ON 0-MINIMAL (m, n) -IDEALS IN AN ORDERED SEMIGROUP

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Abstract: In this paper, we introduce an equivalence relation $\mathcal{B}$ on an ordered semigroup S . Using the relation $\mathcal{B}$ defined, we characterize 0-minimal (m, n) -ideals of S . Several sufficient conditions are presented for 0-minimal left ideals and 0-minimal right ideals of S to be 0-minimal (m, n) -ideals of S . The results obtained generalize the results on semigroups without order.

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1. Preliminaries

Let S be a semigroup-without order-. Kapp [2] introduced an equivalence relation $\mathcal{B}$ on S by, for $a, b \in S$, $a\mathcal{B}b$ if $a = b$ or $a \in bSb$ and $b \in aSa$; using this relation the author characterized 0-minimal bi-ideal of S . Tilidetzke [6] generalized Kapp's results; the author introduced an equivalence relation $\mathcal{B}_m^n$ on S where m, n are non-negative integers by, for $a, b \in S$, $a\mathcal{B}_m^n b$ if $a = b$ or $a \in b^m S b^n$ and $b \in a^m S a^n$; using the relation $\mathcal{B}_m^n$, 0-minimal (m, n) -ideals of S are characterized. The purpose of this paper is to extend Tilidetzke's results based on ordered semigroups. The results obtained both in [2] and [6] become then special cases.

Let us now recall some definitions and results used throughout the paper.

An *ordered semigroup* [1] is defined to be a semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that for $x, y, z \in S$,

$$x \leq y \Rightarrow z \cdot x \leq z \cdot y, x \cdot z \leq y \cdot z.$$

In this paper, we write $(S, \cdot, \leq)$ for an ordered semigroup $(S, \cdot)$ with a partial order $\leq$ and write xy ($x, y \in S$) for $x \cdot y$.

If A and B are nonempty subsets of an ordered semigroup $(S, \cdot, \leq)$, we write

$$AB = \{xy \mid x \in A, y \in B\},$$

$$[A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

For $x \in S$, we write Ax and xA for $A\{x\}$ and $\{x\}A$, respectively.

Lemma 1. ([3]) *The following statements hold for any nonempty subsets A, B of an ordered semigroup $(S, \cdot, \leq)$:*

- (1) $A \subseteq [A]$;
- (2) $A \subseteq B \Rightarrow [A] \subseteq [B]$;
- (3) $[A][B] \subseteq [AB]$;
- (4) $(([A])[B]) = [AB]$;
- (5) $(A \cup B) = [A] \cup [B]$.

A nonempty subset A of an ordered semigroup $(S, \cdot, \leq)$ is called a *left* (respectively, *right*) *ideal* of S if the following conditions hold:

- (i) $SA \subseteq A$ (respectively, $AS \subseteq A$);
- (ii) $A = [A]$, that is, for $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.

If A is both a left and a right ideal of S , then A is called a (two-sided) *ideal* of S .

If there is an element 0 of an ordered semigroup $(S, \cdot, \leq)$ such that $x0 = 0x = 0$ for all $x \in S$ and $0 \leq x$ for all $x \in S$, we call 0 a *zero* element of S .

A left ideal L of an ordered semigroup $(S, \cdot, \leq)$ with zero 0 is said to be *0-minimal* if there is no a left ideal L' of S such that $\{0\} \subset L' \subset L$. For a 0-minimal right ideal of S can be defined similarly.

A nonempty subset A of an ordered semigroup $(S, \cdot, \leq)$ is called a *subsemigroup* of S if $AA \subseteq A$, that is, $xy \in A$ for all $x, y \in A$.

The concept of (m, n) -ideal in semigroups was introduced as a generalization of bi-ideals by Lajos [4]. Sanborisoot and the author [5] defined (m, n) -ideal in ordered semigroups as follows:

Let m, n be non-negative integers. A subsemigroup A of an ordered semigroup $(S, \cdot, \leq)$ is called an (m, n) -ideal of S if the following conditions hold:

- (i) $A^m S A^n \subseteq A$;
- (ii) $(A] = A$, that is, for $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.

In particular, A is called a *bi-ideal* of S if $m = n = 1$. It is clear that if A is a bi-ideal of S , then S is an (m, n) -ideal of S . It was proved in [5] that the principal (m, n) -ideal generated by $a \in S$ is of the form

$$[a]_{m,n} = (a \cup a^2 \cup \dots \cup a^{m+n} \cup a^m S a^n)$$

We now define a relation $\mathcal{B}_m^n$, where m, n are non-negative integers, on an ordered semigroup $(S, \cdot, \leq)$ as follows: For $a, b \in S$, let $a \mathcal{B}_m^n b$ if and only if either $a = b$ or

$$a \leq b^m v b^n \text{ and } b \leq a^m u a^n \text{ for some } u, v \in S,$$

that is,

$$a \in (b^m S b^n] \text{ and } b \in (a^m S a^n].$$

The following three results are comparable, respectively, to Lemma 1.2-1.3 and Corollary 1.4 in [6].

Proposition 2. *The relation $\mathcal{B}_m^n$ is an equivalence relation on an ordered semigroup $(S, \cdot, \leq)$.*

Proof. To show that $\mathcal{B}_m^n$ is transitive, let $a, b, c \in S$ be such that $a \mathcal{B}_m^n b$ and $b \mathcal{B}_m^n c$. There are four cases to consider:

- (a) $a = b, b = c$;
- (b) $a = b, b \in (c^m S c^n], c \in (b^m S b^n]$;
- (c) $b = c, a \in (b^m S b^n], b \in (a^m S a^n]$;
- (d) $a \in (b^m S b^n], b \in (a^m S a^n], b \in (c^m S c^n], c \in (b^m S b^n]$.

If (a), (b) or (c) hold, then, by definition of $\mathcal{B}_m^n$, $a\mathcal{B}_m^n c$. If (d) holds, then we have

$$a \in (b^m S b^n] \subseteq (((c^m S c^n])^m S ((c^m S c^n])^n] \subseteq (c^m S c^n],$$

$$c \in (b^m S b^n] \subseteq (((a^m S a^n])^m S ((a^m S a^n])^n] \subseteq (a^m S a^n],$$

that is, $a\mathcal{B}_m^n c$.

The rest of the proof is easy to see, hence $\mathcal{B}_m^n$ is an equivalence relation on S . $\square$

Proposition 3. *If A is an (m, n) -ideal of an ordered semigroup $(S, \cdot, \leq)$, then*

$$A = \bigcup_{a \in A} B_m^n(a)$$

where $B_m^n(a)$ denotes the $\mathcal{B}_m^n$ -class containing $a \in S$.

Proof. Assume that A is an (m, n) -ideal of an ordered semigroup $(S, \cdot, \leq)$. By Proposition 2, $A \subseteq \bigcup_{a \in A} B_m^n(a)$. For the reverse inclusion, let $x \in \bigcup_{a \in A} B_m^n(a)$. That is $x \in B_m^n(a)$ for some $a \in A$. We now have

$$x \in (a^m S a^n] \subseteq (A^m S A^n] \subseteq (A] = A.$$

Thus $\bigcup_{a \in A} B_m^n(a) \subseteq A$. $\square$

An (m, n) -ideal A of an ordered semigroup $(S, \cdot, \leq)$ with zero 0 is said to be 0-minimal if there is no an (m, n) -ideal A' of S such that $\{0\} \subset A' \subset A$.

Corollary 4. *If B is an (m, n) -ideal of an ordered semigroup $(S, \cdot, \leq)$ with zero 0 such that B is a single non-zero $\mathcal{B}_m^n$ -class union $\{0\}$, then B is a 0-minimal (m, n) -ideal of S .*

Proof. This follows directly from Proposition 3. $\square$

2. Main Results

We begin with the following lemma:

Lemma 5. *Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0. For any $a, b \in S$, $a\mathcal{B}_m^n b$ if and only if $[a]_{m,n} = [b]_{m,n}$.*

Proof. It is clear that if $a\mathcal{B}_m^n b$, then $[a]_{m,n} = [b]_{m,n}$.

Conversely, assume that $[a]_{m,n} = [b]_{m,n}$, that is

$$(a \cup a^2 \cup \dots \cup a^{m+n} \cup a^m S a^n) = (b \cup b^2 \cup \dots \cup b^{m+n} \cup b^m S b^n).$$

Let $a \neq b$. There are four cases to consider:

Case 1: $a \leq b^k$ for some $1 < k \leq m+n$ and $b \leq a^l$ for some $1 < l \leq m+n$. We have $a \in (b^m S b^n)$ and $b \in (a^m S a^n)$.

Case 2: $a \leq b^k$ for some $1 < k \leq m+n$ and $b \in (a^m S a^n)$. We have $a \in (b^m S b^n)$ and $b \in (a^m S a^n)$.

Case 3: $a \in (b^m S b^n)$ and $b \leq a^l$ for some $1 < l \leq m+n$. We have $a \in (b^m S b^n)$ and $b \in (a^m S a^n)$.

Case 4: $a \in (b^m S b^n)$ and $b \in (a^m S a^n)$.

Each of the cases implies that $a\mathcal{B}_m^n b$. □

We now characterize 0-minimal (m, n) -ideals of an ordered semigroup compared to Theorem 1.7 in [6].

Theorem 6. *Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0. Then an (m, n) -ideal A of S is 0-minimal if and only if A is one non-zero $\mathcal{B}_m^n$ -class union $\{0\}$,*

Proof. Assume that an (m, n) -ideal A of S is 0-minimal. Let $a, b \in A \setminus \{0\}$ be such that $a \neq b$. Since $\{0\} \subset [a]_{m,n} \subseteq A$ and the minimality of A , we have $[a]_{m,n} = A$. Similarly, $[b]_{m,n} = A$. Thus $[a]_{m,n} = [b]_{m,n}$. By Lemma 5, $a\mathcal{B}_m^n b$. The opposite direction follows by Corollary 4. □

Theorem 7. *Let m, n be non-negative integers such that $m, n \geq 1$. If I is a 0-minimal (m, n) -ideal of an ordered semigroup $(S, \cdot, \leq)$ with zero 0 such that $(I^2] \neq \{0\}$, then I is a 0-minimal bi-ideal of S .*

Proof. If $\{0\} \subset J \subseteq I$ for some bi-ideal J of S , then $J = I$ since J is an (m, n) -ideal of S . We have I is a 0-minimal bi-ideal of S .

Assume that there is no a bi-ideal J of S such that $\{0\} \subset J \subseteq I$. We have $\{0\} \subset (I^2] \subseteq I$. Since $(I^2]$ is an (m, n) -ideal of S , it follows by the minimality of I that $(I^2] = I$. Since

$$ISI = (I^2]S(I^2] \subseteq (I^2SI^2] \subseteq I$$

and the assumption, we conclude that I is a 0-minimal bi-ideal of S . $\square$

An ordered semigroup $(S, \cdot, \leq)$ with zero is said to be *nilpotent* if $S^l = 0$ for some positive integer l .

The following four propositions are comparable, respectively, to Proposition 2.1-2.4 in [2].

Theorem 8. *Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0. Assume that S contains no non-zero nilpotent (m, n) -ideals. If R (respectively, L) is a 0-minimal right (respectively, left) ideal of S , then $(RL] = \{0\}$ or $(RL]$ is a 0-minimal (m, n) -ideal of S .*

Proof. Assume that R (respectively, L) is a 0-minimal right (respectively, left) ideal of S such that $(RL] \neq \{0\}$. We have

$$(RL](RL] \subseteq (RLRL] \subseteq (RSL] \subseteq (RL],$$

$$(RL]S(RL] \subseteq (RLSRL] \subseteq (RSL] \subseteq (RL].$$

Then $(RL]$ is a bi-ideal of S , and hence $(RL]$ is an (m, n) -ideal of S . To show that $(RL]$ is a 0-minimal (m, n) -ideal of S , let A be an (m, n) -ideal of S such that $\{0\} \subset A \subseteq (RL]$. We have $A^m \neq \{0\}$ and $A^n \neq \{0\}$. Since $(RL] \subseteq R \cap L$, so $A \subseteq R \cap L$. Since $\{0\} \subset (A^m \cup A^m S] \subseteq R$, it follows by the minimality of R that $(A^m \cup A^m S] = R$. Similarly, $(A^n \cup SA^n] = L$. We have

$$\begin{aligned} A \subseteq (RL] &= ((A^m \cup A^m S](A^n \cup SA^n]) \\ &= ((A^m \cup A^m S)(A^n \cup SA^n]) \\ &\subseteq (A^m SA^n] \subseteq A. \end{aligned}$$

Hence $A = (RL]$. $\square$

If $(S, \cdot, \leq)$ is an ordered semigroup, the *center* of S is defined by

$$C(S) = \{x \in S \mid \forall y \in S, xy = yx\}.$$

Theorem 9. *Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0. If R (respectively, L) is a 0-minimal right (respectively, left) ideal of S such that $(R^m L^n] \subseteq C(S)$, then $(R^m L^n] = \{0\}$ or $(R^m L^n]$ is a 0-minimal (m, n) -ideal of S .*

Proof. Assume that R (respectively, L) is a 0-minimal right (respectively, left) ideal of S such that $(R^m L^n] \subseteq C(S)$. Let $(R^m L^n] \neq \{0\}$. Then $R^m \neq \{0\}$ and $L^n \neq \{0\}$. Since $\{0\} \subset (R^m] \subseteq R$, it follows that $(R^m] = R$. Similarly, $(L^n] = L$. We have

$$(R^m L^n] = ((R^m](L^n]) = (RL]$$

is a bi-ideal of S , and hence $(R^m L^n]$ is an (m, n) -ideal of S .

To show that $(R^m L^n]$ is 0-minimal, let A be an (m, n) -ideal of S such that $\{0\} \subset A \subseteq (R^m L^n]$. Since $(RL] \subseteq R \cap L$, it follows that $A \subseteq R$ and $A \subseteq L$. Since $(A \cup AS] \subseteq (R \cup RS] \subseteq R$, we have $(A \cup AS] = R$. Similarly, $(A \cup SA] = L$. Since $(R^m L^n] \subseteq C(S)$, we get

$$\begin{aligned} A \subseteq (R^m L^n] &= (((A \cup AS])^m ((A \cup SA])^n) \\ &\subseteq ((A \cup AS])^m (A \cup SA)^n \\ &\subseteq (A^{m+n} \cup A^m S A^n] \subseteq A. \end{aligned}$$

Thus $A = (R^m L^n]$. Hence $(R^m L^n]$ is a 0-minimal (m, n) -ideal of S . $\square$

An ordered semigroup $(S, \cdot, \leq)$ is said to be (m, n) -regular [5] if for any $a \in S$, $a \leq a^m x a^n$ for some $x \in S$. If every element of S is (m, n) -regular, then S is called an (m, n) -regular ordered semigroup. It is known that S is (m, n) -regular if and only if for any ideal I of S , $I \subseteq (I^m S I^n]$.

We now consider 0-minimal (m, n) -ideals for regular ordered semigroups.

Theorem 10. *Let $(S, \cdot, \leq)$ be an (m, n) -regular ordered semigroup with zero 0. If A (respectively, B) is a 0-minimal $(m, 0)$ -ideal (respectively, $(0, n)$ -ideal) of S such that $(AB] \subseteq A \cap B$, then $(AB] = \{0\}$ or $(AB]$ is a 0-minimal (m, n) -ideal of S .*

Proof. Assume that A (respectively, B) is a 0-minimal $(m, 0)$ -ideal (respectively, $(0, n)$ -ideal) of S such that $(AB] \subseteq A \cap B$. Let $(AB] \neq \{0\}$. We have to show that $(AB]$ is a 0-minimal (m, n) -ideal of S . Since $(AB](AB] \subseteq AB \subseteq (AB]$ and

$$(AB]^m S (AB]^n \subseteq A^m S (AB]^n \subseteq AB^n \subseteq AB \subseteq (AB],$$

we have $(AB]$ is an (m, n) -ideal of S .

To show that $(AB]$ is 0-minimal, let C be an (m, n) -ideal of S such that $\{0\} \subset C \subseteq (AB]$. Then $C \subseteq A, C \subseteq B$. Since $C \subseteq (C^m S C^n]$, it follows that $(C^m S] \neq \{0\}$ and $(S C^n] \neq \{0\}$. Since $(C^m S] \subseteq (A^m S] \subseteq A$, so $(C^m S] = A$. Similarly, $(S C^n] = B$. We have

$$C \subseteq (AB] \subseteq ((C^m S](SC^n]) = (C^m SC^n] \subseteq C.$$

Thus $C = (AB]$, and hence $(AB]$ is 0-minimal. $\square$

Theorem 11. *Let $(S, \cdot, \leq)$ be an (m, n) -regular ordered semigroup with zero 0. If A (respectively, B) is a 0-minimal $(m, 0)$ -ideal (respectively, $(0, n)$ -ideal) of S , then $A \cap B = \{0\}$ or $A \cap B$ is a 0-minimal (m, n) -ideal of S .*

Proof. Assume that A (respectively, B) is a 0-minimal $(m, 0)$ -ideal (respectively, $(0, n)$ -ideal) of S such that $A \cap B = \{0\}$. We shall show that $A \cap B$ is an (m, n) -ideal of S . It is clear that $(A \cap B)(A \cap B) \subseteq A \cap B$. We have

$$(A \cap B)^m S (A \cap B)^n \subseteq (A^m S) B^n \subseteq AB^n \subseteq B,$$

$$(A \cap B)^m S (A \cap B)^n \subseteq A^m (SB^n) \subseteq A^m B \subseteq A,$$

hence $(A \cap B)^m S (A \cap B)^n \subseteq A \cap B$. Therefore $A \cap B$ is an (m, n) -ideal of S . The rest of the proof is similar to the proof of Theorem 10. $\square$

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