ON 0-MINIMAL \((m, n)\)-IDEALS IN AN ORDERED SEMIGROUP

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Abstract: In this paper, we introduce an equivalence relation \(B\) on an ordered semigroup \(S\). Using the relation \(B\) defined, we characterize 0-minimal \((m, n)\)-ideals of \(S\). Several sufficient conditions are presented for 0-minimal left ideals and 0-minimal right ideals of \(S\) to be 0-minimal \((m, n)\)-ideals of \(S\). The results obtained generalize the results on semigroups without order.

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1. Preliminaries

Let \(S\) be a semigroup-without order-. Kapp [2] introduced an equivalence relation \(B\) on \(S\) by, for \(a, b \in S\), \(aBb\) if \(a = b\) or \(a \in bSb\) and \(b \in aSa\); using this relation the author characterized 0-minimal bi-ideal of \(S\). Tilidetzke [6] generalized Kapp’s results; the author introduced an equivalence relation \(B^m_n\) on \(S\) where \(m, n\) are non-negative integers by, for \(a, b \in S\), \(aB^m_nb\) if \(a = b\) or \(a \in b^mSb^n\) and \(b \in a^mSa^n\); using the relation \(B^m_n\), 0-minimal \((m, n)\)-ideals of \(S\) are characterized. The purpose of this paper is to extend Tilidetzke’s results based on ordered semigroups. The results obtained both in [2] and [6] become then special cases.
Let us now recall some definitions and results used throughout the paper.

An ordered semigroup $[1]$ is defined to be a semigroup $(S, \cdot)$ together with a partial order $\leq$ that is compatible with the semigroup operation, meaning that for $x, y, z \in S$,

\[ x \leq y \Rightarrow z \cdot x \leq z \cdot y, \quad x \cdot z \leq y \cdot z. \]

In this paper, we write $(S, \cdot, \leq)$ for an ordered semigroup $(S, \cdot)$ with a partial order $\leq$ and write $xy$ ($x, y \in S$) for $x \cdot y$.

If $A$ and $B$ are nonempty subsets of an ordered semigroup $(S, \cdot, \leq)$, we write $AB = \{xy \mid x \in A, y \in B\}$,

\[ (A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}. \]

For $x \in S$, we write $Ax$ and $xA$ for $A\{x\}$ and $\{x\}A$, respectively.

**Lemma 1.** ([3]) The following statements hold for any nonempty subsets $A, B$ of an ordered semigroup $(S, \cdot, \leq)$:

1. $A \subseteq (A]$;
2. $A \subseteq B \Rightarrow (A] \subseteq (B]$;
3. $(A][B] \subseteq (AB]$;
4. $((A][B]) = (AB]$;
5. $(A \cup B] = (A] \cup (B]$.

A nonempty subset $A$ of an ordered semigroup $(S, \cdot, \leq)$ is called a left (respectively, right) ideal of $S$ if the following conditions hold:

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);
(ii) $A = (A]$, that is, for $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.

If $A$ is both a left and a right ideal of $S$, then $A$ is called a (two-sided) ideal of $S$.

If there is an element 0 of an ordered semigroup $(S, \cdot, \leq)$ such that $x0 = 0x = 0$ for all $x \in S$ and $0 \leq x$ for all $x \in S$, we call 0 a zero element of $S$.

A left ideal $L$ of an ordered semigroup $(S, \cdot, \leq)$ with zero 0 is said to be 0-minimal if there is no a left ideal $L'$ of $S$ such that $\{0\} \subset L' \subset L$. For a 0-minimal right ideal of $S$ can be defined similarly.
A nonempty subset \( A \) of an ordered semigroup \( (S, \cdot, \leq) \) is called a subsemigroup of \( S \) if \( AA \subseteq A \), that is, \( xy \in A \) for all \( x, y \in A \).

The concept of \((m, n)\)-ideal in semigroups was introduced as a generalization of bi-ideals by Lajos [4]. Sanborisoot and the author [5] defined \((m, n)\)-ideal in ordered semigroups as follows:

Let \( m, n \) be non-negative integers. A subsemigroup \( A \) of an ordered semigroup \( (S, \cdot, \leq) \) is called an \((m, n)\)-ideal of \( S \) if the following conditions hold:

(i) \( A^m S A^n \subseteq A \);

(ii) \( \{A\} = A \), that is, for \( x \in A \) and \( y \in S \), \( y \leq x \) implies \( y \in A \).

In particular, \( A \) is called a bi-ideal of \( S \) if \( m = n = 1 \). It is clear that if \( A \) is a bi-ideal of \( S \), then \( S \) is an \((m, n)\)-ideal of \( S \). It was proved in [5] that the principal \((m, n)\)-ideal generated by \( a \in S \) is of the form

\[
[a]_{m,n} = (a \cup a^2 \cup \cdots \cup a^{m+n} \cup a^m S a^n]
\]

We now define a relation \( \mathcal{B}^m_n \), where \( m, n \) are non-negative integers, on an ordered semigroup \( (S, \cdot, \leq) \) as follows: For \( a, b \in S \), let \( a \mathcal{B}^m_n b \) if and only if either \( a = b \) or

\[ a \leq b^m v b^n \text{ and } b \leq a^m u a^n \text{ for some } u, v \in S, \]

that is,

\[ a \in (b^m S b^n] \text{ and } b \in (a^m S a^n]. \]

The following three results are comparable, respectively, to Lemma 1.2-1.3 and Corollary 1.4 in [6].

**Proposition 2.** The relation \( \mathcal{B}^m_n \) is an equivalence relation on an ordered semigroup \( (S, \cdot, \leq) \).

**Proof.** To show that \( \mathcal{B}^m_n \) is transitive, let \( a, b, c \in S \) be such that \( a \mathcal{B}^m_n b \) and \( b \mathcal{B}^m_n c \). There are four cases to consider:

(a) \( a = b, b = c \);

(b) \( a = b, b \in (c^m S c^n], c \in (b^m S b^n] \);

(c) \( b = c, a \in (b^m S b^n], b \in (a^m S a^n] \);

(d) \( a \in (b^m S b^n], b \in (a^m S a^n], b \in (c^m S c^n], c \in (b^m S b^n] \).
If (a), (b) or (c) hold, then, by definition of $B^n_m$, $aB^n_m c$. If (d) holds, then we have
\[ a \in (b^nSb^n] \subseteq (((c^nS c^n])^n S ((c^nS c^n])^n] \subseteq (c^nS c^n], \]
\[ c \in (b^nSb^n] \subseteq (((a^nS a^n])^n S ((a^nS a^n])^n] \subseteq (a^nS a^n], \]
that is, $aB^n_m c$.

The rest of the proof is easy to see, hence $B^n_m$ is an equivalence relation on $S$. \hfill \Box

**Proposition 3.** If $A$ is an $(m, n)$-ideal of an ordered semigroup $(S, \cdot, \leq)$, then
\[ A = \bigcup_{a \in A} B^n_m(a) \]
where $B^n_m(a)$ denotes the $B^n_m$-class containing $a \in S$.

**Proof.** Assume that $A$ is an $(m, n)$-ideal of an ordered semigroup $(S, \cdot, \leq)$. By Proposition 2, $A \subseteq \bigcup_{a \in A} B^n_m(a)$. For the reverse inclusion, let $x \in \bigcup_{a \in A} B^n_m(a)$. That is $x \in B^n_m(a)$ for some $a \in A$. We now have
\[ x \in (a^nS a^n] \subseteq (A^nS A^n] \subseteq (A] = A. \]
Thus $\bigcup_{a \in A} B^n_m(a) \subseteq A$. \hfill \Box

An $(m, n)$-ideal $A$ of an ordered semigroup $(S, \cdot, \leq)$ with zero 0 is said to be 0-minimal if there is no an $(m, n)$-ideal $A'$ of $S$ such that $\{0\} \subset A' \subset A$.

**Corollary 4.** If $B$ is an $(m, n)$-ideal of an ordered semigroup $(S, \cdot, \leq)$ with zero 0 such that $B$ is a single non-zero $B^n_m$-class union $\{0\}$, then $B$ is a 0-minimal $(m, n)$-ideal of $S$.

**Proof.** This follows directly from Proposition 3. \hfill \Box
2. Main Results

We begin with the following lemma:

**Lemma 5.** Let \((S, \cdot, \leq)\) be an ordered semigroup with zero 0. For any \(a, b \in S\), \(aB_m b\) if and only if \([a]_{m,n} = [b]_{m,n}\).

**Proof.** It is clear that if \(aB_m b\), then \([a]_{m,n} = [b]_{m,n}\).

Conversely, assume that \([a]_{m,n} = [b]_{m,n}\), that is

\[
(a \cup a^2 \cup \cdots \cup a^{m+n} \cup a^mSa^n) = (b \cup b^2 \cup \cdots \cup b^{m+n} \cup b^mSb^n).
\]

Let \(a \neq b\). There are four cases to consider:

Case 1: \(a \leq b^k\) for some \(1 < k \leq m+n\) and \(b \leq a^l\) for some \(1 < l \leq m+n\).

We have \(a \in (b^mSb^n)\) and \(b \in (a^m Sa^n)\).

Case 2: \(a \leq b^k\) for some \(1 < k \leq m+n\) and \(b \in (a^m Sa^n)\). We have \(a \in (b^mSb^n)\) and \(b \in (a^m Sa^n)\).

Case 3: \(a \in (b^mSb^n)\) and \(b \leq a^l\) for some \(1 < l \leq m+n\). We have \(a \in (b^mSb^n)\) and \(b \in (a^m Sa^n)\).

Case 4: \(a \in (b^mSb^n)\) and \(b \in (a^m Sa^n)\).

Each of the cases implies that \(aB_m b\). \(\square\)

We now characterize 0-minimal \((m, n)\)-ideals of an ordered semigroup compared to Theorem 1.7 in [6].

**Theorem 6.** Let \((S, \cdot, \leq)\) be an ordered semigroup with zero 0. Then an \((m, n)\)-ideal \(A\) of \(S\) is 0-minimal if and only if \(A\) is one non-zero \(B_m\)-class union \(\{0\}\).

**Proof.** Assume that an \((m, n)\)-ideal \(A\) of \(S\) is 0-minimal. Let \(a, b \in A \setminus \{0\}\) be such that \(a \neq b\). Since \(\{0\} \subset [a]_{m,n} \subset A\) and the minimality of \(A\), we have \([a]_{m,n} = A\). Similarly, \([b]_{m,n} = A\). Thus \([a]_{m,n} = [b]_{m,n}\). By Lemma 5, \(aB_m b\).

The opposite direction follows by Corollary 4. \(\square\)

**Theorem 7.** Let \(m, n\) be non-negative integers such that \(m, n \geq 1\). If \(I\) is a 0-minimal \((m, n)\)-ideal of an ordered semigroup \((S, \cdot, \leq)\) with zero 0 such that \((I^2) \neq \{0\}\), then \(I\) is a 0-minimal bi-ideal of \(S\).

**Proof.** If \(\{0\} \subset J \subset I\) for some bi-ideal \(J\) of \(S\), then \(J = I\) since \(J\) is an \((m, n)\)-ideal of \(S\). We have \(I\) is a 0-minimal bi-ideal of \(S\).

Assume that there is no a bi-ideal \(J\) of \(S\) such that \(\{0\} \subset J \subset I\). We have \(\{0\} \subset (I^2) \subset I\). Since \((I^2)\) is an \((m, n)\)-ideal of \(S\), it follows by the minimality of \(I\) that \((I^2) = I\). Since
\[ ISI = (I^2)S(I^2) \subseteq (I^2SI^2) \subseteq I \]

and the assumption, we conclude that \( I \) is a 0-minimal bi-ideal of \( S \).

An ordered semigroup \((S, \cdot, \leq)\) with zero is said to be \textit{nilpotent} if \( S^l = 0 \) for some positive integer \( l \).

The following four propositions are comparable, respectively, to Proposition 2.1-2.4 in [2].

**Theorem 8.** Let \((S, \cdot, \leq)\) be an ordered semigroup with zero 0. Assume that \( S \) contains no non-zero nilpotent \((m, n)\)-ideals. If \( R \) (respectively, \( L \)) is a 0-minimal right (respectively, left) ideal of \( S \), then \((RL) = \{0\} \) or \((RL)\) is a 0-minimal \((m, n)\)-ideal of \( S \).

**Proof.** Assume that \( R \) (respectively, \( L \)) is a 0-minimal right (respectively, left) ideal of \( S \) such that \((RL) \neq \{0\} \). We have

\[
(RL)(RL) \subseteq (RLRL) \subseteq (RSL) \subseteq (RL),
\]

\[
(RL)S(RL) \subseteq (RLSRL) \subseteq (RSL) \subseteq (RL).
\]

Then \((RL)\) is a bi-ideal of \( S \), and hence \((RL)\) is an \((m, n)\)-ideal of \( S \). To show that \((RL)\) is a 0-minimal \((m, n)\)-ideal of \( S \), let \( A \) be an \((m, n)\)-ideal of \( S \) such that \( \{0\} \subset A \subseteq (RL) \). We have \( A^m \neq \{0\} \) and \( A^n \neq \{0\} \). Since \((RL) \subseteq R \cap L \), so \( A \subseteq R \cap L \). Since \( \{0\} \subset (A^m \cup A^mS) \subseteq R \), it follows by the minimality of \( R \) that \( (A^m \cup A^mS) = R \). Similarly, \( (A^n \cup AS^n) = L \). We have

\[
A \subseteq (RL) = ((A^m \cup A^mS)(A^n \cup AS^n))
\]

\[
= ((A^m \cup A^mS)(A^n \cup AS^n))
\]

\[
\subseteq (A^mSA^n) \subseteq A.
\]

Hence \( A = (RL) \).

If \((S, \cdot, \leq)\) is an ordered semigroup, the \textit{center} of \( S \) is defined by

\[
C(S) = \{ x \in S \mid \forall y \in S, xy = yx \}.
\]

**Theorem 9.** Let \((S, \cdot, \leq)\) be an ordered semigroup with zero 0. If \( R \) (respectively, \( L \)) is a 0-minimal right (respectively, left) ideal of \( S \) such that \((R^mL^n) \subseteq C(S)\), then \((R^mL^n) = \{0\} \) or \((R^mL^n)\) is a 0-minimal \((m, n)\)-ideal of \( S \).
Proof. Assume that $R$ (respectively, $L$) is a 0-minimal right (respectively, left) ideal of $S$ such that $(R^mL^n) \subseteq C(S)$. Let $(R^mL^n) \neq \{0\}$. Then $R^m \neq \{0\}$ and $L^n \neq \{0\}$. Since $\{0\} \subseteq (R^m) \subseteq R$, it follows that $(R^m) = R$. Similarly, $(L^n) = L$. We have

$$(R^mL^n) = ((R^m)(L^n)) = (RL)$$

is a bi-ideal of $S$, and hence $(R^mL^n)$ is an $(m,n)$-ideal of $S$.

To show that $(R^mL^n)$ is 0-minimal, let $A$ be an $(m,n)$-ideal of $S$ such that $\{0\} \subseteq A \subseteq (R^mL^n)$. Since $(RL) \subseteq R \cap L$, it follows that $A \subseteq R$ and $A \subseteq L$. Since $(A \cup AS) \subseteq (R \cup RS) \subseteq R$, we have $(A \cup AS) = R$. Similarly, $(A \cup SA) = L$. Since $(R^mL^n) \subseteq C(S)$, we get

$$A \subseteq (R^mL^n) = (((A \cup AS))^m((A \cup SA))^{n}]$$

$$\subseteq ((A \cup AS)^m(A \cup SA)^n]$$

$$\subseteq (A^{m+n} \cup A^mSA^n] \subseteq A.$$ 

Thus $A = (R^mL^n)$. Hence $(R^mL^n)$ is a 0-minimal $(m,n)$-ideal of $S$. \hfill \square

An ordered semigroup $(S, \cdot, \leq)$ is said to be $(m,n)$-regular \cite{5} if for any $a \in S$, $a \leq a^mxa^n$ for some $x \in S$. If every element of $S$ is $(m,n)$-regular, then $S$ is called an $(m,n)$-regular ordered semigroup. It is known that $S$ is $(m,n)$-regular if and only if for any ideal $I$ of $S$, $I \subseteq (I^mSI^n]$.

We now consider 0-minimal $(m,n)$-ideals for regular ordered semigroups.

**Theorem 10.** Let $(S, \cdot, \leq)$ be an $(m,n)$-regular ordered semigroup with zero 0. If $A$ (respectively, $B$) is a 0-minimal $(m,0)$-ideal (respectively, $(0,n)$-ideal) of $S$ such that $(AB) \subseteq A \cap B$, then $(AB) = \{0\}$ or $(AB)$ is a 0-minimal $(m,n)$-ideal of $S$.

**Proof.** Assume that $A$ (respectively, $B$) is a 0-minimal $(m,0)$-ideal (respectively, $(0,n)$-ideal) of $S$ such that $(AB) \subseteq A \cap B$. Let $(AB) \neq \{0\}$. We have to show that $(AB)$ is a 0-minimal $(m,n)$-ideal of $S$. Since $(AB)(AB) \subseteq AB \subseteq (AB)$ and

$$(AB)^mS(AB)^n \subseteq A^mS(AB)^n \subseteq AB^n \subseteq AB \subseteq (AB),$$

we have $(AB)$ is an $(m,n)$-ideal of $S$.

To show that $(AB)$ is 0-minimal, let $C$ be an $(m,n)$-ideal of $S$ such that $\{0\} \subseteq C \subseteq (AB)$. Then $C \subseteq A, C \subseteq B$. Since $C \subseteq (C^mSC^n]$, it follows that $(C^mS) \neq \{0\}$ and $(SC^n] \neq \{0\}$. Since $(C^mS) \subseteq (A^mS) \subseteq A$, so $(C^mS) = A$. Similarly, $(SC^n] = B$. We have
\[ C \subseteq (AB) \subseteq ((C^m S)(SC^n)] = (C^m SC^n] \subseteq C. \]

Thus \( C = (AB) \), and hence \((AB)\) is 0-minimal. \( \square \)

**Theorem 11.** Let \((S, \cdot, \leq)\) be an \((m, n)\)-regular ordered semigroup with zero 0. If \( A \) (respectively, \( B \)) is a 0-minimal \((m, 0)\)-ideal (respectively, \((0, n)\)-ideal) of \( S \), then \( A \cap B = \{0\} \) or \( A \cap B \) is a 0-minimal \((m, n)\)-ideal of \( S \).

**Proof.** Assume that \( A \) (respectively, \( B \)) is a 0-minimal \((m, 0)\)-ideal (respectively, \((0, n)\)-ideal) of \( S \) such that \( A \cap B = \{0\} \). We shall show that \( A \cap B \) is an \((m, n)\)-ideal of \( S \). It is clear that \((A \cap B)(A \cap B) \subseteq A \cap B \). We have

\[
(A \cap B)^m S(A \cap B)^n \subseteq (A^m S)B^n \subseteq AB^n \subseteq B,
\]

\[
(A \cap B)^m S(A \cap B)^n \subseteq A^m (SB^n) \subseteq A^m B \subseteq A,
\]

hence \((A \cap B)^m S(A \cap B)^n \subseteq A \cap B \). Therefore \( A \cap B \) is an \((m, n)\)-ideal of \( S \). The rest of the proof is similar to the proof of Theorem 10. \( \square \)

**References**


