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ON THE DIOPHANTINE EQUATION $47^x + 49^y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $47^x + 49^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

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1. Introduction

Sroysang [4] proved, in 2012, that $(1, 1, 9)$ is a unique solution (x, y, z) for the Diophantine equation $32^x + 49^y = z^2$ where x, y and z are non-negative integers. Moreover, he [5] proved, in 2012, that $(1, 0, 2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. In this paper, we prove that the Diophantine equation $47^x + 49^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. Many similar problems were solved as follows.

In 2007, Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two non-negative integer solutions for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers.

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In 2011, Suvarnamani, Singta and Chotchaisthit [11] proved that no non-negative integer solution for the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ where x, y and z are non-negative integers.

In 2012-2013, Sroysang [6, 7, 8, 9, 10] proved that (i) $(1, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $8^x+19^y = z^2$, (ii) the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution, (iii) $(0, 1, 3)$ is a unique solution (x, y, z) for the Diophantine equation $7^x + 8^y = z^2$, (iv) $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three non-negative integer solutions for the Diophantine equation $2^x + 3^y = z^2$, and (v) the Diophantine equation $23^x + 32^y = z^2$ has no non-negative integer solution, where x, y and z are non-negative integers.

Recently, Chotchaisthit [2] proved that $(3, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $2^x + 11^y = z^2$ where x, y and z are non-negative integers.

2. Preliminaries

Proposition 2.1. [3] (**the Catalan's conjecture**) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. *The Diophantine equation* $47^x + 1 = z^2$ *has no non-negative integer solution where* x *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers x and z such that $47^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $z^2 = 47^x + 1 \geq 47^1 + 1 = 48$. Then $z \geq 7$. Now, we consider on the equation $z^2 - 47^x = 1$. By Proposition 2.1, we have $x = 1$. Then $z^2 = 48$. This is a contradiction. Hence, the equation $47^x + 1 = z^2$ has no non-negative integer solution. □

Lemma 2.3. [4] *The Diophantine equation* $1 + 49^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

3. Results

Theorem 3.1. *The Diophantine equation* $47^x + 49^y = z^2$ *has no non-negative integer solution where* x, y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and z such that $47^x + 49^y = z^2$. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. Note that z is even. Then $z^2 \equiv 0 \pmod 6$ or $z^2 \equiv 4 \pmod 6$. Moreover, $49^y \equiv 1 \pmod 6$. It follows that $47^x \equiv 5 \pmod 6$ or $47^x \equiv 3 \pmod 6$. Note that $47^x \equiv 5^x \pmod 6$. But $5^x \equiv 1 \pmod 6$ or $5^x \equiv 5 \pmod 6$. This implies that $47^x \equiv 5^x \equiv 5 \pmod 6$. Thus, x is odd. Then $47^x \equiv 3 \pmod 7$ or $47^x \equiv 5 \pmod 7$ or $47^x \equiv 6 \pmod 7$. Note that $49^x \equiv 0 \pmod 7$. It follows that $z^2 \equiv 3 \pmod 7$ or $z^2 \equiv 5 \pmod 7$ or $z^2 \equiv 6 \pmod 7$. This is a contradiction since $z^2 \equiv 0 \pmod 7$ or $z^2 \equiv 1 \pmod 7$ or $z^2 \equiv 2 \pmod 7$ or $z^2 \equiv 4 \pmod 7$. □

Corollary 3.2. *Suppose that u, v and z are non-negative integers such that $7^u + 47^v = z^2$. Then u is odd.*

Proof. Suppose that u is even. Then there is a non-negative integer k such that $u = 2k$. Let $x = v$ and $y = k$. We will divide the number y into two cases.

Case $y = 0$. Then $u = 0$. By Lemma 2.2, the equation $1 + 47^x = z^2$ has no non-negative integer solution. Thus, the equation $7^u + 47^v = z^2$ has no non-negative integer solution. This is a contradiction.

Case $y \geq 1$. By Theorem 3.1, the equation $49^y + 47^x = z^2$ has no non-negative integer solution. Thus, the equation $7^u + 47^v = z^2$ has no non-negative integer solution. This is a contradiction.

Therefore, u is odd. □

Corollary 3.3. *The Diophantine equation $47^x + 49^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $47^x + 49^y = w^4$. Let $z = w^2$. Then $47^x + 49^y = z^2$. By Theorem 3.1, the equation $47^x + 49^y = z^2$ has no non-negative integer solution. This implies that the equation $47^x + 49^y = w^4$ has no non-negative integer solution. □

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