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A NEW PROOF FOR A THEOREM OF STONE AND KATĚTOV

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Abstract: In 1937 M.H. Stone proved that a space is compact if and only if each of its closed subspaces is H-closed, using Boolean Algebra. In 1940, M. Katětov gave a topological proof of the same result. In this note we give a new and simpler proof of this result.

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1. Introduction

All spaces under consideration are Hausdorff. A space is *H-closed* if it is closed in every Hausdorff space in which it can be embedded.

In 1937 Stone [3] proved that a space is compact if and only if each of its closed subspaces is H-closed, using Boolean Algebra. In 1940, M. Katetov [2] gave a topological proof of the same result. Since every closed subspace of a compact space is compact and every compact space is H-closed, it is immediate that if a space is compact, then every closed subspace is H-closed. Here we shall give a new and simpler proof to show that if a space has the property that every closed subspace is H-closed, then the space is compact.

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2. Preliminaries

If $\mathcal{F}$ is a filterbase, *adherence of* $\mathcal{F}$*, denoted as* $adh\mathcal{F}$ is $\cap_{\mathcal{F}}(clF)$ where clF denotes the closure of F. The concept of θ-closure of a set was introduced by Veličko [4] for study of H-closed spaces. If $A \subseteq X$, the θ *-closure of* A, denoted as $cl_\theta(A) = \{x \in X | A \cap clU \neq \emptyset\}$ for each $U \in \Sigma(x)$, where $\Sigma(x)$ represents the set of all open sets containing x. For a filterbase $\mathcal{F}$, θ- *adhenrence* is denoted as $adh_\theta\mathcal{F}$ and $adh_\theta\mathcal{F} = \cap_{\mathcal{F}}(cl_\theta F)$. The concept of θ- adherence and θ-convergence of a filterbase enable to replace an open filterbase with filterbases which are not necessarily open, in the study of H-closed spaces.

Theorem. *If a space* X *has the property that every closed subspace is H-closed, then* X *is compact.*

Proof. Suppose that X has the property that every closed subset is H-closed. Let $\mathcal{W}$ be an ultrafilter on X and let $\mathcal{O}$ be an open ultrafilter on X such that $\mathcal{O}$ contains the open filter $\mathcal{W}^*$ contained in $\mathcal{W}$. Then $adh\mathcal{O} \cap adh\mathcal{W}^* \neq \emptyset$ since $F \cap H \neq \emptyset$ for every $F \in \mathcal{O}$ and $H \in \mathcal{W}^*$ and $adh\mathcal{W}^*$ is H-closed. Moreover, since $\mathcal{O}$ is an open ultrafilter, $adh\mathcal{O}$ is a single point set. Also, $adh\mathcal{O} \cap adh\mathcal{W}^* = adh\mathcal{O} \cap adh_\theta\mathcal{W}$. Therefore, $\mathcal{W}$ converges. The proof is complete. □

References

[1] M.P. Berri, J.R. Porter, R.M. Stephenson, Jr., *A Survey of Minimal Topological Spaces*, General Topology and Its Relations To Modern Analysis and Algebra, Proc. Conf.. Kanpur (1968) (Academia Prague (1971), 93-114.

[2] M. Katětov, Uber H-abgeschlossene und Bikompakte Raume, *Casopis Pest. Mat.*, **69** (1940), 36-49, In German.

[3] M.H. Stone, Applications of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc.*, **41**, No. 3 (1937), 375-481.

[4] N.V. Veličko, H-closed topological spaces, *Math. Sb. (N.S)*, **70**, No. 112 (1966), 98-112, In Russian; *Amer. Math. Soc. Transl.*, **78**, No. 2 (1968), 103-118.