ON THE DIOPHANTINE EQUATION $89^x + 91^y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $89^x + 91^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers.

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1. Introduction

In 2012, Sroysang [6] posed an open problem that, for any positive odd prime numbers $p$ and $q$ such that $q - p = 2$, what is the set of all solutions $(x, y, z)$ for the Diophantine equation $p^x + q^y = z^2$ where $x$, $y$ and $z$ are non-negative integers. Moreover, he [6] also proved that $(1, 0, 2)$ is a unique solution $(x, y, z)$ for the Diophantine equation $3^x + 5^y = z^2$ where $x$, $y$ and $z$ are non-negative integers. In this paper, we prove that the Diophantine equation $89^x + 91^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. For related papers, we list them as follows. Acu [1] showed, in 2007, that the Diophantine equation $2^x + 5^y = z^2$ has only two non-negative integer solutions where $x$, $y$ and $z$ are non-negative integers. The solutions $(x, y, z)$ are $(3, 0, 3)$ and $(2, 1, 3)$. Suvarnamani, Singta and Chotchaisthit [12] showed, in 2011, that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. In 2012,
Chotchaisthit [4] found all non-negative integer solutions for the Diophantine equation of type \(4^x + p^y = z^2\) where \(p\) is a positive prime number. Sroysang [7] showed, in 2012, that \((1, 0, 3)\) is a unique solution \((x, y, z)\) for the Diophantine equation \(8^x + 19^y = z^2\) where \(x, y\) and \(z\) are non-negative integers. Moreover, he [8] showed that the Diophantine equation \(31^x + 32^y = z^2\) has no non-negative integer solution where \(x, y\) and \(z\) are non-negative integers. Chotchaisthit [3] showed, in 2013, that \((3, 0, 3)\) is a unique solution \((x, y, z)\) for the Diophantine equation \(2^x + 11^y = z^2\) where \(x, y\) and \(z\) are non-negative integers. Sroysang [9] showed, in 2013, that \((0, 1, 3)\) is a unique solution \((x, y, z)\) for the Diophantine equation \(7^x + 8^y = z^2\) where \(x, y\) and \(z\) are non-negative integers. Moreover, he [10] showed, in 2013, that the Diophantine equation \(2^x + 3^y = z^2\) has only three non-negative integer solutions where \(x, y\) and \(z\) are non-negative integers. The solutions \((x, y, z)\) are \((0, 1, 2)\), \((3, 0, 3)\) and \((4, 2, 5)\). In the same year, he [11] showed that the Diophantine equation \(23^x + 32^y = z^2\) has no non-negative integer solution where \(x, y\) and \(z\) are non-negative integers.

2. Preliminaries

Proposition 2.1. [5] \((3, 2, 2, 3)\) is a unique solution \((a, b, x, y)\) for the Diophantine equation \(a^x - b^y = 1\) where \(a, b, x\) and \(y\) are integers with \(\min\{a, b, x, y\} > 1\).

Lemma 2.2. The Diophantine equation \(89^x + 1 = z^2\) has no non-negative integer solution where \(x\) and \(z\) are non-negative integers.

Proof. Suppose that there are non-negative integers \(x\) and \(z\) such that \(89^x + 1 = z^2\). If \(x = 0\), then \(z^2 = 2\) which is impossible. Then \(x \geq 1\). Thus, \(z^2 = 89^x + 1 \geq 89^1 + 1 = 90\). Then \(z \geq 10\). Now, we consider on the equation \(z^2 - 89^x = 1\). By Proposition 2.1, we have \(x = 1\). Then \(z^2 = 90\). This is a contradiction. Hence, the equation \(89^x + 1 = z^2\) has no non-negative integer solution.

Lemma 2.3. The Diophantine equation \(1 + 91^y = z^2\) has no non-negative integer solution where \(y\) and \(z\) are non-negative integers.

Proof. Suppose that there are non-negative integers \(y\) and \(z\) such that \(1 + 91^y = z^2\). If \(y = 0\), then \(z^2 = 2\) which is impossible. Then \(y \geq 1\). Thus, \(z^2 = 1 + 91^y \geq 1 + 91^1 = 92\). Then \(z \geq 10\). Now, we consider on the equation \(z^2 - 91^y = 1\). By Proposition 2.1, we have \(y = 1\). Then \(z^2 = 92\). This is a
contradiction. Hence, the equation $1 + 91y = z^2$ has no non-negative integer solution.

\section{Results}

\begin{theorem}
The Diophantine equation $89x + 91y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers.
\end{theorem}

\begin{proof}
Suppose that there are non-negative integers $x, y$ and $z$ such that $89x + 91y = z^2$. By Lemma 2.2 and 2.3, we have $x \geq 1$ and $y \geq 1$. Note that $z$ is even. Then $z^2 \equiv 0$ (mod 6) or $z^2 \equiv 4$ (mod 6). Moreover, $91y \equiv 1$ (mod 6).

It follows that $89x \equiv 5$ (mod 6) or $89x \equiv 3$ (mod 6). Note that $89x \equiv 5x$ (mod 6). But $5x \equiv 1$ (mod 6) or $5x \equiv 5$ (mod 6). This implies that $89x \equiv 5x \equiv 5$ (mod 6). Thus, $x$ is odd.

Then $89x \equiv 3$ (mod 7) or $89x \equiv 5$ (mod 7) or $89x \equiv 6$ (mod 7). Note that $91x \equiv 0$ (mod 7). It follows that $z^2 \equiv 3$ (mod 7) or $z^2 \equiv 5$ (mod 7) or $z^2 \equiv 6$ (mod 7). This is a contradiction since $z^2 \equiv 0$ (mod 7) or $z^2 \equiv 1$ (mod 7) or $z^2 \equiv 2$ (mod 7) or $z^2 \equiv 4$ (mod 7).
\end{proof}

\begin{corollary}
The Diophantine equation $89x + 91y = w^4$ has no non-negative integer solution where $x, y$ and $w$ are non-negative integers.
\end{corollary}

\begin{proof}
Suppose that there are non-negative integers $x, y$ and $w$ such that $89x + 91y = w^4$. Let $z = w^2$. Then $89x + 91y = z^2$. By Theorem 3.1, the equation $89x + 91y = z^2$ has no non-negative integer solution. This implies that the equation $89x + 91y = w^4$ has no non-negative integer solution.
\end{proof}

\section*{References}


