ON THE DIOPHANTINE EQUATION $47^x + 49^y = z^2$

Banyat Sroysang
Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University
Rangsit Center, Pathumthani, 12121, THAILAND

Abstract: In this paper, we prove that the Diophantine equation $47^x + 49^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers.

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1. Introduction

Sroysang [4] proved, in 2012, that $(1, 1, 9)$ is a unique solution $(x, y, z)$ for the Diophantine equation $32^x + 49^y = z^2$ where $x$, $y$ and $z$ are non-negative integers. Moreover, he [5] proved, in 2012, that $(1, 0, 2)$ is a unique solution $(x, y, z)$ for the Diophantine equation $3^x + 5^y = z^2$ where $x$, $y$ and $z$ are non-negative integers. In this paper, we prove that the Diophantine equation $47^x + 49^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. Many similar problems were solved as follows.

In 2007, Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two non-negative integer solutions for the Diophantine equation $2^x + 5^y = z^2$ where $x$, $y$ and $z$ are non-negative integers.
In 2011, Suvarnamani, Singta and Chotchaisthit [11] proved that no non-negative integer solution for the two Diophantine equations \(4^x + 7^y = z^2\) and \(4^x + 11^y = z^2\) where \(x, y\) and \(z\) are non-negative integers.

In 2012-2013, Sroysang [6, 7, 8, 9, 10] proved that (i) \((1, 0, 3)\) is a unique solution \((x, y, z)\) for the Diophantine equation \(8^x + 19^y = z^2\), (ii) the Diophantine equation \(31^x + 32^y = z^2\) has no non-negative integer solution, (iii) \((0,1,3)\) is a unique solution \((x, y, z)\) for the Diophantine equation \(7^x + 8^y = z^2\), (iv) \((0,1,2), (3,0,3)\) and \((4,2,5)\) are only three non-negative integer solutions for the Diophantine equation \(2^x + 3^y = z^2\), and (v) the Diophantine equation \(23^x + 32^y = z^2\) has no non-negative integer solution, where \(x, y\) and \(z\) are non-negative integers.

Recently, Chotchaisthit [2] proved that \((3,0,3)\) is a unique solution \((x, y, z)\) for the Diophantine equation \(2^x + 11^y = z^2\) where \(x, y\) and \(z\) are non-negative integers.

2. Preliminaries

**Proposition 2.1.** [3] (the Catalan’s conjecture) \((3,2,2,3)\) is a unique solution \((a, b, x, y)\) for the Diophantine equation \(a^x - b^y = 1\) where \(a, b, x\) and \(y\) are integers with \(\min\{a, b, x, y\} > 1\).

**Lemma 2.2.** The Diophantine equation \(47^x + 1 = z^2\) has no non-negative integer solution where \(x\) and \(z\) are non-negative integers.

**Proof.** Suppose that there are non-negative integers \(x\) and \(z\) such that \(47^x + 1 = z^2\). If \(x = 0\), then \(z^2 = 2\) which is impossible. Then \(x \geq 1\). Thus, \(z^2 = 47^x + 1 \geq 47^1 + 1 = 48\). Then \(z \geq 7\). Now, we consider on the equation \(z^2 - 47^x = 1\). By Proposition 2.1, we have \(x = 1\). Then \(z^2 = 48\). This is a contradiction. Hence, the equation \(47^x + 1 = z^2\) has no non-negative integer solution. \(\square\)

**Lemma 2.3.** [4] The Diophantine equation \(1 + 49^y = z^2\) has no non-negative integer solution where \(y\) and \(z\) are non-negative integers.

3. Results

**Theorem 3.1.** The Diophantine equation \(47^x + 49^y = z^2\) has no non-negative integer solution where \(x, y\) and \(z\) are non-negative integers.
ON THE DIOPHANTINE EQUATION $47^x + 49^y = z^2$ 281

Proof. Suppose that there are non-negative integers $x, y$ and $z$ such that $47^x + 49^y = z^2$. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. Note that $z$ is even. Then $z^2 \equiv 0 \pmod{6}$ or $z^2 \equiv 4 \pmod{6}$. Moreover, $49^y \equiv 1 \pmod{6}$. It follows that $47^x \equiv 5 \pmod{6}$ or $47^x \equiv 3 \pmod{6}$. Note that $47^x \equiv 5^x \pmod{6}$. But $5^x \equiv 1 \pmod{6}$ or $5^x \equiv 5 \pmod{6}$. This implies that $47^x \equiv 5^x \equiv 5 \pmod{6}$. Thus, $x$ is odd. Then $47^x \equiv 3 \pmod{7}$ or $47^x \equiv 5 \pmod{7}$ or $47^x \equiv 6 \pmod{7}$. Note that $49^y \equiv 0 \pmod{7}$. It follows that $z^2 \equiv 3 \pmod{7}$ or $z^2 \equiv 5 \pmod{7}$ or $z^2 \equiv 6 \pmod{7}$. This is a contradiction since $z^2 \equiv 0 \pmod{7}$ or $z^2 \equiv 1 \pmod{7}$ or $z^2 \equiv 2 \pmod{7}$ or $z^2 \equiv 4 \pmod{7}$.

Corollary 3.2. Suppose that $u, v$ and $z$ are non-negative integers such that $7^u + 47^v = z^2$. Then $u$ is odd.

Proof. Suppose that $u$ is even. Then there is a non-negative integer $k$ such that $u = 2k$. Let $x = v$ and $y = k$. We will divide the number $y$ into two cases.

Case $y = 0$. Then $u = 0$. By Lemma 2.2, the equation $1 + 47^x = z^2$ has no non-negative integer solution. Thus, the equation $7^u + 47^v = z^2$ has no non-negative integer solution. This is a contradiction.

Case $y \geq 1$. By Theorem 3.1, the equation $49^y + 47^x = z^2$ has no non-negative integer solution. Thus, the equation $7^u + 47^v = z^2$ has no non-negative integer solution. This is a contradiction.

Therefore, $u$ is odd.

Corollary 3.3. The Diophantine equation $47^x + 49^y = w^4$ has no non-negative integer solution where $x, y$ and $w$ are non-negative integers.

Proof. Suppose that there are non-negative integers $x, y$ and $w$ such that $47^x + 49^y = w^4$. Let $z = w^2$. Then $47^x + 49^y = z^2$. By Theorem 3.1, the equation $47^x + 49^y = z^2$ has no non-negative integer solution. This implies that the equation $47^x + 49^y = w^4$ has no non-negative integer solution.

References


