3) Define $\phi_2 : A \rightarrow R_l(A)$ as $\phi_2(x) = [\nu_2(x), \mu_2(x)]$. Then ϕ_2 is an isomorphism; i.e, ϕ_2 is bijective,

$$\phi_2(x \wedge y) = \phi_2(x) \wedge \phi_2(y), \phi_2(x \vee y) = \phi_2(x) \vee \phi_2(y)$$

$$\phi_2(x \odot y) = \phi_2(x) \odot \phi_2(y), \phi_2(x \rightarrow y) = \phi_2(x) \rightarrow \phi_2(y).$$

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