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ON A DIFFERENCE SYSTEM FOR SOME MEJER'S FUNCTIONS

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Abstract: We deduce a difference system and prove some of its properties, on base of which we found (see References) new continued fraction for Zeta(3).

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1. Foreword

Let

$$|z| > 1, \quad -3\pi/2 < \arg(z) \le \pi/2, \quad \log(z) = \ln(|z|) + i\arg(z). \tag{1.1}$$

Then $\log(-z) = \log(z) - i\pi$, if $\Re(z) > 0$ and $\log(z) = \log(-z) - i\pi$, if $\Re(z) < 0$. Let $\alpha \in \mathbb{N}_0$, Let

$$\mu = \mu_\alpha(\nu) = (\nu + \alpha)(\nu + 1), \ \tau = \tau_\alpha(\nu) = \nu + (1 + \alpha)/2, \tag{1.2}$$

$$R(\alpha, t, \nu) = \left(\prod_{j=1}^{\nu}(t - j)\right) / \prod_{j=0}^{\nu+\alpha}(t + j), \tag{1.3}$$

$$f_{\alpha,1}(z, \nu) := (\nu!/(\nu + \alpha)!)^2 \sum_{k=0}^{\nu+\alpha}(z)^k \binom{\nu + \alpha}{k}^2 \binom{\nu + k}{\nu}^2, \tag{1.4}$$

$$f_{\alpha,0,2}(z, \nu) = \sum_{t=1}^{+\infty} z^{-t}(R(\alpha, t, \nu))^2, \tag{1.5}$$

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$$
\begin{aligned}
f_{\alpha,0,4}(z,\nu) =& -\sum_{t=1}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^2)\right)(\alpha,t,\nu), & (1.6)\\
f_{\alpha,0,3}(z,\nu) =&(\log(z))f_{\alpha,0,2}(z,\nu)+f_{\alpha,0,4}(z,\nu), & (1.7)\\
f^*_{\alpha,0,k}(z,\nu) =&((\nu+\alpha)!/\nu!^2)^2 f^*_{\alpha,0,k}(z,\nu) & (1.8)
\end{aligned}
$$

where $k = 1,\ 2,\ 3,\ 4,\ \nu \in \mathbb{N}_0$. Let

$$
\begin{aligned}
X_{\alpha,0,k}(z;\nu) =& \begin{pmatrix} f_{\alpha,0,k}(z,\nu)\\ \delta f_{\alpha,0,k}(z,\nu)\\ \delta^2 f_{\alpha,0,k}(z,\nu)\\ \delta^3 f_{\alpha,0,k}(z,\nu)\end{pmatrix},\\
X^*_{\alpha,0,k}(z;\nu) =& \frac{((\nu+\alpha)!)^2}{(\nu!)^2} X_{\alpha,0,k}(z;\nu), & (1.9)
\end{aligned}
$$

for $k = 1,\ 2,\ 3,\ |z| > 1,\ \nu \in \mathbb{N}_0$. Let further

$$X_{\alpha,0,k}(z;-\nu-1-\alpha) = X_{\alpha,0,k}(z;\nu), \tag{1.10}$$

where $\nu \in \mathbb{N}_0$. The following results are obtained in [11] $-$ [13]

Theorem 1. *The column $X_{\alpha,0,k}(z;\nu)$ satisfies to the equation*

$$\nu^5 X_{\alpha,0,k}(z;\nu-1) = A^*_{\alpha,0}(z;\nu) X_{\alpha,0,k}(z;\nu), \tag{1.11}$$

for $\nu \in M^*_\alpha = (-\infty, -1-\alpha] \cup [1, +\infty)) \cap \mathbb{Z}$, $k = 1,\ 2,\ 3,\ |z| > 1$, and for some matrix $A^*_{\alpha,0}(z;\nu)$ with property:

$$-\nu^5(\nu+\alpha)^5 E_4 = A^*_{\alpha,0}(z;-\nu-\alpha) A^*_{\alpha,0}(z;\nu), \tag{1.12}$$

where E_4 is the 4×4 unit matrix, $z \in \mathbb{C}$, $\nu \in \mathbb{C}$. Let us consider the row

$$\bar{R}_{\alpha,0}(\nu) = (r_{\alpha,0,1}(\nu),\ r_{\alpha,0,2}(\nu),\ r_{\alpha,0,3}(\nu),\ r_{\alpha,0,4}(\nu)), \tag{1.13}$$

where

$$
\begin{gathered}
r_{\alpha,0,1}(\nu) = \mu_\alpha(\nu)^2, \quad r_{\alpha,0,2}(\nu) = -2(1-\alpha)\mu_\alpha(\nu), \qquad (1.14)\\
r_{\alpha,0,3}(\nu) = (1-\alpha)^2 - 2\mu_\alpha(\nu), \quad r_{\alpha,0,4}(0,\nu) = 2(1-\alpha).
\end{gathered}
$$

The following Theorem is proved in [13] (Lemma 11.3.1).

Theorem 2. *The row* $\bar{R}_{\alpha,0}(\nu)$ *has the following property:*

$$\bar{R}_{\alpha,0}(\nu-1)A^*_{\alpha,0}(1;\nu)=\nu^5 R_{\alpha,0}(\nu),\ \text{where } \nu\in\mathbb{C}. \tag{1.15}$$

We use Theorem 1 and Theorem 2 with $\alpha=1$ in [16] $-$ [17] where we obtain continued fraction for $\zeta(3)$ parametrized by a family of points on projective line. The calculation of the matrix $A^*_{\alpha,0}(1;\nu)$ and proofs of Theorem 1 and Theorem 2 for arbitrary $\alpha\in\mathbb{N}_0$ are long and connected with heavy calculations. Therefore we decided to present calculations of the matrix $A^*_{\alpha,0}(z;\nu)$ and proofs of Theorem 1 and Theorem 2 for $\alpha=1$. We do this in Section 2 $-$ Section 5.

2. Some Relations for the Functions from Section 1 in the Case $\alpha = 1$

Let $M_\alpha=\mathbb{Z}\diagdown(-1-\alpha,0)\cap\mathbb{Z}$. If $\alpha=1$, then $M_\alpha=\mathbb{Z}\diagdown\{-1\|$ and,in view of (1.2), (1.14),

$$\tau=\tau_1(\nu)=\nu+1,\quad \mu=\mu_1(\nu)=(\nu+1)^2=\tau^2, \tag{2.1}$$

$$r_{1,0,1}(\nu)=\tau^4,\quad r_{1,0,2}(\nu)=r_{1,0,4}(\nu)=0,\quad r_{1,0,3}(\nu)=-2\tau^2. \tag{2.2}$$

Lemma 2.1. *Let* ν, α *and* w *be variables,*

$$P(\alpha,w,\nu)=\nu^3(\nu+\alpha)^2-2\nu^2(\nu+\alpha)(2\nu+\alpha)w \\ +\nu(2\nu+\alpha)(4\nu+3\alpha)w^2-2(2\nu+\alpha)(3\nu+2\alpha)w^3, \tag{2.3}$$

$$Q\alpha,w,\nu)=(2\nu+\alpha)(\nu(8\nu+5\alpha)+2(3\nu+2\alpha)w), \tag{2.4}$$

$$T^\wedge(\alpha,w,\nu)=(w+\nu)^2P(\alpha,w,\nu)+w^4Q(\alpha,w,\nu)-\nu^5(w-\nu-\alpha)^2. \tag{2.5}$$

Then $T^\wedge(\alpha,w,\nu)=0$.

Proof. I view of (2.3), (2.4) and definition in (2.5), if $(T^\wedge(\alpha,w,\nu)\neq 0$, then $\deg_\alpha(T_0^\wedge(\alpha,w,\nu))\leq 2$, and, consequently, it is sufficient to check the equality (2.5) for $\alpha=-2\nu$, $\alpha=-\nu$ and $\alpha=w-\nu$. If $\alpha=-2\nu$, then

$$P(\alpha,w,\nu)=\nu^5,\quad Q(\alpha,w,\nu)=0,\quad (w-\nu-\alpha)^2=(w+\nu)^2,$$

and (2.5) holds. If $\alpha=-\nu$, then

$$P(\alpha,w,\nu)=\nu^2w^2(\nu-2w),\quad Q(\alpha,w,\nu)=\nu^2(3\nu+2w),\quad (w-\nu-\alpha)^2=w^2,$$

$$(w+\nu)^2\nu^2w^2(\nu-2w)+w^4\nu^2(3\nu+2w)=$$

$$w^2\nu^2((w+\nu)(\nu^2-w\nu-2w^2)+w^2(3\nu+2w))=$$

$$w^2\nu^2(\nu(nu^2-w^2)-2w^2(w+\nu))+w^2(3\nu+2w))=w^2\nu^5.$$

If $\alpha=w-\nu$, then

$$P(\alpha,w,\nu)=$$

$$\nu^3w^2-2\nu^2(\nu+w)w^2+\nu(\nu+w)(\nu+3w)w^2-2(\nu+w)(\nu+2w)w^3=$$

$$\nu^3w^2+(\nu+w)(-2\nu^2+\nu^2+3w\nu-2w\nu-4w^2)w^2=$$

$$\nu^3w^2+(\nu+w)(-\nu^2+w\nu-4w^2)w^2=$$

$$\nu^3w^2+\nu(w^2-\nu^2)w^2+(\nu+w)(-4w^2)w^2=-(3w^2\nu+4w^2)w^2=-(3\nu+4w)w^4,$$

$$Q(\alpha,w,\nu)=(\nu+w)(3\nu^2+7w\nu+4w^2)=(\nu+w)(\nu+w)(3\nu+4w)=$$

$$w^4(\nu+w)^2(3\nu+4w),\ (w-\nu-\alpha)^2=0,$$

and (2.5) holds. □

In view of (2.3), (2.3), we put $P(w,\nu)=P(1,w,\nu)$, $Q(w,\nu)=Q(1,w,\nu)$ So, the equality (2.5) take the form

$$(w+\nu)^2P(w,\nu)+w^4Q(w,\nu)-\nu^5(w-\nu-1)^2=0. \tag{2.6}$$

Let

$$D_1(z,\nu,w)=D_1^*(z,\tau,w)=z(w-\nu-1)^2(w+\nu+1)^2-w^4=z(w^2-\tau^2)^2-w^4,$$

and w is independent variable. Clearly,

$$D_1(z,\nu,w)=D_1^*(z,\tau,w)=D_1^*(z,-\tau,w)=D_1(z,-\nu-2,w) \tag{2.7}$$

for $\nu\in\mathbb{Z}$. In view of (2.2),

$$D_1(z,\nu,w)=D_1^*(z,\tau,w)=(z-1)w^4+z(r_{1,0,1}(\nu)+r_{1,0,3}(\nu)w^2). \tag{2.8}$$

In view of (1.4 – (1.9), (1.11, (1.13, let $A^*(z;\nu)=A^*_{1,0}(z;\nu)$,

$$f_k(z,\nu)=f_{1,0,k}(z,\nu),\ f_k^ast(z,\nu)=f^*_{1,0,k}(z,\nu),\ r_j(\nu)=r_{1,0,j}(\nu)$$

$$X_k(z,\nu)=X_{1,0,k}(z,\nu),\ X_k^*(z,\nu)=X^*_{1,0,k}(z,\nu),\bar{R}(\nu)=\bar{R}_{1,0}(\nu)$$

where $k = 1, 2, 3$, and $j = 1, 2, 3, 4$. According to the general properties of Mejer's functions (see [2], chapter 5),

$$D_1(z, \nu, \delta)y = 0, \tag{2.9}$$

for $y = f_k(z, \nu)$, where $\nu \in \mathbb{N}_0$, $k \in \mathrm{K}_0 = \{1, 2, 3\}$,

$$(\delta + \nu + 1)^2 f_k(z, \nu) = (\delta - \nu - 2)^2 f_k(z, \nu + 1), \tag{2.10}$$

where $\nu \in \mathbb{N}_0$, $k \in \mathrm{K}_0$. Therefore

$$(\delta + \nu)^2 f_k(z, \nu - 1) = (\delta - \nu - 1)^2 f_k(z, \nu), \tag{2.11}$$

$$(\delta + \tau - 1)^2 f_k(z, \nu - 1) = (\delta - \tau)^2 f_k(z, \nu), \tag{2.12}$$

where $\nu \in \cap\mathbb{N}$, $k \in \mathrm{K}_0$. Let

$$f_k(z, -\nu - 2) = f_k(z, \nu), \text{ where } \nu \in \mathbb{N}_0,\ k \in \mathrm{K}_0 = \{1, 2, 3\}. \tag{2.13}$$

So, $f_k(z, \nu)$ is defined for $\nu \in M_1 = \mathbb{Z}\diagdown\{-1\}$ Moreover, if $\nu \in (-\infty, -2] \cap \mathbb{Z}$, then $\nu_1 = -\nu - 2 \in [0, +\infty)) \cap \mathbb{Z}$, and, in view of (2.13),

$$f_k(z, \nu) = f_k(z, \nu_1) = f_k(z, -\nu - 2).$$

Therefore (2.13) holds for all the $\nu \in M_1$ Furthermore,

$$\delta^s f_k(z, -\nu - 2) = \delta^s f_k(z, \nu), \tag{2.14}$$

where $s \in \mathbb{N}_0$, $\nu \in M_1$, $k \in \mathrm{K}_0$. In view of (2.7), (2.14), the equality (2.9) holds for $\nu \in M_1$. Moreover, if $\nu \in (-\infty, -2] \cap \mathbb{Z}$, then,$\nu_1 = -\nu - 2 \in \mathbb{N}_0$, and $\nu_1 + 1 = -\nu - 1 \in \mathbb{N}$. Therefore, in view of (2.10), (2.14),

$$(\delta - \nu - 1)^2 f_k(z, \nu) = (\delta + \nu_1 + 1)^2 f_k(z, \nu_1) = \tag{2.15}$$

$$(\delta - \nu_1 - 2)^2 f_k(z, \nu_1 + 1) =$$

$$(\delta + \nu)^2 f_k(z, -\nu - 1) = (\delta + \nu)^2 f^{\vee}_{\alpha,0,k}(z, \nu - 1);$$

if $\nu \in (-\infty, -3] \cap \mathbb{Z}$, then $\nu + 1 \in (-\infty, -2] \cap \mathbb{Z}$, and, in view (2.15)

$$(\delta - \nu - 2)^2 f_k(z, \nu + 1) = (\delta + \nu + 1)^2 f_k(z, \nu). \tag{2.16}$$

consequently, the equality (2.11) holds for $\nu \in M_1^* = \mathbb{Z}\diagdown\{-1, 0\}$, and the equality (2.10) holds for $\nu \in \mathbb{Z}\diagdown\{-2, -1\}$.

In view of (2.11), (2.5), (2.15), (2.8), (2.9),

$$\nu^5(\delta+\nu)^2 f_k(z,\nu-1) = \nu^5(\delta-\nu-1)^2 f_k(z,\nu) = \tag{2.17}$$

$$((\delta+\nu)^2 P(\delta,\nu) + \delta^4 Q(\delta,\nu)) f_k(z,\nu) =$$

$$((\delta+\nu)^2 P(\delta,\nu) + Q(\delta,\nu)\delta^4) f_k(z,\nu) =$$

$$((\delta+\nu)^2 P_{\alpha,0}(\delta,\nu) + Q_{\alpha,0}(\delta,\nu) z(\delta+\nu+1)^2(\delta-\nu-1)^2) f^{\vee}_{\alpha,0,k}(z,\nu),$$

where $\nu \in M_1^*$. Clearly,

$$z(\delta+\nu+1)^2 = (\delta+\nu)^2 z,\ Q(\delta,\nu)(\delta+\nu)^2 = \tag{2.18}$$

$$(\delta+\nu)^2 Q(\delta,\nu),\ Q(\delta,\nu)z = zQ(\delta+1,\nu).$$

(We consider left and right parts of (2.18) as operators, for example, z denotes there the operator of multiplication by z.) Therefore, in view of (2.17), (2.18)

$$(\delta+\nu)^2\nu^5 f_k(z,\nu-1) = (\delta-\nu-1)^2\nu^5 f_k(z,\nu) = \tag{2.19}$$

$$((\delta+\nu)^2 P(\delta,\nu) + Q(\delta,\nu)(\delta+\nu)^2 z(\delta-\nu-1)^2) f_k(z,\nu) =$$

$$((\delta+\nu)^2 P(\delta,\nu) + (\delta+\nu)^2 Q(\delta,\nu) z(\delta-\nu-1)^2) f_k(z,\nu) =$$

$$((\delta+\nu)^2 (P(\delta,\nu) + zQ(\delta+1,\nu)(\delta-\nu-1)^2)) f_k(z,\nu).$$

where $\nu \in M_1^*$. If $\nu \in \mathbb{Z}\diagdown\{-2,-1\}$, then $-\nu-3 \in \mathbb{Z}\diagdown\{-2,-1\} \subset M_1$, and, clearly, $\nu_1 = -\nu-2 \in M_1^*$. Therefore, in view of (2.19), (2.14)

$$(\delta-\nu-2)^2(-\nu-2)^5 f_k(z,\nu+1) = \tag{2.20}$$

$$(\delta-\nu-2)^2(-\nu-2)^5 f_k(z,-\nu-3) = (\delta+\nu_1)^2\nu_1^5 f_k(z,\nu_1-1) =$$

$$(\delta+\nu_1)^2(P(\delta,\nu_1) + zQ(\delta+1,\nu_1)(\delta-\nu_1-1)^2)) f_k(z,\nu_1) =$$

$$(\delta-\nu-2)^2\times$$

$$(P(\delta,-\nu-2) + zQ(\delta+1,-\nu-2)(\delta+\nu+1)^2)) f_k(z,-\nu-2) =$$

$$(\delta-\nu-2)(P(\delta,-\nu-2) + zQ(\delta+1,-\nu-2)(\delta+\nu+1)^2)) f_k(z,\nu).$$

Let

$$\mathrm{W}_k^{\vee}(z,\nu) = \nu^5 f_k(z,\nu-1) - \tag{2.21}$$

$$(P(\delta,\nu) + zQ(\delta+1,\nu)(\delta-\nu-1)^2)) f_k(z,\nu),$$

where $\nu \in M_1^* = \mathbb{Z}\diagdown\{-1,0\}$, $k \in \mathrm{K}_0$, $|z|>1$. Let further

$$\mathrm{W}_k^{\wedge}(z,\nu) = (-\nu-2)^5 f_k(z,\nu+1) - \tag{2.22}$$

$$(P(\delta, -\nu - 1 - \alpha) + zQ(\delta + 1, -\nu - 1 - \alpha)(\delta + \nu + 1)^2))f_k(z, \nu),$$

where $\nu \in \mathbb{Z} \diagdown \{-2, -1\}$, $k \in \mathrm{K}_0$.

In view of (2.14), (2.21), (2.22), if $\nu \in \mathbb{Z} \diagdown \{-2, -1\}$, $k \in \mathrm{K}_0$, then

$$\mathrm{W}_k^{\vee}(z, -\nu - 2) = (-\nu - 2)^5 f_k(z, -\nu - 3) - \tag{2.23}$$

$$(P(\delta, -\nu - 2) + zQ(\delta + 1, -\nu - 2)(\delta + \nu + 11)^2))f_k(z, -\nu - 2) =$$

$$(-\nu - 2)^5 f_k(z, \nu + 1) - P(\delta, -\nu - 2) f^{\vee}_{\alpha,0,k}(z, \nu) -$$

$$zQ(\delta + 1, -\nu - 2)(\delta + \nu + 1)^2) f^{\vee}_{\alpha,0,k}(z, \nu) = \mathrm{W}_k^{\wedge}(z, \nu).$$

and, if $\nu \in \mathbb{Z} \diagdown \{-1, -0\}$, $k \in \mathrm{K}_0$, then

$$\mathrm{W}_k^{\wedge}(z, -\nu - 2) = \nu^5 f_k(z, -\nu - 1) - \tag{2.24}$$

$$(P(\delta, \nu) + zQ_{\alpha,0}(\delta + 1, \nu)(\delta - \nu - 1)^2))f_k(z, -\nu - 2) =$$

$$\nu^5 f^{\vee}_{\alpha,0,k}(z, \nu - 1) -$$

$$(P_{\alpha,0}(\delta, \nu) + zQ_{\alpha,0}(\delta + 1, \nu)(\delta - \nu - 1)^2))f_k(z, \nu) = \mathrm{W}_k^{\vee}(z, \nu).$$

Clearly, (2.19) can be rewritten in the form

$$(\delta + \nu)^2 \mathrm{W}_k^{\vee}(z, \nu) = 0, \tag{2.25}$$

where $\nu \in M_1^* = \mathbb{Z} \diagdown \{-1, 0\}$, $k \in \mathrm{K}_0$, $|z| > 1$, and (2.20) can be rewritten in the form

$$(\delta - \nu - 2)^2 \mathrm{W}_k^{\wedge}(z, \nu) = 0, \tag{2.26}$$

where $\nu \in \mathbb{Z} \diagdown \{-2, -1\}$, $k \in \mathrm{K}_0$, $|z| > 1$.

We want to prove the following equalities:

$$\mathrm{W}_k^{\vee}(z, \nu) = 0, \text{ where } \nu \in M_1^*, \ k \in \mathrm{K}_0, \ |z| > 1, \tag{2.27}$$

$$\mathrm{W}_k^{\wedge}(z, \nu) = 0, \text{ where } \nu \in \mathbb{Z} \diagdown \{-2, -1\}, \ k \in \mathrm{K}_0, \ |z| > 1, \tag{2.28}$$

In view of (1.4), (2.21), $\mathrm{W}_1^{\vee}(z, \nu) \subset \mathbb{C}[z]$ for fixed $\nu \in M_1^*$. If $\nu \in \mathbb{N}$, then null-space of $(\delta + \nu)^2$ as linear operator on $\mathbb{C}[z]$ coincides with $\{0\}$. Consequently, the equality (2.27) holds for $\nu \in \mathbb{N}$, $k = 1$. If $\nu \in (-\infty, -3] \cap \mathbb{Z}$, then, clearly, we have $\nu_1 = -\nu - 2 \in \mathbb{N}$; hence, in view of (2.27) and (2.23), $\mathrm{W}_1^{\wedge}(z, \nu) = 0$. So, (2.28) holds for $k = 1$, if $\nu \in (-\infty, -3] \cap \mathbb{Z}$. In view of (2.4),

$$Q(w, -\nu - 2) = -(2\nu + 3)((\nu + 2)(8\nu + 11) - (6\nu + 8)w), \tag{2.29}$$

$$Q(\delta + 1, -\nu - 2) z^{\nu + 1} = -(\nu + 2)(2\nu + 3)^2 z^{\nu + 1}. \tag{2.30}$$

If $\nu \in \mathbb{N}_0$, then in view of (1.4), (2.21), (2.30), $\deg_z \mathrm{W}_k^{\wedge}(z,\nu) = \nu+2$; the null-space of the operator $(\delta-\nu-2)^2$ (as linear operator on $\mathbb{C}[z]$ coincides with $\mathbb{C}z^{\nu+2}$; therefore,in view of (2.30) to establish (2.28) in this case, we must check the equality

$$(-\nu-2)^5\frac{1}{(\nu+2)^2}\binom{2\nu+3}{\nu+2}^2 = -\frac{1}{(\nu+1)^2}\binom{2\nu+1}{\nu+1}^2(2\nu+2)^2(2+\nu)(2\nu+3)^2,$$

which is equivalent to the equality

$$(\nu+2)^3\frac{(2\nu+3)^2(2\nu+2)^2}{(\nu+2)^2(\nu+1)^2} = \frac{1}{(\nu+1)^2}(2\nu+2)^2(2+\nu)(2\nu+3)^2,$$

and the last equality, clearly, holds. So, (2.28) holds for $k=1$.

If $\nu \in (-\infty,-2]\cap\mathbb{Z}$, then $\nu_1 = -\nu-2 \in \mathbb{N}_0$, and since the equality (2.28) holds for $k=1$, it follows from (2.24) that $\mathrm{W}^{\vee}_{\alpha,0,1}(z,\nu)=0$. Consequently, the equality (2.27) holds for $k=1$.

In view of (1.5),(2.22), $\mathrm{W}_2^{\wedge}(z,\nu) \in \mathbb{C}[[z^{-1}]]$, where $\nu \in \mathbb{N}_0$, $|z|>1$ and, as usually, $\mathbb{C}[[x]]$ denotes the linear space (and also ring) of all the formal power series over $\mathbb{C}$ with variable x; furthermore, it follows from (1.3) that, if $t \in [1,\nu]\cap\mathbb{Z}$, then $R(\alpha,t,\nu)=0$; therefore $z^{\nu+1}\mathrm{W}_2^{\wedge}(z,\nu) \in \mathbb{C}[[z^{-1}]]$, where $\nu \in \mathbb{N}_0$, $|z|>1$.

If $\nu \in \mathbb{N}_0$, then null-space of the operator $(\delta-\nu-2)^2$ (as operator on linear over $\mathbb{C}$ space $\mathbb{C}[[z^{-1}]]$) coincides with $\{0\}$. Therefore, in view of (2.26), the equality (2.28) holds for $k=2$, $\nu\in\mathbb{N}_0$ and $|z|>1$.

If $\nu \in (-\infty,-2)\cap\mathbb{Z}\,|z|>1$, then $\nu_1=-\nu-2\in\mathbb{N}_0$; since (2.28) holds for $k=2$, it follows from (2.24) that $\mathrm{W}_2^{\vee}(z,\nu)=0$.

So, the equality (2.27) holds for $k=2$, $\nu\in(-\infty,-2]\cap\mathbb{Z}\,|z|>1$.

In view of (2.4),

$$Q(\delta+1,\nu)z^{-\nu-1} = (2\nu+1)(\nu(8\nu+5)+2(3\nu+2)(-nu)))z^{-\nu-1} = \nu(2\nu+1)^2)z^{-\nu-1}. \tag{2.31}$$

In view of (1.3),

$$(R(\nu+1,\nu))^2 = \frac{(\nu!)^4}{((2\nu+2)!)^2}, (R(\nu,\nu-1))^2 = \frac{((\nu-1)!)^4}{((2\nu)!)^2} \tag{2.32}$$

According to (2.21), (1.5) and (1.3), $z^{\nu}\mathrm{W}_2^{\vee}(z,\nu)\in\mathbb{C}[[z^{-1}]]$, for $\nu\in\mathbb{N}$ and for $|z|>1$. Hence, in view of (2.21), (1.5), (2.31), (2.32), to establish (2.27) in this case, we must check the equality

$$z^{-\nu}\nu^5(R(\nu,\nu-1))^2 = z^{-\nu}(R(\nu+1,\nu))^2(2\nu+2)^2)\nu(2\nu+1)^2,$$

i.e. the equality $\nu^5\dfrac{((\nu-1)!)^4}{((2\nu!)^2}=\dfrac{((\nu!)^4}{((2\nu+2)!)^2}(2\nu+2))^2\nu(2\nu+1)^2,$

which, evidently, holds. So, the equality (2.27) holds for $k=2$.

Therefore, if $\nu\in(-\infty,-3]\cap\mathbb{Z}$, then $\nu_1=-\nu-2\in\mathbb{N}$; since (2.27) holds for $k=2$, it follows from (2.23) that $\mathrm{W}_2^{\wedge}(z,\nu)=0$. So, the equality (2.28) holds for $k=2$.

In view of (1.7), $\mathrm{W}_3^{\wedge}(z,\nu)\in\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]$, for $\nu\in\mathbb{N}_0$, $|z|>1$, moreover $z^{\nu+1}\mathrm{W}_3^{\wedge}(z,\nu)\in\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]$, for $\nu\in\mathbb{N}_0$, $|z|>1$. We can interpret $\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]$ as direct sum $\mathbb{C}[[z^{-1}]]\oplus_{\mathbb{C}}\mathbb{C}[[z^{-1}]]$ with linear operator δ, which acts according to the formula

$$\delta\begin{pmatrix}T_1\\T_2\end{pmatrix}=\begin{pmatrix}\delta T_1+T_2\\\delta T_2\end{pmatrix},\tag{2.33}$$

where $T_k\in\mathbb{C}[[z^{-1}]]$ for $k=1,\,2$. For those, who want more detailed explanation, we add, that if $h(z)=T_1(z)+(\log(z))T_2(z)$, where functions $T_1(z)$ and $T_2(z)$ are regular for $|z|>1$, then the function $h(z)$ can be analytically prolonged to $H(Z)=H((r,\varphi))=T_1(r\exp(i\varphi))+(i\varphi+\log(r))T_2(r\exp(i\varphi))$, where $Z=(r,\varphi)$ lie on Rimannian surface of $Log(z)$, H is uniquely defined by h, $r>1$, $\varphi\in\mathbb{R}$. Then $H((r,\varphi+2\pi))-H((r,\varphi))=2i\pi T_2(r\exp(i\varphi))$, and functions T_1, T_2 are uniquely defined by h.

If $\nu\in\mathbb{N}_0$, then null-space of the operator $(\delta-\nu-2)^2$ (as operator, which acts on $\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]$ according to (2.33) coincides with $\{0\}$. Therefore, in view of (2.26), the equality (2.28) holds for $k=3$, $\nu\in\mathbb{N}_0$ and $|z|>1$.

If $\nu\in(-\infty,-2]\cap\mathbb{Z}\,|z|>1$, then $\nu_1=-\nu-2\in\mathbb{N}_0$; since (2.28) holds for $k=3$, it follows from (2.24) that $\mathrm{W}_3^{\vee}(z,\nu)=0$.

So, (2.27) holds for $k=3$, $\nu\in(-\infty,-1-\alpha]\cap\mathbb{Z}\,|z|>1$. Let $H(w)\in\mathbb{C}[w]$ Then, according to the Leibnitz formula

$$H(\delta)((\log(z)f(z))=(\log(z))H(\delta)f(z)+\left(\frac{dH}{dw}\right)\Big|_{w=\delta}f(z).$$

Since (2.27) holds if $k=2$, $\nu\in\mathbb{N}$, $|z|>1$, it follows from (2.21), (1.7) that

$$\mathrm{W}_3^{\vee}(z,\nu)=(\log(z))\mathrm{W}_2^{\vee}(z,\nu)+\nu^5f_4(z,\nu-1)-\tag{2.34}$$

$$(P(\delta,\nu)+zQ(\delta+1,\nu)(\delta-\nu-1)^2))f_4(z,\nu)-$$

$$\left(\frac{dP(w,\nu)}{dw}\Big|_{w=\delta}+z\frac{dQ(w+1,\nu)(w-\nu-1)^2}{dw}\Big|_{w=\delta}\right)f_2(z,\nu)=$$

$$\nu^5f_4(z,\nu-1)-(P(\delta,\nu)+zQ_(\delta+1,\nu)(\delta-\nu-1)^2))f_4(z,\nu)-$$

$$\left(\frac{d}{dw}P(w,\nu)\Big|_{w=\delta}+z\frac{dQ(w+1,\nu)(w-\nu-1)^2}{dw}\Big|_{w=\delta}\right)f_2(z,\nu)$$

for $\nu\in\mathbb{N}$, $|z|>1$. In view of (2.25) to establish the equality (2.27) in the case $k=3$, $\nu\in\mathbb{N}$, $|z|>1$, we must check the equality

$$-\nu^5\frac{\partial(R(t,\nu-1))^2}{\partial t}\Big|_{t=\nu}z^{-\nu}- \tag{2.35}$$

$$\left(-\frac{\partial(R(t,\nu))^2}{\partial t}\Big|_{t=\nu+1}\right)zQ(\delta+1,\nu)(\delta-\nu-1)^2)z^{-\nu-1}-$$

$$(R(\nu+1,\nu))^2z\frac{dQ(w+1,\nu)(w-\nu-1)^2}{dw}\Big|_{w=\delta}z^{-\nu+1}=0.$$

In view of (1.3), (1.5), (1.6), (1.7), (2.32), (2.4),

$$-\nu^5\frac{\partial(R(t,\nu-1))^2}{\partial t}\Big|_{t=\nu}z^{-\nu}- \tag{2.36}$$

$$\left(-\frac{\partial(R(t,\nu))^2}{\partial t}\Big|_{t=\nu+1}\right)zQ_(\delta+1,\nu)(\delta-\nu-1)^2)z^{-\nu-1}=$$

$$2\nu^5(((\nu-1)!)^4/((2\nu)!)^2\left(\left(\sum_{k=1}^{\nu-1}\frac{1}{k}\right)-\sum_{k=\nu}^{2\nu}\frac{1}{k}\right)-$$

$$2((\nu!)^4/((2\nu+2)!)^2)\left(\left(\sum_{k=1}^{\nu}\frac{1}{k}\right)-\sum_{k=\nu+1}^{2\nu+2}\frac{1}{k}\right)\nu(2\nu+1)^2)(2\nu+2)^2z^{-\nu}=$$

$$\nu(2(\nu!)^4/((2\nu)!)^2)(-2/\nu+1/(2\nu+1)+1/(2\nu+2)z^{-\nu}=$$

$$-(2(\nu!)^4/(2\nu)!)^2)(4\nu^2+9\nu+4)/((2\nu+1)(2\nu+2)).$$

In view of (2.4),

$$\frac{dQ(w+1,\nu)(w-\nu-1)^2}{dw}=(2\nu+1)(2(3\nu+2))(w-\nu-1)^2+$$

$$(2\nu+1)(\nu(8\nu+5)+2(3\nu+2)(w+1))2(w-\nu-1),$$

$$\frac{dQ(w+1,\nu)(w-\nu-1)^2}{dw}\Big|_{w=\delta}z^{-\nu-1}=8(2\nu+1)(3\nu+2)(\nu+1)^2- \tag{2.37}$$

$$4(2\nu+1)(\nu+1)(-6\nu^2-4\nu+8\nu^2+5\nu)=4(2\nu+1)(\nu+1)(4\nu^2+9\nu+4)z^{-\nu-1},$$

$$(R(\nu+1,\nu))^2 z \frac{dQ(w+1,\nu)(w-\nu-1)^2}{dw}\bigg|_{w=\delta} z^{-\nu-1} = \tag{2.38}$$
$$((\nu!)^4/(2\nu+2)!)^2)2(2\nu+1)(2\nu+2)(4\nu^2+9\nu+4)z^{-\nu}.$$

According to (2.36) and (2.38), the equality (2.35) holds, and, consequently, the equality (2.27) holds for $k=3$, $|z|>1$.

If $\nu \in (-\infty,-3]\cap\mathbb{Z}$, then then $\nu_1=-\nu-2\in\mathbb{N}$; since (2.27) holds for $k=3$, it follows from (2.23) that $\mathrm{W}^{\wedge}_{\alpha,0,3}(z,\nu)=0$.

So, (2.28) holds for $k=3$, $|z|>1$.

3. Calculations of the Matrix $A^*(z;\nu)$

In view of (2.8), let $b_j(z;\nu)=-z(z-1)^{-1}r_j(z;\nu)$ for $j=1, 2, 3, 4$,and let Let $B_(z;\nu)$ be 4×4-matrix, which has $b_j(z;\nu)$ as j-th element of its last row, has 1 as $(i+1)$-th element of its i-th row for $i=1, 2, 3$ and has 0 on all other places. Then, in view of (2.9), (2.10),

$$X_k(z;-\nu-2)=X_k(z;\nu),\ \delta X_k(z;\nu)=B(z;\nu)X_k(z;\nu), \tag{3.1}$$

where $k\in\mathrm{K}_0$, $|z|>1$, $\nu\in M_1$. In view of (2.4),

$$Q(w+1,\nu)=(2\tau-1)(8\tau^2-5\tau+1+(6\tau-2)w),$$

Therefore

$$Q(w+1,\nu)(w-\nu-1)^2=q_0(\nu)+q_1(\nu)w+q_2(\nu)w^2+q_3(\nu)w^3,$$

where

$$q_0(\nu)=(2\tau-1)(8\tau^4-5\tau^3+\tau^2),\ q_1(\nu)=-2\tau(2\tau-1)(5\tau^2-4\tau^2+1),$$
$$q_2(\nu)=-(2\tau-1)(4\tau^2+\tau-1),\ q_3(\nu)=2(2\tau-1)(3\tau-1),$$
$$P(w,\nu)=p_0(\nu)+p_1(\nu)w+p_2(\nu)w^2+p_3(\nu)w^3,$$

where

$$p_0(\nu)=(\tau-1)^3\tau^2=\nu^3(\nu+1)^2,$$
$$p_1(\nu)=-2(\tau-1)^2(2\tau-1)\tau=-2\nu^2(2\nu+1)(\nu+1),$$
$$p_2(\nu)=(\tau-1)(4\tau-1)(2\tau-1)=\nu(4\nu+3)(2\nu+1),$$
$$p_3(\nu)=-2(2\tau-1)(3\tau-1)=-2(2\nu+1)(3\nu+2).$$

Let $a^*_{1,k}(z;\nu) = q_{k-1}(\nu)z + p_{k-1}(\nu)$ for $k = 1,\ 2\,3\,4$,

$$\mathbf{a}^*_{1,1}(z;\nu) = \tau^2((2\tau-1)(8\tau^2-5\tau+1)z + (\tau-1)^3) =$$

$$(\nu+1)^2((2\nu+1)(8\nu^2+11\nu+4)z+\nu^3,\ \mathbf{a}^*_{1,2}(z;\nu) =$$

$$-2\tau(2\tau-1)((5\tau^2-4\tau+1)z+(\tau-1)^2) = -2(\nu+1)(2\nu+1)((5\nu^2+6\nu+2)z+\nu^2),$$

$$\mathbf{a}^*_{1,3}(z;\nu) = (2\tau-1)((-4\tau^2-\tau+1)z+(\tau-1)(4\tau-1)) = (2\nu+1)\times$$

$$((-4\nu^2-9\nu-4)z+\nu(4\nu+3)),\ \mathbf{a}^*_{1,4}(z;\nu) = 2(2\tau-1)(3\tau-1)(z-1) =$$

$$2(2\nu+1)(3\nu+2)(z-1),\ \bar{a}^*_1(z;\nu) = (a^*_{1,1}(z;\nu),\ a^*_{1,2}(z;\nu),\ a^*_{1,3}(z;\nu),\ a^*_{1,4}(z;\nu)).$$

Then, in view of (2.21), (2.27), (1.9), $\nu^5 f_k(z;\nu-1) = \bar{a}^*_1(z;\nu)X_k(z;\nu)$, and therefore $\delta^{r-1}\nu^5 f_k(z;\nu-1) = \bar{a}^*_r(z;\nu)X_k(z;\nu)$ for $r = 2,\ 3,\ 4$, where, in view of (3.1), $\bar{a}^*_r$, can be calculated recurrently, as follows:

$$\bar{a}^*_r(z;\nu) = (a^*_{r,1}(z;\nu),\ a^*_{r,2}(z;\nu),\ a^*_{r,3}(z;\nu),\ a^*_{r,4}(z;\nu)) = \tag{3.2}$$

$$\delta\bar{a}^*_{r-1}(z;\nu) + a^*_{r-1}(z;\nu)B(z;\nu).$$

So, we have

$$\mathbf{a}^*_{2,1}(z;\nu) = \tau^2((2\tau-1)(8\tau^2-5\tau+1)z - 2(2\tau-1)(3\tau-1)\tau^4 z =$$

$$\tau^2(2\tau-1)(-6\tau^3+10\tau^2-5\tau+1)z = -\tau^2(2\tau-1)(\tau-1)(6\tau^2-4\tau+1)z =$$

$$-(\nu+1)^2(2\nu+1)\nu(6\nu^2+8\nu+3)z,\ \mathbf{a}^*_{2,2}(z;\nu) = -2\tau(2\tau-1)(5\tau^2-4\tau+1)z+$$

$$\tau^2((2\tau-1)(8\tau^2-5\tau+1)z+\tau^2(\tau-1)^3 = \tau(2\tau-1)(8\tau^3-15\tau^2+9\tau-2)z+$$

$$\tau^2(\tau-1)^3 = \tau(2\tau-1)(\tau-1)(8\tau^2-7\tau+2)z+\tau^2(\tau-1)^3 = (\nu+1)^2\nu^3+$$

$$(\nu+1)(2\nu+1)\nu(8\nu^2+9\nu+3,\ \mathbf{a}^*_{2,3}(z;\nu) = (2\tau-1)\times$$

$$((-4\tau^2-\tau+1)z-2\tau((5\tau^2-4\tau+1)z+(\tau-1)^2)+4\tau^2(3\tau-1)z) =$$

$$(2\tau-1)(2\tau^3-3\tau+1)z-2\tau(\tau-1)^2) = (2\tau-1)(\tau-1)\times$$

$$((2\tau^2+2\tau-1)z-2\tau(\tau-1)) = (2\nu+1)\nu((2\nu^2+6\nu+3)z-2(\nu+1)\nu),$$

$$\mathbf{a}^*_{2,4}(z;\nu) = (2\tau-1)((-4\tau^2-\tau+1)z+(\tau-1)(4\tau-1))+2(2\tau-1)(3\tau-1)z =$$

$$(-4\tau^2+5\tau-1)z+(\tau-1)(4\tau-1))(2\tau-1) = -(2\tau-1)(\tau-1)(4\tau-1)(z-1) =$$

$$-(2\nu+1)\nu(4\nu+3)(z-1),\ \mathbf{a}^*_{3,1}(z;\nu) = -\tau^2(2\tau-1)(\tau-1)\times$$

$$((6\tau^2-4\tau+1)z+(4\tau-1)\tau^2 z) = (4\tau^3-7\tau^2+4\tau-1)\tau^2(\tau-1)z =$$

$$\tau^2(2\tau-1)(\tau-1)^2(4\tau^2-3\tau+1)z=(2\nu+1)\nu^2(\nu+1)^2(4\nu^2+5\nu+2),$$

$$\mathbf{a}^*_{3,2}(z;\nu)=\tau(2\tau-1)(\tau-1)((8\tau^2-7\tau+2)z-\tau(6\tau^2-4\tau+1)z)=$$

$$-\tau(2\tau-1)(\tau-1)(6\tau^3-12\tau^2+8\tau-2)z=-\tau(2\tau-1)(\tau-1)^2(6\tau^2-6\tau+2)z=$$

$$-(\nu+1)(2\nu+1)\nu^2(6\nu^2+6\nu+2)z,\ \mathbf{a}^*_{3,3}(z;\nu)=(\tau-1)(2\tau-1)\times)$$

$$(2\tau^2+2\tau-1)z-2(4\tau-1)\tau^2z+\tau(8\tau^2-7\tau+2)z+\tau^2(\tau-1)^2)=$$

$$(\tau-1)(-(2\tau-1)(\tau-1)(3\tau-1)z+\tau^2(\tau-1)^2)=$$

$$(\tau-1)^2(-(2\tau-1)(3\tau-1)z+\tau^2(\tau-1))=\nu^2(-(2\nu+1)(3\nu+2)z+\nu(\nu+1)^2),$$

$$\mathbf{a}^*_{3,4}(z;\nu)=(2\tau-1)(\tau-1)(-(4\tau-1)z+(2\tau^2+2\tau-1)z-2(\tau-1)\tau)=$$

$$2(2\tau-1)(\tau-1)^2\tau(z-1)=2(2\nu+1)\nu^2(\nu+1)(z-1),\ \mathbf{a}^*_{4,1}(z;\nu)=$$

$$\tau^2(2\tau-1)(\tau-1)^2z(4\tau^2-3\tau+1-2\tau^3)=(2\tau-1)\tau^2(\tau-1)^3\times$$

$$(-2\tau^2+2\tau^2-1)=-\nu^3(\nu+1)^2(2\nu+1)(2\nu^2+2\nu+1),\ \mathbf{a}^*_{4,2}(z;\nu)=(2\tau-1)\times$$

$$(\tau-1)^2(-\tau(6\tau^2-6\tau+2)z+\tau^2(4\tau^2-3\tau+1)z)=(2\tau-1)(\tau-1)^2\tau)z\times$$

$$(4\tau^3-9\tau^2+7\tau-2)=(2\tau-1)(\tau-1)^3\tau(4\tau^2-5\tau+2)=(2\nu+1)\nu^3(\nu+1)\times$$

$$(4\nu^2+3\nu+1)z,\ \mathbf{a}^*_{4,3}(z;\nu)=(\tau-1)^2(2\tau-1)z(-2\tau^3+6\tau^2-5\tau+1)=$$

$$(\tau-1)^3(2\tau-1)z(-2\tau^2+4\tau-1)=\nu^3(2\nu+1)(-2\nu^2+1),$$

$$\mathbf{a}^*_{4,4}(z;\nu)=(2\tau-1)(\tau-1)^2(2\tau z-(3\tau-1))z+$$

$$\tau^2(\tau-1)^3=(\tau-1)^3(-(2\tau-1)z+\tau^2)=\nu^3((\nu+1)^2-(2\nu+1)z).$$

and,hence, for the 4×4-matrix $A^*(z;\nu)$, which has $\bar{a}^*_i(z;\nu)$ as its i-th row for $i=1,\ 2,\ 3,\ 4$ we have the equality (1.11).

4. Proof of Theorem 1 for $\alpha = 1$

Remark. *Theorem 1 would be evident if (what is impossible) the considered 3 Mejer's functions compose a fundamental system of solutions of equation (2.9), which has degree 4. Theorem 1 is useful also for test of our calculations.*

To prove equalities (1.12) and (1.15) it is sufficient consider ν as variable.

Let $v^*_{i,k}(\nu) = a^*_{i,k}(1;\nu) - a^*_{i,k}(0;\nu)$ where $\{i,k\} \subset \{1, 2, 3, 4\}$. Let $S(\nu)$ and $V^*(\nu)$ are the matrices, which have respectively $a^*_{i,k}(0,\nu)$ and $v^*_{i,k}(\nu)$ on intersection of their i-th row and k-th column. Then, according to results of previous section, $A^*(z;\nu) = S(\nu) + zV^*(\nu)$. Let N_4 be 4×4-matrix which has 1 as $(i+1)$-th element of its i-th row for $i = 1, 2, 3$ and has 0 as all other its elements. Then $S(\nu) = p_0(\nu)E_4 + p_1(\nu)N_4 + p_2(\nu)N_4^2 + p_3(\nu)N_4^3$. We denote further by $D(x,y,z,u)$ the 4×4 diagonal matrix, which has x, y, z, u as its respectively first, second, third and fourth diagonal element, and we denote by E_n the $n\times n$ unit matrix. Clearly, if $i \in [0, 3] \cap \mathbb{Z}$, then

$$p_i(\nu) = \nu^{3-i}(\nu+1)^{\max(2-i,0)}p^*_i(\nu) \text{ for } p^*_0(\nu) = 1,\ p^*_1(\nu) = -2(2\nu+1),$$

$$p^*_2(\nu) = (4\nu+3)(2\nu+1),\ p^*_3(\nu) = -2(3\nu+2)(2\nu+1). \text{ Then}$$

$$S^*(\nu) = (\nu)^{-3}(\nu+1)^{-2}(D_1(\nu(\nu+1))^{-1}S(\nu)D_1(\nu(\nu+1)) = \tag{4.1}$$

$$E_4 + \sum_{k=1}^{3} p^*_k(\nu)(N_4)^k = \begin{pmatrix} S^*_{1,1}(\nu) & S^*_{1,2}(\nu) \\ 0E_2 & S^*_{1,1}(\nu) \end{pmatrix},$$

with

$$S^*_{1,1}(\nu) = \begin{pmatrix} 1 & p^*_1(\nu) \\ 0 & 1 \end{pmatrix},$$

$$S^{**}_{1,2}(\nu) = \begin{pmatrix} p^*_2(\nu) & (\nu+1)p^*_3(\nu) \\ p^*_1(\nu) & p^*_2(\nu) \end{pmatrix}.$$

Lemma 4.1. *The following equality holds:*

$$S(\nu)S(-\nu-1) = -\nu^5(\nu+1)^5E_4. \tag{4.2}$$

Proof. Let

$$D_1(\nu) = D(1,\nu,\nu^2,\nu^3),\ D_2(\nu) = D(\nu^2,\nu,1,1). \tag{4.3}$$

Clearly,$D_1(\nu)D_1(\nu+1) = D_1(\nu(\nu+1)) = D_1(-\nu-1)D_1(-\nu)$ and ,$\nu^3(\nu+1)^2(-\nu-1)^3(-\nu)^2 = -(\nu(\nu+1))^5$. Therefore, in view of (4.1), it is sufficient to prove the equality $S^*(\nu)S^*(-\nu-1) = E_4$. Since $N_4^k = 0N_4$ for $k \in [4,\infty) \cap \mathbb{Z}$, we have

$$S^*(\nu)S^*(-\nu-1) = \mathbf{s}_0(\nu)E_4 + \mathbf{s}_1(\nu)N_4 + \mathbf{s}_2(\nu)N_4^2 + \mathbf{s}_3(\nu)N_4^3,$$

with

$$\mathbf{s}_0(\nu) = 1,\ \mathbf{s}_1^*(\nu) = p_1^*(-\nu-1) + p_1^*(\nu) = 4\nu+2-4\nu-2 = 0,$$

$$\mathbf{s}_2(\nu) = p_2^*(-\nu-1) + p_1^*(\nu)p_1^*(-\nu-1) + p_2^*(\nu) =$$

$$-2\nu^3(\nu+1)^2(-\nu-1)^2(-2\nu-1)(-\nu) + (-2)(-\nu-1)^3(-\nu)^2\nu^2(2\nu+1)(\nu+1) =$$

$$(4\nu+1)(2\nu+1) - 4(2\nu+1)^2 + (2\nu+3)(2\nu+1) = 0,\ \mathbf{s}_3(\nu) = -\nu p_3^*(-\nu-1) +$$

$$p_1^*(\nu)p_2^*(-\nu-1) + p_2^*(\nu)p_1^*(-\nu-1) + (\nu+1)p_3^*(\nu) =$$

$$2\nu(3\nu+1)(2\nu+1) - 2(\nu+1)(3\nu+2)(2\nu+1) - 2(2\nu+1)(4\nu+1)(2\nu+1) +$$

$$2(2\nu+1)(4\nu+3)(2\nu+1) =$$

$$2(2\nu+1)(3\nu^2+\nu-3\nu^2-5\nu-2) + 2(2\nu+1)^2)(4\nu+3-4\nu-1) = 0. \quad \square$$

Lemma 4.2. *The following equality holds*

$$V^*(\nu)V^*(-\nu-1) = 0E_4. \tag{4.4}$$

Proof. Let

$$V_1^*(\nu) = \begin{pmatrix} 8\nu^2+11\nu+4 & -10\nu^2-12\nu-4 & -4\nu^2-9\nu-4 & 6\nu+4 \\ -6\nu^2-8\nu-3 & 8\nu^2+9\nu+3 & 2\nu^2+6\nu+3 & -4\nu-3 \\ 4\nu^2+5\nu+2 & -6\nu^2-6\nu-2 & -3\nu-2 & 2\nu+2 \\ -2\nu^2-2\nu-1 & 4\nu^2+3\nu+1 & -2\nu^2+1 & -1 \end{pmatrix}.$$

In view of (4.3),

$$V^*(\nu) = (2\nu+1)D_1(\nu)V_1^*(\nu)D_2(\nu+1). \tag{4.5}$$

$$\text{Let } C_{h1} = \begin{pmatrix} E_2 & C_{h1,1,2} \\ 0E_2 & E_2 \end{pmatrix}$$

with

$$C_{h1,1,2} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix},$$

$$C_{h2}(\nu) = \begin{pmatrix} E_2 & 0E_2 \\ C_{h2,2,1}(\nu) & E_2 \end{pmatrix}$$

with

$$C_{h2,2,1}(\nu) = \begin{pmatrix} -1 & 2 \\ -2\nu - 2 & 3\nu + 3 \end{pmatrix},$$

Let further $V^*_{h1}(\nu)$ be a result of replacement of all elements in first and second rows in V^*_1 by zeros, $V^*_{h2}(\nu)$ be a result of replacement of all elements in first and second columns in V^*_1 by zeros, and $V^*_{h3}(\nu)$ be a result of replacement of all elements in first and second rows and columns in V^*_1 by zeros. Then, clearly,

$$C_{h1}V^*_1(\nu) = V^*_{h1}(\nu), V^*_1(\nu)C_{h2}(\nu) = V^*_{h2}(\nu), \tag{4.6}$$

$$C_{h1}V^*_1(\nu)C_{h2}(\nu) = V^*_{h3}(\nu) = \begin{pmatrix} 0E_2 & 0E_2 \\ 0E_2 & V^*_{1,2,2}(\nu) \end{pmatrix}$$

with

$$V^*_{1,2,2}(\nu) = \begin{pmatrix} -3\nu - 2 & 2\nu + 2 \\ -2\nu^2 + 1 & -1 \end{pmatrix}.$$

In view of (4.5)

$$V^*(\nu)V^*(-\nu - 1) = -(2\nu + 1)^2 \times$$

$$D_1(\nu)V^*_1(\nu)D_2(\nu + 1)D_1(-\nu - 1))V^*_1(-\nu - 1)D_2(-\nu).$$

Let $D_6(\nu) = D(1, -1, 1, -\nu)$. Clearly,

$$D_2(\nu)D_1(-\nu) = \nu^2 D_6(\nu) \tag{4.7}$$

Therefore, in view of (4.6), to prove (4.4) it is sufficient to prove the equality $V^*_{h1}(\nu)D_6(\nu + 1)V^*_{h2}(-\nu - 1) = 0E_4$. Further we have

$$V^*_{h1}(\nu)D_6(\nu + 1)V^*_{h2}(-\nu - 1) = V^*_{h3}(\nu)(C_{h2}(\nu))^{-1}D_6(\nu + 1)(C_{h1})^{-1}V^*_{h3}(\nu),$$

$$C_{h3}(\nu) = D_6(\nu + 1)(C_{h1})^{-1} = \begin{pmatrix} C_{h3,1,1}(\nu) & C_{h3,1,2} \\ 0E_2 & C_{h3,2,2}(\nu) \end{pmatrix}$$

with

$$C_{h3,1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C_{h3,1,2} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix},$$

$$C_{h3,2,1} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}, \; C_{h3,2,2} = \begin{pmatrix} 1 & 0 \\ 0 & -\nu - 1 \end{pmatrix}.$$

Therefore

$$C_{h4}(\nu) = (C_{h2}(\nu))^{-1} D_6(\nu+1)(C_{h1}(\nu))^{-1} = \begin{pmatrix} C_{h3,1,1}(\nu) & C_{h3,1,2} \\ C_{h4,2,1} & C_{h4,2,2}(\nu) \end{pmatrix}$$

with

$$C_{h4,2,1} = -C_{h2,2,1}C_{h3,1,1} + C_{h3,2,1}, \; C_{h4,2,2} = -C_{h2,2,1}C_{h3,1,2} + C_{h3,2,2} =$$

$$-\begin{pmatrix} -1 & 2 \\ -2\nu-2 & 3\nu+3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -\nu-1 \end{pmatrix} = 0E_2.$$

Clearly,

$$V_{h3}^* \begin{pmatrix} * & * \\ * & 0E_2 \end{pmatrix} V_{h3}^* = 0E_4,$$

where $*$ denotes any 2×2 matrices (i.e. exact expression of them has not influence on final result; of course, these matrices can be different each from other for $*$, which stand on different places; the $*$ plays for 2×2 matrices here the same role as $O(1)$ plays for bounded values). □

Lemma 4.3. *Let* $G^*(\nu) = (-1/(2\nu+1))S(\nu)V^*(-\nu-1)$, *and* $H^*(\nu) = (1/(2\nu+1))V^*(\nu)S(-\nu-1)$. *Then*

$$G^*(\nu) = H^*(\nu). \tag{4.8}$$

Proof. Clearly, $D_2(\nu)D_1(\nu) = \nu^2 D_7(\nu)$, $D_2(-\nu)D_1(\nu) = \nu^2 D_8(\nu)$ with

$$D_7(\nu) = D(1,\, 1,\, 1,\, \nu),\; D_8(\nu) = D(1,\, -1,\, 1,\, \nu) \tag{4.9}$$

In view of (4.1), (4.5),

$$G^*(\nu) = -\nu^3(\nu+1)^2 \times$$

$$D_1(\nu^2+\nu)S^*(\nu)D_1(1/(\nu^2+\nu))D_1(-\nu-1)V_1^*(-\nu-1))D_2(-\nu) =$$

$$-\nu^3(\nu+1)^2 D_1(\nu^2+\nu)S^*(\nu)D_1(-1/\nu))V_1^*(-\nu-1))D_2(-\nu),$$

$$H^*(\nu) = (\nu+1)^3\nu^2 \times$$

$$D_1(\nu)V_1^*(\nu)D_2(\nu+1))D_1(\nu^2+\nu))S^*(-\nu-1)D_1(1/(\nu^2+\nu)).$$ Let

$$G_1^*(\nu) = \nu^3(\nu+1)^2 D_1(\nu+1)S^*(\nu)D_1(-1/\nu)V_1^*(-\nu-1)),\; H_1^*(\nu) =$$

$$(\nu+1)^3\nu^2 V_1^*(\nu)D_2(\nu+1))D_1(\nu^2+\nu))S^*(-\nu-1)(D_1(\nu^2+\nu)D_2(-\nu))^{-1}.$$

In view of (4.9), $H_1^*(\nu)=\nu^2(\nu+1)^5\times$

$$V_1^*(\nu)D_7(\nu+1))D_1(\nu))S^*(-\nu-1)(D_1(\nu+1)D_8(\nu))^{-1}.$$

Let $G_2^*(\nu)=C_{h1}G_1^*(\nu)C_{h2}(-\nu-1)$, $H_2^*(\nu)=C_{h1}H_1^*(\nu)C_{h2}(-\nu-1)$. Then the equality (4.8) holds if and only if $-G_2^*(\nu)=H_2^*(\nu)$.

Remark 4.1. If

$$U=\begin{pmatrix} * & * \\ U_{2,1} & U_{2,2}\end{pmatrix},\ V=\begin{pmatrix} * & V_{1,2} \\ * & V_{2,2}\end{pmatrix} \text{ are any } 4\times 4 \text{ matrices}$$

$$\text{over } \mathbb{C}(\nu) \text{ then } V_{h3}(\nu)U=\begin{pmatrix} 0E_2 & 0E_2 \\ V_{1,2,2}^*(\nu)U_{2,1} & V_{1,2,2}^*(\nu)U_{2,2}\end{pmatrix},$$

$$VV_{h3}(-\nu-1)=\begin{pmatrix} 0E_2 & V_{2,2}V_{1,2,2}^*(-\nu-1) \\ 0E_2 & V_{2,2}V_{1,2,2}^*(-\nu-1)\end{pmatrix}.$$

Let $C_{h5}(\nu)=(D_7(\nu+1))^{-1}(C_{h2}(\nu))^{-1}D_7(\nu+1)D_1(\nu)$.

$$C_{h5}(\nu)=\begin{pmatrix} * & * \\ C_{h5,2,1}(\nu) & C_{h5,2,2}(\nu)\end{pmatrix}$$

with

$$C_{h5,2,1}(\nu)=\begin{pmatrix} 1 & -2\nu \\ 2 & -3\nu\end{pmatrix},$$

$$C_{h5,2,2}(\nu)=\nu^2\begin{pmatrix} 1 & 0 \\ 0 & \nu\end{pmatrix},$$

$$\text{and } C_{h6}(\nu)=C_{h5}(\nu)S^{**}(-\nu-1)=\begin{pmatrix} * & * \\ C_{h6,2,1}(\nu) & C_{h6,2,2}(\nu)\end{pmatrix}$$

with

$$C_{h6,2,1}(\nu)=C_{h5,2,1}(\nu)S_{1,1}^*(-\nu-1)=\begin{pmatrix} 1 & -2\nu \\ 2 & -3\nu\end{pmatrix}\begin{pmatrix} 1 & 4\nu+2 \\ 0 & 1\end{pmatrix}=$$

$$\begin{pmatrix} c_{h8,3,1}(\nu) & c_{h8,3,2}(\nu) \\ c_{h6,4,1}(\nu) & c_{h6,4,2}(\nu)\end{pmatrix}=\begin{pmatrix} 1 & 2\nu+2 \\ 2 & 5\nu+4\end{pmatrix}$$

$$C_{h6,2,2}(\nu)=C_{h5,2,1}(\nu)S_{1,2}^{**}(-\nu-1)+C_{h5,2,2}(\nu)S_{1,1}^{**}(-\nu-1)=$$

$$\begin{pmatrix} 1 & -2\nu \\ 2 & -3\nu\end{pmatrix}(2\nu+1)\begin{pmatrix} 4\nu+1 & (-\nu)(-6\nu-2) \\ 2 & (4\nu+1)\end{pmatrix}+$$

$$\nu^2 \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix} \begin{pmatrix} 1 & 4\nu+2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c_{h6,3,3}(\nu) & c_{h6,3,4}(\nu) \\ c_{h6,4,3}(\nu) & c_{h6,4,4}(\nu) \end{pmatrix},$$

where

$$c_{h6,3,3}(\nu) = (2\nu+1)(4\nu+1-4\nu) + \nu^2 = (\nu+1)^2,\ c_{h6,4,3}(\nu)) = (2\nu+1)\times$$

$$(2\nu+2), c_{h6,3,4}(\nu) = (2\nu+1)((6\nu^2+2\nu-8\nu^2-2\nu+2\nu^2) = 0,\ c_{h6,4,4}(\nu) =$$

$$(2\nu+1)(12\nu^2+4\nu-12\nu^2-3\nu)+\nu^3 = \nu(\nu+1)^2.\text{ Consequently,}$$

$$C_{h7}(\nu) = C_{h6}(\nu)(\nu+1)^3(D_1(nu+1)D_8(\nu))^{-1} = \begin{pmatrix} * & * \\ C_{h7,2,1}(\nu) & C_{h7,2,2}(\nu) \end{pmatrix}$$

with

$$C_{h7,2,1}(\nu) = \begin{pmatrix} c_{h7,3,1}(\nu) & c_{h7,3,2}(\nu) \\ c_{h7,4,1}(\nu) & c_{h7,4,2}(\nu) \end{pmatrix} = \begin{pmatrix} (\nu+1)^3 & -2(\nu+1)^3 \\ 2(\nu+1)^3 & -(5\nu+4)(\nu+1)^2 \end{pmatrix},$$

$$C_{h7,2,2}(\nu) = \begin{pmatrix} c_{h7,3,3}(\nu) & c_{h7,3,4}(\nu) \\ c_{h7,4,3}(\nu) & c_{h7,4,4}(\nu) \end{pmatrix} = \begin{pmatrix} (\nu+1)^3 & 0 \\ (4\nu+2)(\nu+1)^2 & (\nu+1)^2 \end{pmatrix}.$$

Hence

$$C_{h8}(\nu) = C_{h7}(\nu)C_{h2}(-\nu-1) = \begin{pmatrix} * & * \\ C_{h8,2,1}(\nu) & C_{h8,2,2} \end{pmatrix},$$

with

$$C_{h8,2,2} = C_{h7,2,2},\ C_{h8,2,1}(\nu) = \begin{pmatrix} c_{h8,3,1}(\nu) & c_{h8,3,2}(\nu \\ c_{h8,4,1}(\nu) & c_{h8,4,2}(\nu) \end{pmatrix},$$

where

$$c_{h8,3,1}(\nu) = c_{h7,3,1}(\nu) - c_{h7,3,3}(\nu) + 2\nu c_{h7,3,4}(\nu) = (\nu+1)^3 - (\nu+1)^3 = 0,$$

$$c_{h8,3,2}(\nu) = c_{h7,3,2}(\nu) + 2c_{h7,3,3}(\nu) - 3\nu c_{h7,3,4}(\nu) = -2(\nu+1)^3 + 2(\nu+1)^3 = 0,$$

$$c_{h8,4,1}(\nu) = c_{h7,4,1}(\nu) - c_{h7,4,3}(\nu) + 2\nu c_{h7,4,4}(\nu) =$$

$$2(\nu+1)^3 - (4\nu+2)(\nu+1)^2 + 2\nu(\nu+1)^3 = 0,$$

$$c_{h8,4,2}(\nu) = c_{h7,4,2}(\nu) + 2c_{h7,4,3}(\nu) - 3\nu c_{h7,4,4}(\nu) = -(5\nu+4)(\nu+1)^2 +$$

$$2(4\nu+2)(\nu+1)^2 - 3\nu(\nu+1)^2 = 0.$$

So,

$$C_{h8,2,1} = 0E_2,\ \text{and } H_2^*(\nu) = \begin{pmatrix} 0E_2 & 0E_2 \\ 0E_2 & H_{2,2,2}^* \end{pmatrix}$$

with

$$H^*_{2,2,2} = \nu^2(\nu+1)^4 V^*_{1,2,2}(\nu)C^*_1(\nu),$$

where

$$C^*_1(\nu) = (\nu+1)\begin{pmatrix} 1 & 0 \\ 4\nu+2 & 1 \end{pmatrix}.$$

On the other hand,

$$C_{h9}(\nu) = \nu^3 D_1(-1/\nu)(C_{h1})^{-1} =$$

$$\begin{pmatrix} * & C_{h9,1,2}(\nu) \\ * & C_{h9,2,2} \end{pmatrix}$$

with

$$C_{h9,1,2} = \begin{pmatrix} 3\nu^3 & 2\nu^3 \\ 2\nu^2 & \nu^2 \end{pmatrix}, \, C_{h9,2,2} = \begin{pmatrix} \nu & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore

$$C_{h10}(\nu) = S^*(\nu)C_{h9}(\nu) = \begin{pmatrix} * & C_{h10,1,2}(\nu) \\ * & C_{h10,2,2} \end{pmatrix},$$

with

$$C_{h10,1,2}(\nu) = S^*_{1,2}(\nu)C_{h9,2,2}(\nu) + S^*_{1,1}(\nu)C_{h9,1,2}(\nu) =$$

$$(2\nu+1)\begin{pmatrix} 4\nu+3 & (\nu+1)(-6\nu-4) \\ -2 & 4\nu+3 \end{pmatrix}\begin{pmatrix} \nu & 0 \\ 0 & -1 \end{pmatrix} +$$

$$\begin{pmatrix} 1 & -2(2\nu+1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3\nu^3 & 2\nu^3 \\ 2\nu^2 & \nu^2 \end{pmatrix} = \begin{pmatrix} c_{h10,1,3}(\nu) & c_{h10,1,4}(\nu) \\ c_{h10,2,3}(\nu) & c_{h10,2,4}(\nu) \end{pmatrix},$$

where

$$c_{h10,1,3}(\nu) = (2\nu+1)(4\nu+3)\nu + 3\nu^3 - 4\nu^2(2\nu+1) =$$

$$\nu(3\nu^2+6\nu+3) = 3\nu(\nu+1)^2,$$

$$c_{h10,1,4}(\nu) = (2\nu+1)(\nu+1)(6\nu+4) + 2\nu^3 - 2\nu^2(2\nu+1) =$$

$$(10\nu^2+14\nu+4)(\nu+1) =$$

$$(\nu+1)^2(10\nu+4), \, c_{h10,2,3}(\nu) = -2\nu(2\nu+1) + 2\nu^2 = -2\nu(\nu+1),$$

$$c_{h10,2,4}(\nu) = -(2\nu+1)(4\nu+3) + \nu^2 = -7\nu^2 - 10\nu - 3 = -(\nu+1)(7\nu+3),$$

$$C_{h10,2,2}(\nu) = S^*_{1,1,1}(\nu)C_{h9,2,2}(\nu) = \begin{pmatrix} 1 & -2(2\nu+1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \nu & 0 \\ 0 & -1 \end{pmatrix} =$$

$$\begin{pmatrix} c_{h10,3,3}(\nu) & c_{h10,3,4}(\nu) \\ c_{h10,4,3}(\nu) & c_{h10,4,4}(\nu) \end{pmatrix} = \begin{pmatrix} \nu & 2(2\nu+1) \\ 0 & -1 \end{pmatrix}.$$

Consequently,

$$C_{h11}(\nu) = D_1(\nu+1)C_{h10}(\nu) = \begin{pmatrix} * & C_{h11,1,2}(\nu) \\ * & C_{h11,2,2} \end{pmatrix}$$

with

$$C_{h11,1,2}(\nu) = \begin{pmatrix} 3\nu(\nu+1)^2 & (10\nu+4)(\nu+1)^2 \\ -2\nu(\nu+1)^2 & -(\nu+1)^2(7\nu+3) \end{pmatrix},$$

$$C_{h11,2,2}(\nu) = (\nu+1)^2 \begin{pmatrix} \nu & 2(2\nu+1) \\ 0 & -(\nu+1) \end{pmatrix}.$$

Hence

$$C_{h12}(\nu) = C_{h1}C_{h11}(\nu) = \begin{pmatrix} * & C_{h12,1,2}(\nu) \\ * & C_{h11,2,2} \end{pmatrix}$$

with

$$C_{h12,1,2}(\nu) = \begin{pmatrix} c_{h12,1,3}(\nu) & c_{h12,1,4}(\nu) \\ c_{h12,2,3}(\nu) & c_{h12,2,4}(\nu) \end{pmatrix},$$

$$C_{h12,2,2}(\nu) = \begin{pmatrix} c_{h12,3,3}(\nu) & c_{h12,3,4}(\nu) \\ c_{h12,4,3}(\nu) & c_{h12,4,4}(\nu) \end{pmatrix},$$

where

$$c_{h12,1,3}(\nu) = c_{h11,1,3}(\nu) - 3c_{h11,3,3}(\nu) - 2c_{h11,4,3}(\nu) = \\ 3\nu(\nu+1)^2 - 3\nu(\nu+1)^2 = 0,$$

$$c_{h12,1,4}(\nu) = c_{h11,1,4}(\nu) - 3c_{h11,3,4}(\nu) - 2c_{h11,4,4}(\nu) = \\ (10\nu+4)(\nu+1)^2 - 6(2\nu+1)(\nu+1)^2 + 2(\nu+1)^3 = 0,$$

$$c_{h12,2,3}(\nu) = c_{h11,2,3}(\nu) + 2c_{h11,3,3}(\nu) + c_{h11,4,3}(\nu) = \\ -2\ nu(\nu+1)^2 + 2\nu(\nu+1)^2 = 0,$$

$$c_{h12,2,4}(\nu) = c_{h11,4,4}(\nu) + 2c_{h11,3,4}(\nu) + c_{h11,4,4}(\nu) = \\ -(\nu+1)^2(7\nu+3) + 4(2\nu+1)(\nu+1)^2 - (\nu+)^3 = 0,$$

So,

$$C_{h12,1,2} = 0E_2, \text{ and } G_2^*(\nu) = \begin{pmatrix} 0E_2 & 0E_2 \\ 0E_2 & G_{2,2,2}^* \end{pmatrix}$$

with

$$G_{2,2,2}^* = \nu^2(\nu+1)^4(\nu)C_2^*(\nu)V_{1,2,2}^*(-\nu-1),$$

where

$$C_2^*(\nu) = \begin{pmatrix} \nu & 2(2\nu+1) \\ 0 & -(\nu+1) \end{pmatrix}.$$

Finally, we have $C_3^*(\nu) = V_{1,2,2}^*(\nu)C_1^*(\nu) = (\nu+1)\times$

$$\begin{pmatrix} -3\nu-2 & 2\nu+2 \\ -2\nu^2+1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4\nu+2 & 1 \end{pmatrix} = \begin{pmatrix} c_{3,1,1}^* & c_{3,1,2}^* \\ c_{3,2,1}^* & c_{3,2,2}^* \end{pmatrix},$$

where

$$c_{3,1,1}^* = (nu+1)(-3\nu-2) + (2\nu+2)(4\nu+2)) = (\nu+1)(8\nu^2+9\nu+2),$$

$$c_{3,1,2}^* = 2(\nu+1)^2,\ c_{3,2,1}^* = (\nu+1)(-2\nu^2+1) - 4\nu - 2) = (\nu+1)(-2(\nu+1)^2+1),$$

$$c_{3,2,2}^* = -(\nu+1),\ C_4^*(\nu) = C_2^*(\nu)V_{1,2,2}^*(-\nu-1) =$$

$$\begin{pmatrix} \nu & 2(2\nu+1) \\ 0 & -(\nu+1) \end{pmatrix} \begin{pmatrix} 3\nu+1 & -2\nu \\ -2(\nu+1)^2+1 & -1 \end{pmatrix} = \begin{pmatrix} c_{4,1,1}^* & c_{4,1,2}^* \\ c_{4,2,1}^* & c_{4,2,2}^* \end{pmatrix},$$

where

$$c_{4,1,1}^* = \nu(3\nu+1) + (4\nu+2)(-2(\nu+1)^2+1) = -(\nu+1)(8\nu^2+12\nu+4-3\nu+-2),$$

$$c_{4,1,2}^* = -2\nu^2 - 2(2\nu+1) = -2(\nu+1)^2,\ c_{4,2,1}^* = -(\nu+1)(-2(\nu+1)^2+1).$$

$$c_{4,2,2}^* = (\nu+1).$$

The equality (1.12) directly follows from (4.2),(4.4) and(4.8). □

5. Proof of Theorem 2 for $\alpha = 1$

In view of (2.1), (2.2)

$$\tau_1(\nu-1) = \tau - 1, \quad \mu_1(\nu-1) = (\tau-1)^2,$$

$$r_{1,0,1}(\nu-1) = (\tau-1)^4,\ r_{1,0,2}(\nu) = r_{1,0,4}(\nu) = 0,\ r_{1,0,3}(\nu) = -2(\tau-1)^2.$$

Therefore

$$\sum_{k=0}^{4} r_{1,0,k}(\nu-1)\mathbf{a}_{k,1}^*(1;\nu) =$$

$$(\tau-1)^4\tau^2((2\tau-1)(8\tau^2-5\tau+1) + \tau - 1)^3) -$$

$$2(\tau-1)^2\tau^2(2\tau-1)(\tau-1)^2(4\tau^2-3\tau+1) = (\tau-1)^4\tau^2\times$$

$$((8\tau^2-5\tau+1-8\tau^2+6\tau-2)(2\tau-1) + \tau - 1)^3 = (\tau-1)^5\tau^2\times$$

$$(\tau-1)^2 + 2\tau - 1) = (\tau-1)^5\tau^4,$$

$$\sum_{k=0}^{4} r_{1,0,k}(\nu-1)\mathbf{a}^*_{k,2}(1;\nu) =$$

$$(\tau-1)^4(-2\tau(2\tau-1)((5\tau^2-4\tau+1)+(\tau-1)^2))-$$

$$2(\tau-1)^2(-\tau(2\tau-1)(\tau-1)^2(6\tau^2-6\tau+2)) =$$

$$(\tau-1)^4(\tau(2\tau-1)(2\tau^2-4\tau+2-2(\tau-1)^2 = 0$$

$$\sum_{k=0}^{4} r_{1,0,k}(\nu-1)\mathbf{a}^*_{k,3}(1;\nu) =$$

$$(\tau-1)^4((2\tau-1)((-4\tau^2-\tau+1)+(\tau-1)(4\tau-1)))-$$

$$2(\tau-1)^2((\tau-1)^2(-(2\tau-1)(3\tau-1)+\tau^2(\tau-1))) =$$

$$(\tau-1)^5((2\tau-1)(-4\tau+1)+(-2\tau^2+4\tau-1)) = -2\tau^2(\tau-1)^5,$$

$$\sum_{k=0}^{4} r_{1,0,k}(\nu-1)\mathbf{a}^*_{k,4}(1;\nu) = 0+0+0+0 = 0. \qquad \square$$

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