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IDENTIFICATION OF FRACTIONAL-ORDER DYNAMICAL SYSTEMS BASED ON NONLINEAR FUNCTION OPTIMIZATION

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Abstract: In general, real objects are fractional-order systems and also dynamical processes taking place in them are fractional-order processes, although in some types of systems the order is very close to an integer order. So we consider dynamical system whose mathematical description is a differential equation in which the orders of derivatives can be real numbers. With regard to this, in the task of identification, it is necessary to consider also the fractional order of the dynamical system. In this paper we give suitable numerical solutions of differential equations of this type and subsequently an experimental method of identification in the time domain is given. We will concentrate mainly on the identification of parameters, including the orders of derivatives, for a chosen structure of the dynamical model of the system. Under mentioned assump-

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tions, we would obtain a system of nonlinear equations to identify the system. More suitable than to solve the system of nonlinear equations is to formulate the identification task as an optimization problem for nonlinear function minimization. As a criterion we have considered the sum of squares of the vertical deviations of experimental and theoretical data and the sum of squares of the corresponding orthogonal distances. The verification was performed on systems with known parameters and also on a laboratory object.

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1. Introduction

In the tasks of analyzing the properties of the system and in the tasks of designing the appropriate control of the system it is very important to know the mathematical model of the mentioned system. The principal aim of the identification is to design the structure and parameters of the mathematical model of the system [1, 2, 3]. In this paper we will focus only on the dynamical processes, where the mathematical models are differential equations. Many identification methods for the dynamical systems have been developed and also many auxiliary tools like e.g. Matlab, widely used in the academy and in practice.

Some of the identification methods are based on analysis of the processes taking place in the system and subsequent design of the model applying the so called deductive principle technique [3, 4, 5]. Another methods, so called experimental identification methods, or black-box identification methods, are based on measured or simulated inputs and outputs of the system in time domain or in frequency domain. It is such experimental methods that we will focus on also in this article.

The standard systems and also standard control systems used so far were all considered as integer-order systems, regardless of the reality. In their analysis and design, the Laplace transform was used heavily for simplicity. But as it results from the recent research works, the majority of real objects in general are fractional-order (FO) systems. Even if the fractional-order calculus is an about 300 year old topic, the theory of fractional-order derivatives was developed mainly in the 19^{th} century and because of the higher complexity and the absence of adequate mathematical tools, fractional-order dynamical systems were only treated marginally in the theory and practice of control systems, e.g. [4, 6]. Their analysis requires familiarity with fractional-order

derivatives and integrals [7, 8, 9]. In the last decades there has been, besides the theoretical research of FO derivatives and integrals [10, 11, 12, 13], a growing number of applications of FO calculus in many different areas such as, for example, long electrical lines, electrochemical processes, dielectric polarization, colored noise, viscoelastic materials, chaos and of course in control theory as well [8, 14, 15, 16, 17, 18, 19, 20]. This is a confirmation of the statement that real objects are generally fractional-order, however, for many of them the fractionality is very low. Such systems are mainly electronic systems composed of quality electronic elements.

Fractional-order models are more adequate for the description of dynamical systems than integer-order models. But appropriate methods for the analytical or numerical calculations of fractional-order differential equations (FODE) are needed in such cases [10, 11, 12, 13, 14] and also methods for the identification of such systems in time domain or in frequency domain [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

In work [33] was considered system whose mathematical description was a differential equation in which the orders of derivatives can be real numbers and an experimental method of identification based on combination of the method of differentiation of step responses and of the method of passive search. In recent years new optimization methods [34, 35, 36] and new high-performance languages and tools for technical computing have been developed. Thus, in this contribution we will concentrate mainly on the identification of parameters (including the order of derivatives) for a chosen structure of the model and we compare two different criterions - the first is the sum of squares of the vertical deviations of experimental and theoretical data and the second is the sum of squares of the corresponding orthogonal distances [37, 38, 39, 40]. The verification of the correctness of the proposed identification method will be done by using this method for identification of the systems with known parameters and also on a laboratory object.

2. Brief Description of the Methods for the Solution of FODE

Let us consider systems whose mathematical descriptions are three- or two-term FODE with constant coefficients in which the orders of derivatives α, β can be real numbers:

$$a_2 y^{(\alpha)}(t) + a_1 y^{(\beta)}(t) + a_0 y(t) = u(t), \tag{1}$$

$$a_1 y^{(\beta)}(t) + a_0 y(t) = u(t), \tag{2}$$

a_2, a_1, a_0 are arbitrary constants. In the case a_2=0 we have two-term FODE, $u(t)$ is the input signal into the dynamical system and $y(t)$ is the output of the system described by FODE (1), (2) with zero initial conditions. Now we give the basic principle of the methods for solving such equations.

2.1. The Analytical Solution - A

The analytical solution [12, 13] of the initial-value problem (2) for the input signal $u(t)$ equal to the unit step function is:

$$y(t) = \frac{1}{a_1} t^{\beta} E_{\beta,\beta+1}\left(-\frac{a_0}{a_1} t^{\beta}\right), E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{3}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function in two parameters [13, 14, 41] and $\Gamma(z)$ is the Gamma function [12, 13, 14]. For special cases β=1 and 2 we obtain the following classical analytical solutions of equation (2):

$$y(t) = \frac{1}{a_0}\left(1 - e^{-\frac{a_0}{a_1} t}\right), y(t) = \frac{1}{a_0}\left(1 - \cos\left(\sqrt{\frac{a_0}{a_1}} t\right)\right), \tag{4}$$

that are, in these special cases, identical to the solution (3). Using the procedure for the calculation of the Mittag-Leffler function [41] the procedure for (3) is very simple [42]:

```
a1=2; a0=0.8; Beta=1.4; tmax=25; T=0.01; t=[0:T:tmax];
y=1/a1*(t.^Beta).*mitlef(Beta, Beta+1,-a0/a1*(t.^Beta));
plot(t,y,'black'); xlabel('Time [sec]'); ylabel('y(t)'); grid;

function y=mitlef(alpha,beta,z,N)
if nargin<4, N=100; end
m1=max(size(z)); m2=min(size(z)); if m2>1, z=z(1,:); end
k=repmat((1:N)',1,m1); t=repmat(z,N,1); t=t.^(k-1);
a=repmat(gamma(alpha*((1:N)'-1)+beta),1,m1); y=sum(t./a);
```

Disadvantages of such approach as above are high time requirements and inaccuracies occurring in solving more complicated FODE and of course higher order of FODE [42]. The analytical solutions of the FODE (1) or other types of FODE are presented in [12, 13] and their comparison with numerical solution can be found in work [14].

2.2. The Numerical Solution - B

For discretization of the FODE (1) or (2) and derivation of the numerical solution of the FODE [14] we can use e.g. the Grunwald-Letnikov (GL) definition of the FO differ-integral operator [13, 14]:

$$ {}_aD_t^r f(t) = f^{(r)}(t) = \lim_{T\to 0} \frac{1}{T^r} \sum_{j=0}^{[\frac{t-a}{T}]} b_j^{(r)} f(t-jT), \tag{5} $$

where $[x]$ means the integer part of x, $b_j^{(r)}$ are binomial coefficients [14], r is the order of differentiation - generally a real number, T is the time step of calculation. By using the relation (5) we obtain discrete form of FODE (1), (2) and we can derive [14], [42] the following explicit recurrent relation for the numerical solution e.g. of the initial-value problem (2):

$$ y_k = \frac{u_k - a_1 T^{-\beta} \sum_{i=1}^{k} b_j y_{k-j}}{a_1 T^{-\beta} + a_0}, k = 1, 2, \ldots, $$
$$ b_0^{(\beta)} = 1, b_j^{(\beta)} = (1 - (1 + (\pm\beta))/j) b_{j-1}^{(\beta)}. \tag{6} $$

For more details and another types of FODEs see e.g. [13, 14]. We can obtain the Matlab procedure for this numerical solution either (a) by direct programming of relation (6) or (b) more effectively using the procedure for a new general FIR digital fractional order differentiator/integrator "dfod2" based on power series expansion of the backward difference (Euler) rule [44]:

(a)

```
a1=2; a0=0.8; Beta=1.4; j=1; y(j)=0; tmax=25; T=0.01;
N=tmax/T+1; t=[0:T:tmax]; u=1; b(1)=1;
for j=1:N, b(j+1)=(1-(1+Beta)/j)*b(j); end
for m=2:N,
    S1=0; for j=2 : m, S1=S1+b(j)*y(m-j+1); end
    y(m)=(u-a1/T^Beta*S1)/(a1/T^Beta + a0);
end; plot(t,y); grid;
```

(b)

```
a1=2; a0=0.8; Beta=1.4; tmax=25; T=0.01; n=1500;
G=1/(a0+a1*dfod2(n,T,Beta));t=[0:T:tmax];[y,t]=step(G,tmax);
plot(t,y); grid; xlabel('Time [sec]'); ylabel(' y(t)');

function sysdfod=dfod2(n,T,r)
```

```
if r>0 bc=cumprod([1,1-(( r+1)./[1:n])]);
       sysdfod=filt(bc,[T^r],T); end
if r<0 bc=cumprod([1,1-((-r+1)./[1:n])]);
       sysdfod=filt([T^-r],bc,T); end
```

This approach is very effective also for more complicated and higher order of FODE and was used for verification of analytical solutions in [14]. But to reach enough accuracy, especially in integer- and higher- order systems, it requires short time step and then it is very time-consuming [42] and causes problems in identification [46, 47].

2.3. The Numerical Solution Based on the Laplace Inversion, I - C

In principle, finding the original $y(t)$ to the Laplace transform $Y(s)$, can be realized by solving the Bromwich integral [43, 45]. Exact inversion is possible only if the poles of $Y(s)$ are known. Since we wish to avoid root-finding procedures, in [45] the variable t was removed from e^{st} by the transformation $z = st$ and then approximated e^z by a rational function - Padé approximation. In [45] are published coefficients (poles z_i and residues K_i) for various orders of approximation and then for numerical Laplace inversion:

$$y(t) = \frac{1}{t} \sum_{i=1}^{M/2} \mathrm{Re}\left[K_i Y\left(\frac{z_i}{t}\right)\right]. \tag{7}$$

Application of the formula (7) at time t:

a) Let the function $Y(s)$ be given as $Ys1$.

b) Select appropriate even M and take z_i and K_i [45].

c) Divide each z_i by t, substitute z_i/t for each s in $Y(s)$.

d) Retain only the real part and divide by t.

Programming the method for the computer is easy with Matlab:

```
a1=2; a0=0.8; Beta=1.4; tmax=25; T=0.01; radt=0.01:T:tmax;
Ys1='1/(a1*s^Beta+a0)/s';
for kk=2:length(radt)            % loop for time t
    t=radt(kk); f1(kk)=0;
    for mm=1:length(z)           % M/2, z-predefined poles, [45]
        s=z(mm)/t; Y=eval(Ys1);
```

```
        f1(kk)=f1(kk)+real(K(mm)*Y)/t;
    end;                              % K-predefined residues, [45]
end; plot(radt,f1); grid; xlabel('Time [sec]'); ylabel('y(t)');
```

The above steps give the approximation to $y(t)$. The user must provide his own function $Y(s)$ in term of variable s. In the case of FODE (2) the function $Y(s)$ consists of the transfer function of such dynamic system, described by this FODE, multiplied by the input signal into the system also in term of variable s:

$$Ys1 = Y(s) = F(s)U(s) = \frac{1}{a_1 s^{\beta} + a_0} \frac{1}{s}. \tag{8}$$

Because of division by t in (7), the time function for $t = 0$ cannot be calculated by the program. This value can be obtained by means of initial value theorem, etc.

This method is suitable for the calculation of time responses where the intervals are not very long. Since the error grows with t, for longer time interval higher M, N are needed. However, this method for the inverse Laplace transform is approximated by the Taylor expansion with limited number of term. The order of approximation is determined by the available table of coefficients and a change of the order is not as easy, it requires a change in the program.

2.4. The Numerical Solution Based on the Laplace Inversion, II - D

The principle of the method for the numerical inversion of Laplace transforms described in [43] is based also on the approximation of the function e^{st} with functions containing hyperbolic cosine or sine functions and then subsequent expansion as a convergent MacLaurin series, the convergence of which can be accelerated by Euler transformation. After modifications [43] we finally get:

$$y_s(t,a) = \frac{e^a}{t}\left[\frac{1}{2}Y\left(\frac{a}{t}\right) + \sum_{n=1}^{ns+1+nd}(-1)^n \mathrm{Re}\left\{Y\left(\frac{a}{t} + jn\frac{\pi}{t}\right)\right\}\right] \tag{9}$$

where a, n_s, n_d are the parameters of the method. Their implicit values are $a = 6, n_s = 20, n_d = 19$ and more details are given in [43]. This method differs from other similar algorithms with the same purpose. The required accuracy of results can be enhanced without changing the program, only at the cost of a longer computation time. The achievable accuracy may be very high. The algorithms can be easily and very effectively realized with Matlab [42, 43]:

```
Ys1='1/(2*s^1.4+0.8)/s'; tmin=0.01; tmax=25; T=0.01;
a=6; ns=80; nd=20; N=round((tmax-tmin)/T)+1;
[t,yt]=ILT1(Ys1,tmin,tmax,N,a,ns,nd); plot(t,yt), grid

function [radt,yt]=ILT1 (Ys,tini,tend,nnt,a,ns,nd);
Yv=strrep(strrep(strrep(Ys,'*','.*'),'/','./'),'^','.^');
if nargin==4 a=6; ns=20; nd=19; end;
radt=linspace(tini,tend,nnt); if tini==0 radt=radt(2:1:nnt); end;
for n=1:ns+1+nd A(n)=a+(n-1)*pi*j;B(n)=-exp(a)*(-1)^n;end;
n=1:nd;
c1=gamma(nd+1); m1=gamma(nd+2-n); m2=gamma(n);
bdif=fliplr(cumsum(c1./m1./m2))./2^nd;
B(ns+2:ns+1+nd)=B(ns+2:ns+1+nd).*bdif; B(1)=B(1)/2;
for kt=1:nnt tt=radt(kt); s=A/tt; bt=B/tt; btY=bt.*eval(Yv);
yt(kt)=sum(real(btY)); end;
```

This numerical inversion of Laplace transforms does not require any other operations, as, for example, finding the poles of $Y(s)$ and partial fraction expansion. The method has two important errors: *static* and *dynamic*. The *static* error is caused by the approximation of the exponential function e^{st} and it depends on the selection of constant a. The *dynamic* error is caused by truncating the infinite series and it depends on the selection of the number of terms n_s and n_d used in the basic series and for Euler transformation respectively [43]. If we want e.g. to obtain the harmonic waveform in the time interval from zero to $t = n_{per}T$ (for n_{per} complete periods) we have to use $n_s > 2n_{per}$.

2.5. Verification of the Methods for the Solution of FODE

Only for demonstration purposes let us now consider a FO system described by two-term FODE (2) with coefficients $a_1 = 2, a_0 = 0.8, \beta = 1.6$ and $u(t) = 1$. The step responses of the system calculated by all four methods are shown in Fig. 1 ($t_{max} = 25, T = 0.1, n = 500, M = 24, a = 6, n_s = 80, n_d = 20$). We can see from Fig. 1 that the output of such FO system has periodic damped behavior contrary of the first order system that has always aperiodic behavior. The agreement of all four methods A, B, C, D is very good for time step $T <= 0.1$s. For this purpose insignificant deviations are only at the beginning of the curve obtained by method B. The more detailed verification of the described methods for different cases of parameters and types of FODE (1) is presented in [42].

After multiple usage of the Matlab functions tic, toc for this calculations

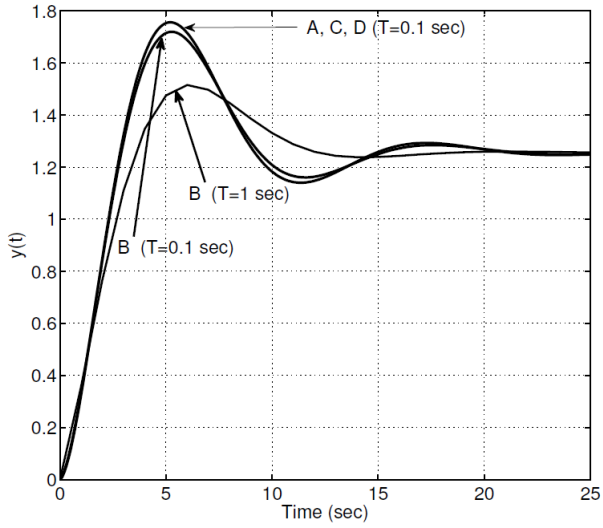

Figure 1: Step responses of the FO system

($t_A = 0.018, t_{Bb} = 0.017, t_C = 0.29, t_D = 0.038$ (s)) we can say that the methods A, B have approximately the same computational time requirements, the method C has more than ten times and the method D two times higher time consumption than methods A, B.

While the time step T does not influence the accuracy of the methods A, C, D, for the method B the time step has significant effect. The accuracy of the methods B tends to the accuracy of the method A, C, D with decreasing time step. The value $T = 0.1$ or $T = 0.01$ is often the maximum and e.g. $T = 1$ (Fig. 1) is inconvenient for serious calculations. The time requirements of all methods increase with decreasing T proportionally. More problems with the divergence of the A method occur when solving higher-order FODEs while methods B, C, D work correctly if the simulated system is stable [42]. The methods A, C, D work especially quickly for higher T and it is also possible to calculate only one or a few values $y(t)$ of the output of the system. The calculations are then not so time- consuming [42] like the methods B, what is very important in identification [46, 47]. We can calculate the values $y(t)$ only for the same points of the independent variable like those measured in experiment.

3. Identification of Fractional-Order Dynamical Systems

In this case we understand the task of identification as the determination of the mathematical model of the dynamical system and the appropriate parameters of this model. That means that for the known structure of the model in the form of e.g. (1) or (2), the task is reduced to determine the unknown parameters $a_2, a_1, a_0, \alpha, \beta$.

3.1. Simplified Approach of Identification

Assume that the input of the system (1) with known values α and β is acted upon by arbitrary function $u(t)$, and $y(t)$ is the output function of the system. If the input and output functions $u_e(t), y_e(t)$ are known functions of time, obtained for example by measurement, then for given discrete times $t_i (i = 2, 3, \ldots, N)$ we can determine the values of the derivatives of the output function according to the formula (5). For each time t_i the equation (1) must be satisfied.

If we consider now two-term FODE (2), after substituting the corresponding values into the equation (2) for three discrete times, we obtain a system of three non-linear equations for estimation of three unknown coefficients a_1, a_0, β. For three-term FODE (1) we would obtain a system of five non-linear equations, etc. To solve this system of equations using Matlab routine fsolve, we need the system in annulled form:

$$\begin{aligned} a_1 y_e^{(\beta)}(t_1) + a_0 y_e(t_1) - u_e(t_1) &= 0 \\ a_1 y_e^{(\beta)}(t_2) + a_0 y_e(t_2) - u_e(t_2) &= 0 \\ a_1 y_e^{(\beta)}(t_3) + a_0 y_e(t_3) - u_e(t_3) &= 0 \end{aligned} \tag{10}$$

This system must be specified in an M-file function and the values of the derivatives of the output function must be calculated at every iteration for actual order β. The selection of above-mentioned three discrete times is very important for the accuracy of identification. For example, after dividing the considered time interval into three subintervals, we can use their midpoints as selected times. Such method is usable if the signal is not very noisy, otherwise we need to use more measured values.

3.2. Optimization-Based Identification with System of Nonlinear Equations

In order to be able to use more measured values $u_e(t_i), y_e(t_i)$ and to make the identification more accurate in cases with noisy signals, we consider for the estimate of the vector of unknown parameters $\boldsymbol{a} = [a_2, a_1, a_0, \alpha, \beta]$ as an error criterion the following minimized functional:

$$F(\boldsymbol{a}) = \sum_{i=0}^{N} \left(a_2 y_{i,e}^{(\alpha)} + a_1 y_{i,e}^{(\beta)} + a_0 y_{i,e} - u_{i,e} \right)^2 \approx min. \tag{11}$$

A necessary condition for the minimum of functional (11) to be achieved is:

$$\frac{\partial F(\boldsymbol{a})}{\partial \boldsymbol{a}} = 0. \tag{12}$$

If we now consider two-term FODE (2), after applying manipulations (12) on functional (11) we obtain a system of three non-linear equations for calculations of the vector of unknown parameters $\boldsymbol{a} = [a_1, a_0, \beta]$:

$$\begin{aligned}
a_1 \sum \left(y_i^{(\beta)} \right)^2 \quad &+ a_0 \sum y_i^{(\beta)} y_i \quad - \sum y_i^{(\beta)} \quad u_i = 0 \\
a_1 \sum \quad y_i^{(\beta)} \, y_i \quad &+ a_0 \sum y_i^2 \quad - \sum y_i \quad u_i = 0 \\
a_1 \sum \left(y_i^{(\beta)} \right)^2 \ln y_i &+ a_0 \sum y_i^{(\beta)} y_i \ln y_i - \sum y_i^{(\beta)} \left(\ln y_i \right) u_i = 0
\end{aligned} \tag{13}$$

To solve this system of equations using Matlab routine fsolve, the system must be specified in an M-file function and the values of the derivatives of the output function must be calculated for actual order β at every iteration.

3.3. Identification of FO Dynamical Systems Based on Nonlinear Function Optimization

We can formulate the task of experimental identification as optimization problem [35, 46, 47, 48]. The values of unknown parameters $\boldsymbol{a} = [a_2, a_1, a_0, \alpha, \beta]$ must be so estimated that the integral or sum of absolute or quadratic differences between measured real system and modeled output is minimized. The test function, or the minimized function $F(\boldsymbol{a})$ for the tuning of the model parameters of the FO dynamical system is the integral or sum square error:

$$F(\boldsymbol{a}) = \sum_{i=0}^{N} \Big(y_{i,e} - y_{i,m}(a_2, a_1, a_0, \alpha, \beta) \Big)^2 \approx min. \tag{14}$$

or sum square error of orthogonal differences (Fig. 2), where y_e is the experimentally measured output of the dynamical system, y_m is the output value of the model defined by equations (3) or (6), (7), (9), N is the number of measured or modeled points. A necessary condition for the minimum of function F is again defined by equation (12). The computing problems occurred by increasing the number of estimated parameters. So in our previous work [33] we have used the method of differentiation of step responses for the calculation of parameters a_2, a_1, a_0 and of the method of passive search for the calculation of parameters α, β. In work [46] we have calculated parameters α, β using optimization method. In recent years new optimization methods [34, 35, 36] and new high-performance languages and tools for technical computing have been developed. Thus, we will concentrate mainly on the identification of all parameters by optimization.

3.4. Criterions Used for Identification

As we mentioned above, we consider two different criterions. The first classical criterion is the sum of squares of the vertical deviations of experimental and theoretical data (eq. (15), (a) in Fig. 2), as it is used in the classical least squares method (CLSM) for e.g. finding the line ($y = a \cdot x + b$) to a given set of points by minimizing the sum of squares of the residuals of the points $(y_{e,i} - y_{m,i})$ and determining the vector of unknown parameters $\boldsymbol{a} = [a, b]$. The second is the sum of squares of the corresponding orthogonal distances $\sqrt{(}(x_{i,e} - x_{i,mo})^2 + (y_{i,e} - y_{i,mo})^2$ (eq. (16), (b) in Fig. 2) or so called total least squares method (TLSM) [37, 38] or the method of least circles [39] or orthogonal least squares method:

$$F_1(\boldsymbol{a}) = \sum_{i=0}^{N} \Big(y_{i,e} - y_{i,m} \Big)^2 = \sum_{i=0}^{N} \Big(y_{i,e} - a x_i - b \Big)^2 \approx min, \tag{15}$$

$$F_2(\boldsymbol{a}) = \sum_{i=0}^{N} \Big((x_{i,e} - x_{i,mo})^2 + (y_{i,e} - y_{i,mo})^2 \Big)^{1/2} \approx min. \tag{16}$$

Both criterions and methods have advantages and disadvantages. Now we will show, what the arguments are for using orthogonal instead of the classical regression based on the least squares method and our experience with those criterions.

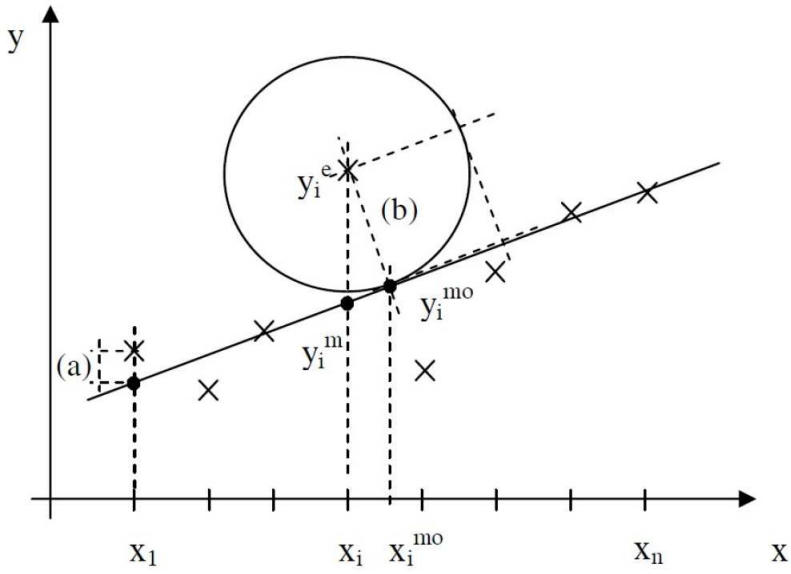

Figure 2: Classical least squares criterion and orthogonal criterion (TLSM) or least circles criterion or least orthogonal squares criterion

3.5. Experience with Criterions and our Results

The answer to the question *"Why the squares? Why not just get the smallest sum of distances of the data points from the line?"* [39] is not that *"it is easy to compute"* parameters a and b of the line by minimizing the sum of squares of distances of y (vertical distances), but that it is principally impossible to minimize only the sum of distances. The criterion must be the sum of absolute distances or squares of distances or another even power of distances. The nature of this is close to the similar fact, that the sum of differences of the data from their average is equal to zero.

It is difficult to agree with the declaration, that the shortest (orthogonal) distance is the most natural basis for any fitting. For the majority of users of fitting methods the principal interest is the variance (distance) between the experimental and theoretical values of both independent variables $(y_{i,e} - y_{i,m})$ for the same value of dependent variable (x_i), see Fig. 2. The total least squares method uses the distance between the experimental dependent variable $(y_{i,e})$ and theoretical value of dependent variable $(y_{i,mo})$ for different value (not x_i) of independent variable $(x_{i,mo})$, see Fig. 2.

The argument, that the sum of orthogonal distances is invariant with respect

to the choice of the system of coordinates (e.g. non-rectangular coordinates) is interesting only for a negligible number of interested persons.

In TLSM there are no conjugate regression lines, which appear after swapping x and y because in the case of orthogonal regression the fitting $y = f(x)$ gives exactly the same line as the fitting $x = f^{-1}(y)$. The classical regression approach leads to a different result expressed by a conjugate regression line as we can see in the Nievergelt's example [38, 39] with experimental data $x = [1; 3; 4; 5; 7]$ and $y_e = [4; 2; 6; 8; 5]$, see Fig. 3. The classical least squares regression y versus x gives the regression line $y = 0.45x + 3.2$ with the sum of squares (SS) equal to 15.95. After swapping x and y, CLSM gives the conjugate regression line $x = 0.45y + 1.75$ with the sum of squares again equal to SS=15.95. However, in the case of orthogonal distance regression we obtain the same line $y = x + 1$ ($x = y - 1$) independent of the order of x and y, but with SS=22. However, the orthogonal distances are better for the TLSM, the vertical distances and the sums of their squares are always better in the case of CLSM in comparison with the TLSM, which is a very important fact for the users interested in this area.

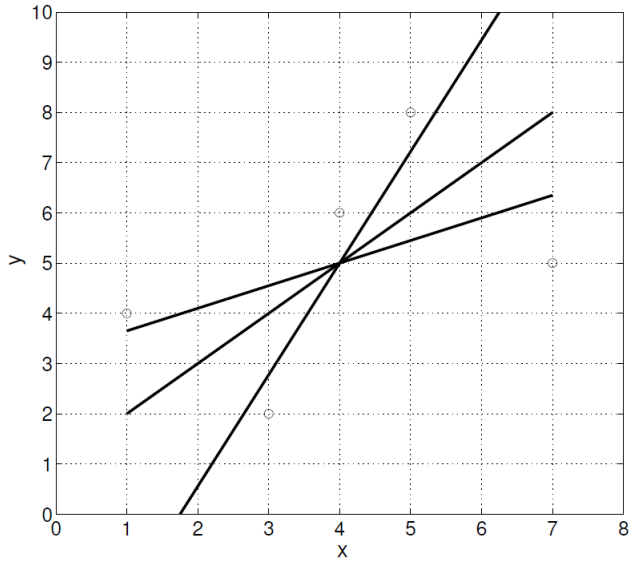

Figure 3: Nievergelt's example: classical regression versus orthogonal regression

The criticized so-called "*scissors*" formed by the conjugate regression lines obtained by the CLSM indicate wrong experimental data or unsuitable mathe-

matical model. At first sight it is clear, that the amount of experimental data is insufficient in the mentioned example and their behavior is not linear. It is more visible by creating the line graph. In our opinion more convenient in this case is the fitting of this experimental data by e.g. a polynomial of the third order (Fig. 4 a) with $SS = 0.53$. In the case of the TLSM we obtain the fitting curve depicted in Fig. 4 b, c with $SS = 10.3$ and $SS = 7.3$ (local extreme). The CLSM better represents the physical nature of the measured data and the sum of squares is better again in comparison with the TLSM.

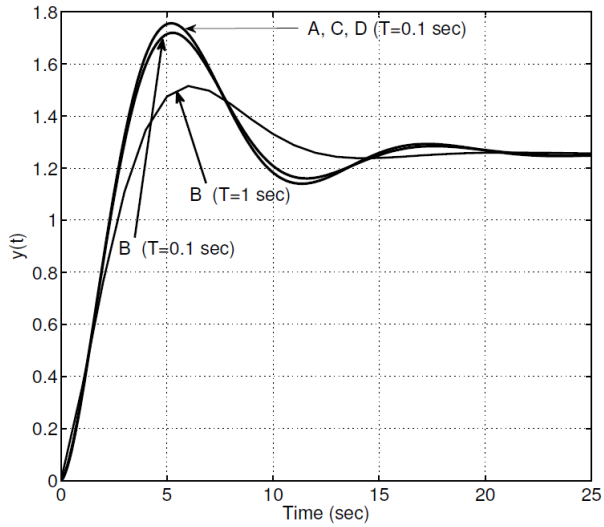

Figure 4: Nievergelt's example: classical regression versus orthogonal regression

The previous examples show that the criterion of least circles or TLSM (Fig. 2) is not really optimal even though the radii of the circles are equal to minimal distances between the points $(x_i, y_{i,e})$ and the fitting line $(x_{i,mo}, y_{i,mo})$. Against the argument that there are no problems in TLSM with determination of what is an independent variable and what is a dependent variable we can say, that in majority of technical matters we have no problem with the determination of that.

The total least squares method [37, 38, 39] is only one of the fitting methods, but in our opinion with fewer advantages compared to the CLSM. It is convenient mainly in linear cases. For nonlinear or polynomial dependencies their application can lead to irrational results. The TLSM is more demanding on computing time in comparison with CLSM. Even in the case of line the

problem of finding their optimal parameters is a nonlinear problem (in parameters). In the next parts we will verify both criterions by identification of the parameters of fractional-order differential equations.

3.6. Verification of the Methods

Let us now consider a FO system described by two-term FODE (2) with $a_1 = 2.0, a_0 = 0.8, \beta = 1.6$ and $u_e(t) = 1$. The step response of this system has periodic behavior and is shown in Fig. 5. These modeled results were calculated by method A from Section 2 for time step $T = 0.1$ and used as experimental data in identification.

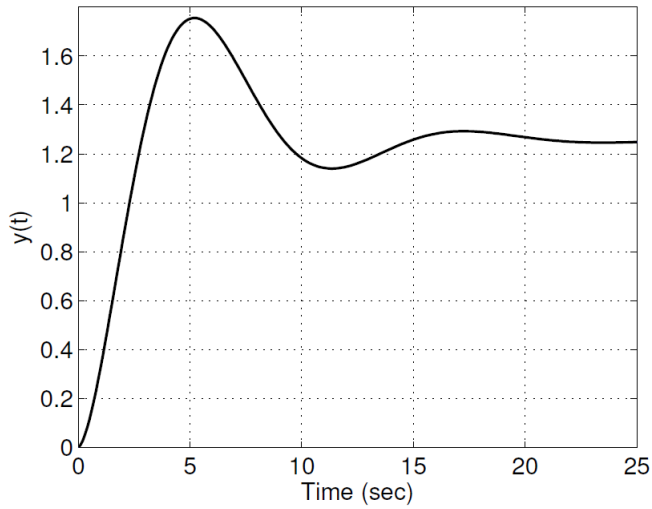

Figure 5: Unit step response of the FO system (2)

Using the simplified identification method (Section 3.1) with corresponding system of three non-linear equations (10) and using Matlab routine fsolve, for the initial guess $a_{1,0} = 0.3, a_{0,0} = 1.5, \beta_0 = 0.6$ and $T = 0.1, t_1 = 1, t_2 = 3, t_3 = 6$ seconds, we obtain very precise values of the coefficients $a_1 = 2.000, a_0 = 0.800, \beta = 1.600$. But only one percentage change of the value $y_e(t_2)$ causes big change of calculated coefficients $a_1 = 1.7826, a_0 = 0.7569, \beta = 1.5497$ and demonstrates high sensitivity of this methods to noise.

Using now the optimization-based approach of identification (Section 3.2) with system of nonlinear equations to the same system (Fig. 5) with corresponding system of three non-linear equations (13) and using Matlab routine fsolve,

for the initial guess $a_{1,0} = 0.3, a_{0,0} = 1.5, \beta_0 = 0.6$ and $T = 0.1$ second, we again obtain very precise values of the coefficients $a_1 = 2.000, a_0 = 0.800, \beta = 1.600$. The same one percentage change of the value y_e causes much lower change in calculated coefficients $a_1 = 2.0184, a_0 = 0.8061, \beta = 1.6021$ as it was in previous case. We have considered 120 experimental data in this case of identification.

To verify the identification method based on nonlinear function optimization (Section 3.3) we consider the same system (2) with coefficients $a_1 = 2, a_0 = 0.8, \beta = 1.6$. The experimental data were calculated now by method A. We can estimate the vector of unknown parameters $\boldsymbol{a} = [a_1, a_0, \beta]$ with the following algorithms for Matlab with criterion (14):

```
x0=[-2,-3,0.5];
[x,fval]=fmincon(@(x)mf(x,y,t,N),x0,[],[],[],[],[-9,-9,0.2],[9,9,3],[])

function f = mf(x,y,t,N)
a1=x(1); a0=x(2); Beta=x(3);
yM=1/a1*(t.^Beta).*mitlef(Beta,Beta+1,-a0/a1*(t.^Beta));
f0=0; for m=2:N, f0=f0+(y(m)-yM(m))^2; end f=f0; return
```

We have compared also the results obtained by both criterions - sum of squares of vertical distances and orthogonal distances. Using the criterion (15) and e.g. optimization method for nonlinear function minimization fmincon from Matlab for the intervals $a_1 \in \langle -9; 9\rangle$, $a_0 \in \langle -9; 9\rangle$, $\beta \in \langle 0.2; 3\rangle$ initial values $a_1 = -2, a_0 = -3, \beta = 0.5$ and time step 0.01 sec (method B from chapter 2), very precise values of the coefficients $a_1 = 2.000, a_0 = 0.800, \beta = 1.600$ were calculated with SS=1.448e-7. Using the orthogonal criterion (16) and again optimization method for nonlinear function minimization for the same intervals, initial values and time step, we obtained the following values of the coefficients $a_1 = 1.9986, a_0 = 0.8004, \beta = 1.5976$ with SS=0.0151S. Here are larger deviations in the parameters of the model compared to real parameters and also the value SS is larger. More obvious here is the advantage of comparing $y_{i,e}$ with $y_{i,m}$ - both at the same time t_i for CLSM, and disadvantage of comparing the values $y_{i,m}$ at time t_i with $y_{i,mo}$ at a different time $t_{i,mo}$ for TLSM.

Results obtained with the optimization-based method with system of nonlinear equations are very similar to those obtained by the method based on pure optimization under the same conditions. But multiple usage of the Matlab functions tic, toc for the previous example and mentioned two methods with the required times 0.3 and 0.1 seconds respectively proved that the optimization method with system of nonlinear equations is three times more time-consuming than the method based on pure optimization.

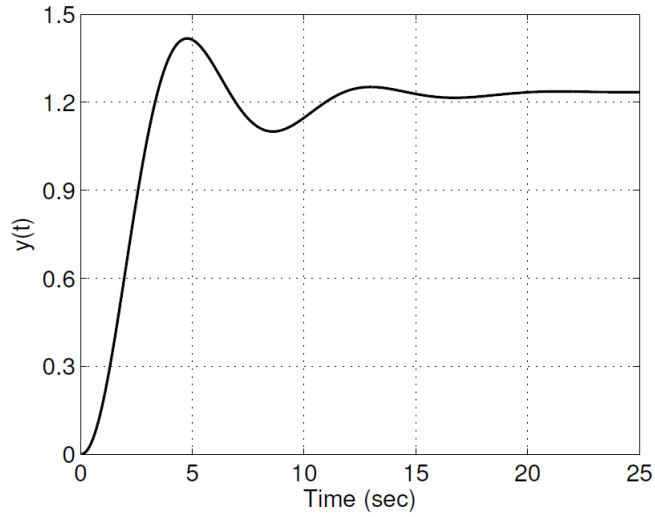

Figure 6: Unit step response of the FO system (1)

We also tried to use and compare the optimization method with so called self-organizing migrating algorithm (SOMA) utilizing the principles of artificial intelligence [35]. Until now the results obtained by using the SOMA algorithm showed the possibility to use them for the FO system parameters identification. A disadvantage of this algorithm is higher computational time. The calculations moreover confirmed that the identification of five or more parameters by optimization of criterion (15) with the SOMA algorithm is not only more time consuming but also more sensitive to initial values and intervals of identified parameters.

In general, for the identification we can use the experimental data $y_{i,e}$ and theoretical data $y_{i,m}(a_2, a_1, a_0, \alpha, \beta)$ calculated by one of the method A, B, C, D described in Section 2. In a previous example, very precise values of the coefficients a_1, a_0, β were calculated also by using methods A, C, D for different values of time steps T=0.1 and T=2 with significantly less computing time for T=2 sec, while the coefficients obtained by identification using the method B are significantly different (unacceptable) for such large time steps.

Let us consider now a fractional-order system [14] described by equation (1) with coefficients $a_2 = 2, a_1 = 1.3, a_0 = 0.8, \alpha = 2.2, \beta = 0.9$, three-term FODE. The step response of this system has periodic behavior and is shown in Fig. 6. Using the criterion (15) and optimization method for nonlinear function minimization fmincon from Matlab for the next intervals $a_2 \in \langle -1; 9 \rangle$, $a_1 \in$

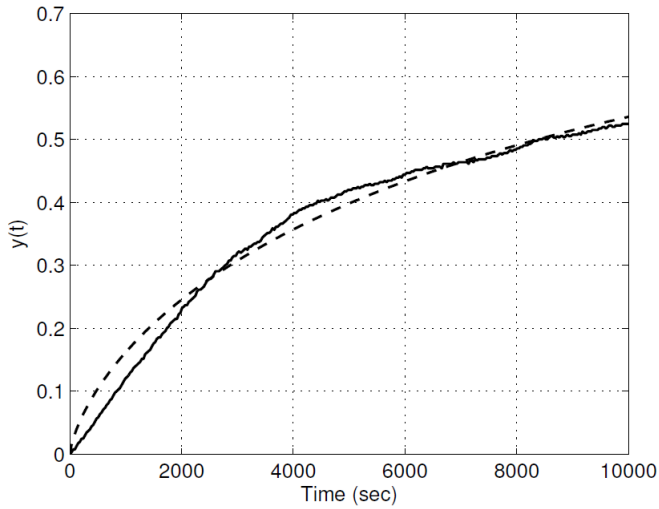

Figure 7: Unit step responses of the real object and their FO model

$\langle -1;9\rangle$, $a_0 \in \langle -1;9\rangle$, $\alpha \in \langle -1;9\rangle$, $\beta \in \langle -1;9\rangle$ initial values $a_2 = -2, a_1 = -3, a_0 = -5, \alpha = 3, \beta = 2$ and time step 0.01 sec, the values of the very good coefficients $a_2 = 1.9999, a_1 = 1.3001, a_0 = 0.8000, \alpha = 2.2001, \beta = 0.9001$ were calculated.

In the papers [46, 49, 50] were identified parameters of an experimental heating furnace with the step response of this system shown in Fig. 7 (solid line). It is rather difficult to identify this system with the first-order model, however it looks so. A little better result can be obtained considering this plant as a second-order system. The deficiency of these two models is visible at first sight. Assuming it as a two-term FO differential equation, using criterion (15) and optimization method for nonlinear function minimization fmincon, the parameters take on the values $a_1 = 782.0007, a_0 = 1.0649, \beta = 0,7057$ and the value of the criterion (15) is 0.1243.

This identification method (chapter 3.3) was successfully used also for determining the parameters of the electronic realization of the FO system [51].

4. Conclusions

The above results show the possibility of using new optimization methods and new high-performance languages and tools for parameter identification in fractional- and integer-order dynamical systems with only a few parameters. On

the other hand these methods are very time-consuming and better initial values of parameters are needed. In such cases we can also use combined method to speed up the identification by using e.g. the method of differentiation of step responses for identification of some parameters and optimization method for identification of the rest of unknown coefficients.

With respect to time consumption the optimization method with a system of nonlinear equations is more time-consuming than the method based on pure optimization, moreover, it requires complicated mathematical derivation to obtain a system of nonlinear equations. Until now the results obtained by using the so called self-organizing migrating algorithm for optimization-based identification confirm that they are very time-consuming and also more sensitive to initial values and intervals.

For the known systems the agreement of calculated and real parameters was very good. In the case of laboratory object identification it is necessary to use more experimental data and to pay attention to the accuracy of the output quantity measurement in the identified system, and to the time step of the measurement from the point of view of used algorithms for the solution of the FODE.

Concerning the method for the numerical solutions of fractional-order differential equations suitable for the optimization based identification in time domain, obtained results confirmed that the most suitable are methods C and especially D described in Sections 2.3, 2.4. Disadvantages of the method A (chapter 2.1) compared to the above methods are high time requirements and inaccuracies occurring in solving more complicated FODE and the higher order of FODE. While the time step T does not influence the accuracy of the methods A, C, D, for the method B the time step has significant effect. With methods A, C, Dit is also possible to calculate only one or a few values of the output of the system. The calculations are then much less time-consuming compared to the method B, which makes them preferable for identification. The output values can be calculated only for the same points of the independent variable like those measured in experiment.

Testing of both optimizing criterions - the sum of squares of the vertical deviations of experimental and theoretical data (CLSM) and the sum of squares of the corresponding orthogonal distances of experimental and theoretical data (TLSM) confirmed that the TLSM is only one of the fitting methods, but in our opinion with insignificant advantages compared to the CLSM. It is convenient mainly in linear cases. For nonlinear or polynomial dependencies their application can lead to irrational results. The TLSM is more demanding on computing time in comparison with CLSM. Even in the case of line the problem of finding

their optimal parameters is a nonlinear problem in parameters.

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