International Journal of Pure and Applied Mathematics
Volume 89 No. 2 2013, 283-286
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v89i2.12

ijpam.eu

ON THE DIOPHANTINE EQUATION $89^x + 91^y = z^2$

Banyat Sroysang

Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University
Rangsit Center, Pathumthani, 12121, THAILAND

Abstract: In this paper, we prove that the Diophantine equation $89^{\mathsf{x}} + 91^{\mathsf{y}} = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

AMS Subject Classification: 11D61
Key Words: exponential Diophantine equation

1. Introduction

In 2012, Sroysang [6] posed an open problem that, for any positive odd prime numbers p and q such that $q - p = 2$, what is the set of all solutions (x, y, z) for the Diophantine equation $p^{\mathsf{x}} + q^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers. Moreover, he [6] also proved that $(1, 0, 2)$ is a unique solution (x, y, z) for the Diophantine equation $3^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers. In this paper, we prove that the Diophantine equation $89^{\mathsf{x}} + 91^{\mathsf{y}} = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. For related papers, we list them as follows. Acu [1] showed, in 2007, that the Diophantine equation $2^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$ has only two non-negative integer solutions where x, y and z are non-negative integers. The solutions (x, y, z) are $(3, 0, 3)$ and $(2, 1, 3)$. Suvarnamani, Singta and Chotchaisthit [12] showed, in 2011, that the two Diophantine equations $4^{\mathsf{x}} + 7^{\mathsf{y}} = z^2$ and $4^{\mathsf{x}} + 11^{\mathsf{y}} = z^2$ have no non-negative integer solution where x, y and z are non-negative integers. In 2012,

Received: September 27, 2013

Chotchaisthit [4] found all non-negative integer solutions for the Diophantine equation of type $4^x + p^y = z^2$ where p is a positive prime number. Sroysang [7] showed, in 2012, that $(1, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers. Moreover, he [8] showed that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. Chotchaisthit [3] showed, in 2013, that $(3, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $2^x + 11^y = z^2$ where x, y and z are non-negative integers. Sroysang [9] showed, in 2013, that $(0, 1, 3)$ is a unique solution (x, y, z) for the Diophantine equation $7^x + 8^y = z^2$ where x, y and z are non-negative integers. Moreover, he [10] showed, in 2013, that the Diophantine equation $2^x + 3^y = z^2$ has only three non-negative integer solutions where x, y and z are non-negative integers. The solutions (x, y, z) are $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$. In the same year, he [11] showed that the Diophantine equation $23^x + 32^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

2. Preliminaries

Proposition 2.1. [5] $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. *The Diophantine equation* $89^x + 1 = z^2$ *has no non-negative integer solution where* x *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers x and z such that $89^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $z^2 = 89^x + 1 \geq 89^1 + 1 = 90$. Then $z \geq 10$. Now, we consider on the equation $z^2 - 89^x = 1$. By Proposition 2.1, we have $x = 1$. Then $z^2 = 90$. This is a contradiction. Hence, the equation $89^x + 1 = z^2$ has no non-negative integer solution. □

Lemma 2.3. *The Diophantine equation* $1 + 91^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 91^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 1 + 91^y \geq 1 + 91^1 = 92$. Then $z \geq 10$. Now, we consider on the equation $z^2 - 91^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z^2 = 92$. This is a contradiction. Hence, the equation $1 + 91^{\mathsf{y}} = z^2$ has no non-negative integer solution. □

3. Results

Theorem 3.1. *The Diophantine equation* $89^{\mathsf{x}} + 91^{\mathsf{y}} = z^2$ *has no non-negative integer solution where* x, y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and z such that $89^{\mathsf{x}} + 91^{\mathsf{y}} = z^2$. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. Note that z is even. Then $z^2 \equiv 0 \pmod 6$ or $z^2 \equiv 4 \pmod 6$. Moreover, $91^{\mathsf{y}} \equiv 1 \pmod 6$. It follows that $89^{\mathsf{x}} \equiv 5 \pmod 6$ or $89^{\mathsf{x}} \equiv 3 \pmod 6$. Note that $89^{\mathsf{x}} \equiv 5^{\mathsf{x}} \pmod 6$. But $5^{\mathsf{x}} \equiv 1 \pmod 6$ or $5^{\mathsf{x}} \equiv 5 \pmod 6$. This implies that $89^{\mathsf{x}} \equiv 5^{\mathsf{x}} \equiv 5 \pmod 6$. Thus, x is odd. Then $89^{\mathsf{x}} \equiv 3 \pmod 7$ or $89^{\mathsf{x}} \equiv 5 \pmod 7$ or $89^{\mathsf{x}} \equiv 6 \pmod 7$. Note that $91^{\mathsf{x}} \equiv 0 \pmod 7$. It follows that $z^2 \equiv 3 \pmod 7$ or $z^2 \equiv 5 \pmod 7$ or $z^2 \equiv 6 \pmod 7$. This is a contradiction since $z^2 \equiv 0 \pmod 7$ or $z^2 \equiv 1 \pmod 7$ or $z^2 \equiv 2 \pmod 7$ or $z^2 \equiv 4 \pmod 7$. □

Corollary 3.2. *The Diophantine equation* $89^{\mathsf{x}} + 91^{\mathsf{y}} = w^4$ *has no non-negative integer solution where* x, y *and* w *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $89^{\mathsf{x}} + 91^{\mathsf{y}} = w^4$. Let $z = w^2$. Then $89^{\mathsf{x}} + 91^{\mathsf{y}} = z^2$. By Theorem 3.1, the equation $89^{\mathsf{x}} + 91^{\mathsf{y}} = z^2$ has no non-negative integer solution. This implies that the equation $89^{\mathsf{x}} + 91^{\mathsf{y}} = w^4$ has no non-negative integer solution. □

References

[1] D. Acu, On a Diophantine equation $2^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$, *Gen. Math.*, **15** (2007), 145–148.

[2] E. Catalan, Note extraite dune lettre adressee a lediteur, *J. Reine Angew. Math.*, **27** (1844), 192.

[3] S. Chotchaisthit, On the Diophantine equation $2^{\mathsf{x}} + 11^{\mathsf{y}} = z^2$, *Maejo Int. J. Sci. Technol.*, **7** (2013), 291–293.

[4] S. Chotchaisthit, On the Diophantine equation $4^{\mathsf{x}} + p^{\mathsf{y}} = z^2$ where p is a prime number, *Amer. J. Math. Sci.*, **1** (2012), 191–193.

[5] P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, **27** (2004), 167−195.

[6] B. Sroysang, On the Diophantine equation $3^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 605−608.

[7] B. Sroysang, More on the Diophantine equation $8^{\mathsf{x}} + 19^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 601−604.

[8] B. Sroysang, On the Diophantine equation $31^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 609−612.

[9] B. Sroysang, On the Diophantine equation $7^{\mathsf{x}} + 8^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 111−114.

[10] B. Sroysang, More on the Diophantine equation $2^{\mathsf{x}} + 3^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 133−137.

[11] B. Sroysang, On the Diophantine equation $23^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 231−234.

[12] A. Suvarnamani, A. Singta, S. Chotchaisthit, On two Diophantine equations $4^{\mathsf{x}} + 7^{\mathsf{y}} = z^2$ and $4^{\mathsf{x}} + 11^{\mathsf{y}} = z^2$, *Sci. Technol. RMUTT J.*, **1** (2011), 25−28.