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CURVILINEAR SCHEMES AND THE ASSOCIATED X-RANKS

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Abstract: Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. For all $e_1 \geq \cdots \geq e_k > 0$ consider the set of all curvilinear subschemes of X_{reg} with k connected components of degree $e_1, \dots, e_k$. For any $P \in \mathbb{P}^n$ we define two X-ranks with respect to these curvilinear schemes.

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1. The Statement

Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Fix a finite non-decreasing sequence of positive integers $\underline{e} = e_1 \geq \cdots \geq e_k > 0$. Let $X(\underline{e})$ be the set of all curvilinear schemes $Z \subset X_{reg}$ with k connected components, say $Z = Z_1 \sqcup \cdots \sqcup Z_k$, with $\deg(Z_i) = e_i$ for all i (we impose that each $Z \in X(\underline{e})$ is contained in X_{reg}). The algebraic set $X(\underline{e})$ is an irreducible quasi-projective variety.

Fix $P \in \mathbb{P}^n$. The X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle\ \rangle$ denote the linear span. The X-widerank $w_X(P)$ is the minimal integer t such that for each proper closed subset $T \subsetneq X$ there is $S \subset X \setminus T$ with $\sharp(S) = t$ and $P \in \langle S \rangle$. The $\underline{e}$-X-rank $r_{X,\underline{e}}(P)$ is the minimal degree of a subscheme $Z \subset X_{reg}$ with Z disjoint union of an element of

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$X(\underline{e})$ and of a finite subset of X_{reg}. The $\underline{e}$-X-widerank $w_{X,\underline{e}}(P)$ is the minimal integer t with the following property: for each proper closed subset $T \subsetneq X$ there is a degree t subscheme $Z \subset X_{reg}$ with Z disjoint union of an element of $X(\underline{e})$ and of a finite subset of X_{reg}. For a definition (the open rank) related to the widerank of a Veronese variety, see [3] (the widerank and the open rank are the same numbers, but the open rank is defined only for polynomials which cannot be defined in a smaller number of variables). The difference with respect to the curvilinear rank defined in [2] is that we prescribe the degrees of all unreduced connected components. As in [1] the proofs are just an adaptation of the proof of [5], Proposition 4.1.

Remark 1. Look at the definitions of $\underline{e}$-X-rank and $\underline{e}$-X-widerank. Allowing that the finite subset, S, with $Z = A \sqcup S$, $A \in X(\underline{e})$, is contained in X (not just in X_{reg}) we get two other interesting definitions. It is however essential that in the definition of $X(\underline{e})$ we only take subschemes of X_{reg}.

In this note we prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Assume that either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > \deg(X)$. *Fix a partition $\underline{e} = e_1 \geq \cdots \geq e_k > 0$ of an integer $\leq n - \dim(X)$. Fix $P \in \mathbb{P}^n \setminus X$. Then $w_{X,\underline{e}}(P) \leq n - \dim(X) + 1$.*

2. The Proof

For any linear subspace $V \subset \mathbb{P}^n$ let $\ell_V : \mathbb{P}^n \to \mathbb{P}^{n-\dim(V)-1}$ denote the linear projection from V. Let $X_V \subset \mathbb{P}^{n-\dim(V)-1}$ denote the closure of the variety $\ell_V(X \setminus X \cap V)$. Write ℓ_Q and X_Q if $V = \{Q\}$ is a point.

Lemma 1. *Take X, $\underline{e}$ and P as in Theorem 1. Then $r_X(P) \leq n - \dim(X) + 1$.*

Proof. Since either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > \deg(X)$, the restriction to X of the linear projection from P is separable, i.e. P is not a strange point of X.

(a) Assume $\dim(X) = 1$.

First assume $n = 2$. In this case we have $k = 1$ and $e_1 = 1$. Let $L \subset \mathbb{P}^2$ be a general line through P. Since L is general and $P \notin T$, we have $L \cap T = \emptyset$. Since P is not a strange point of X, the set $(L \cap X)_{red}$ has cardinality $\deg(X) \geq 2$. Any two points of $L \cap X$ gives $r_{X,\underline{e}}(P) \leq 2$.

From now on we assume $n \ge 3$. Fix a general $E \in X(\underline{e})$.

Claim 1. *We claim that our assumption on* $\mathrm{char}(\mathbb{K})$ *and the generality of* E *imply* $\dim(\langle E\rangle) = \deg(E) - 1$.

Proof of Claim 1. The case $k = 1$ is just [4], theorem 15. Now assume $k \ge 2$. Take a connected component E' of E. Apply the inductive assumption to the curve $X_{\langle E'\rangle} \subset \mathbb{P}^{n-\deg(E')}$ obtained from X by the linear projection from $\langle E'\rangle$.

If $P \in \langle E\rangle$, then $r_{X_{\mathsf{reg}},\underline{e}}(P) \le e_1 + \cdots + e_k$. Hence we may assume $P \notin \langle E\rangle$, i.e. $\dim(\langle E \cup \{P\}\rangle) = \deg(E)$. Write $M(E,P) := \langle E \cup \{P\}\rangle$.

Claim 2. *For general* E *we have* $\mathrm{Sing}(X) \cap M(E,P) = \emptyset$.

Proof of Claim 2. Claim 2 is trivially true if $\mathrm{Sing}(X) = \emptyset$. Hence we may assume $\mathrm{Sing}(X) \ne \emptyset$. For any $Q \in \mathrm{Sing}(X)$ let B_Q (resp. B'_Q) be the set of all $E \in X(\underline{e})$ such that $Q \in \langle E \cup \{P\}\rangle$ (resp. $Q \in \langle E\rangle$). Since $X(\underline{e})$ is irreducible, it is sufficient to prove that $\dim(B_Q) \le k-1$ for all $Q \in \mathrm{Sing}(X)$. Fix $Q \in \mathrm{Sing}(X)$. Apply Claim 1 to the curve X_Q. Now we check that $\dim(B_Q) \le k-1$. We apply Claim 1 to the curve $X_{\langle\{Q,P\}\rangle} \subset \mathbb{P}^{n-2}$.

Let $H \subset \mathbb{P}^n$ be a general hyperplane containing $\langle E \cup \{P\}\rangle$. By Claim 2 we have $H \cap \mathrm{Sing}(X) = \emptyset$. Since the restriction to $X \setminus X \cap M(E,P)$ of $\ell_{M(E,P)}$ is separable, and H is general, we get that the scheme $X \cap H \setminus X \cap M(E,P)$ is reduced. Since X is connected, we have $h^1(\mathcal{I}_X) = 0$. Hence the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X\cap H}(1) \to 0$$

gives that the scheme $X \cap H$ spans H. and in particular it contains P in its linear span.

Claim 3. *For a general* $H \supset M(E,P)$ *every connected component* A *of* $X \cap H$ *such that* $A_{red} \subset M(E,P)$ *satisfies* $A \subset M(E,P)$.

Proof of Claim 3. Let $U_1, \dots, U_x$ be the connected components of the scheme $X \cap M(E,P)$. Set $Q_i := (U_i)_{red}$. Since $M(E,P) \cap \mathrm{Sing}(X) = \emptyset$, we have $Q_i \in X_{reg}$ for all i. Let f_i be the multiplicity of Q_i in U_i. Since U_i is the connected component of $M(E,P) \cap X$ containing Q_i, we have $\langle (f_i+1)Q_i\rangle \nsubseteq M(E,P)$. Since $Q_1, \dots, Q_x$ is a finite set and H is general, we get $\langle (f_i+1)Q_i\rangle \nsubseteq H$, i.e. Claim 3 is true.

Since the scheme $X \cap H$ spans H, E spans $\langle E\rangle$ and the scheme $X \cap H \setminus X \cap M(E,P)$ is reduced, Claim 3 means that H is spanned by E and by a finite set $F \subset X_{reg}$ with $\sharp(F) \le n - \deg(E)$. Since $P \in \langle F \cup E\rangle$, we get $r_{X,\underline{e}}(P) \le n$.

(b) Now assume $m := \dim(X) > 1$. Let $T \subsetneq X$ be a proper closed subset of X. Let $M \subset \mathbb{P}^n$ be a general codimension $b-1$ linear subspace of $\mathbb{P}^n$

containing P. Since $P \notin X$ and X is not a strange variety with P in its strange locus, the scheme $X \cap M$ is an integral curve spanning M ([1], page 6; an easy application of a characteristic free part of Bertini's theorem) and P is not a strange point of X. For general M the closed set $T \cap M$ is a finite set. We have $(X \cap M)_{reg} \subseteq X_{reg} \cap M$. By the curve case just proved there is a curvilinear scheme $Z \in (X \cap M)_{reg} \setminus X \cap T \cap M$ with associated sequence equivalent to $\underline{e}$, with $\deg(Z) \le n - m + 1$ and with $P \in \langle W \rangle$. Since $W \subset W_r eg$ and $W \cap T = \emptyset$, we get We have $w_{X,\underline{e}}(P) \le n - m + 1$. □

Proof of Theorem 1. Fix a proper closed subset $T \subsetneq X$. We may assume that $T \supset \mathrm{Sing}(X)$. We repeat all steps of the proof of Lemma 1. At the end we need to find a scheme $Z \subset X_{reg}$ with the additional condition that $Z \cap T = \emptyset$. Since either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > \deg(X)$, the restriction to X of the linear projection from P is separable, i.e. P is not a strange point of X.

(a) First assume $\dim(X) = 1$. Fix a finite set $T \subset X$ containing $\mathrm{Sing}(X)$.

(a.1) Assume $n = 2$. In this case we have $k = 1$ and $e_1 = 1$. Let $L \subset \mathbb{P}^2$ be a general line through P. Since L is general and $P \notin T$, we have $L \cap T = \emptyset$. Since P is not a strange point of X, the set $(L \cap X)_{red}$ has cardinality $\deg(X) \ge 2$. Since $L \cap T = \emptyset$, any two points of $L \cap X$ give $w_{X,\underline{e}}(P) \le 2$.

(a.2) Now assume $n \ge 3$. For each $O \in T$ let D_O be the line $\langle \{O, P\} \rangle$. We get a finite family $\mathcal{F} = \{D_O\}_{O \in T}$ of lines. Fix a general $E \subset X(\underline{e})$. Since T is finite, we have $T \cap E = \emptyset$.

Claim I. *For a general $E \in X(\underline{e})$ we have $\langle E \rangle \cap T = \emptyset$.*

Proof of Claim I. For any $O \in T$ set $B_O := \{E \in X(\underline{e}) : O \in \langle E \rangle\}$. Since $X(\underline{e})$ is irreducible of dimension k and T is finite, it is sufficient to prove that $\dim(B_O) \le k - 1$ for each $O \in T$, i.e. that $O \notin \langle E \rangle$ for a general $E \in X(\underline{e})$. This is true by Claim 1 of the proof of Proposition **??** applied to the curve $X_O \subset \mathbb{P}^{n-1}$.

Fix a general $E \in X(\underline{e})$. Claim I gives $E \cap T = \emptyset$. If $P \in \langle E \rangle$, then we get $w_{X,\underline{e}}(P) \le \deg(E) < n$. Hence we may assume $P \notin \langle E \rangle$. Set $M(E, P) := \langle E \cup \{P\} \rangle$.

Claim II. *For general E we have $M(E, P) \cap D_O = \emptyset$ for all $O \in T$.*

Proof of Claim II. P and O are fixed. We apply Claim I to the curve $\ell_P(X) \subset \mathbb{P}^{n-1}$ with respect to the set $\ell_P(T)$.

Fix a general hyperplane $H \subset \mathbb{P}^n$ containing $M(E, P)$. Since $D_O \cap \langle E \rangle = \emptyset$ (Claim II), we have $H \cap T = \emptyset$.

Then the proof of Lemma 1 goes verbatim.

(b) Now assume $m := \dim(X) > 1$. Let $T \subsetneq X$ be a proper closed subset of X. Let $M \subset \mathbb{P}^n$ be a general codimension $b-1$ linear subspace of $\mathbb{P}^n$ containing P. Since $P \notin X$ and X is not a strange variety with P in its strange locus, the scheme $X \cap M$ is an integral curve spanning M ([1], page 6; an easy application of a characteristic free part of Bertini's theorem) and P is not a strange point of X. For general M the closed set $T \cap M$ is a finite set. We have $(X \cap M)_{reg} \subseteq X_{reg} \cap M$. By the curve case just proved there is a curvilinear scheme $Z \in (X \cap M)_{reg} \setminus X \cap T \cap M$ with associated sequence equivalent to $\underline{e}$, with $\deg(Z) \leq n-m+1$ and with $P \in \langle W \rangle$. Since $W \subset W_reg$ and $W \cap T = \emptyset$, we get We have $w_{X,\underline{e}}(P) \leq n-m+1$. $\square$

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