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TITCHMARSH'S THEOREM FOR THE JACOBI TRANSFORM IN THE SPACE $L^2_{(\alpha,\beta)}(\mathbb{R}^+)$

R. Daher[1], H. Lahlali[2], M. El Hamma[3 §]

[1,2,3]Department of Mathematics
Faculty of Sciences Aïn Chock
University of Hassan II
Casablanca, MOROCCO

Abstract: Using a generalized translation operator, we obtain a generalization of Titchmarsh's theorem of the Jacobi transform for functions satisfying the ψ-Jacobi Lipschitz condition.

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1. Introduction and Preliminaries

In [3], we proved an analog of Titchmarsh's theorem for the Jacobi transform in the space $L^2_{(\alpha,\beta)}(\mathbb{R}^+)$ of functions satisfying the Jacobi-Lipschitz condition. In this paper we prove the generalization of this theorem of functions satisfying the ψ-Jacobi Lipschitz condition. For this purpose, we use a generalized translation operator.

Now, we collect some basic facts on the Jacobi transform, and more details about the Jacobi transform can be found in [1] and [5].

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§Correspondence author

The Jacobi function $\varphi_\lambda(t) = \varphi_\lambda^{(\alpha,\beta)}(t)$ of order (α,β) $(\alpha \geq -\frac{1}{2},\ \alpha > \beta \geq -\frac{1}{2})$ is the unique C^∞ function on $\mathbb{R}$ with equals 1 at 0 and satisfies the differential equation

$$(\mathrm{D}_{\alpha,\beta} + \lambda^2 + \rho^2)\varphi_\lambda(t) = 0,$$

where $\rho = \alpha + \beta + 1$ and

$$\mathrm{D}_{\alpha,\beta} = \frac{d^2}{dt^2} + ((2\alpha+1)\coth t + (2\beta+1)\tanh t)\frac{d}{dt}.$$

Lemma 1.1. *The following inequalities are valid for a Jacobi function $\varphi_\lambda(t)$ $(\lambda, t \in \mathbb{R}^+)$.*

1. $|\varphi_\lambda(t)| \leq 1,$
2. $1 - \varphi_\lambda(t) \leq t^2(\lambda^2 + \rho^2),$
3. *there is a constant $c > 0$ such that*

$$1 - \varphi_\lambda(t) \geq c$$

for $\lambda t \geq 1$.

Proof. See [6, Lemmas 3.1-3.2] □

Consider the Hilbert space $\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+) = \mathrm{L}^2(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$ with the norm

$$\|f\|_{2,(\alpha,\beta)} = \left(\int_0^\infty |f(x)|^2 \Delta_{(\alpha,\beta)}(x)dx\right)^{1/2},$$

where

$$\Delta_{(\alpha,\beta)}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}.$$

The c-function is defined by (see [4])

$$c(\lambda) = \frac{2^\rho \Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta))},$$

where $\alpha \geq -\frac{1}{2}$ and $\alpha > \beta \geq -\frac{1}{2}$.

The Jacobi transform of a function $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$ is defined by

$$\widehat{f}(\lambda) = \int_0^\infty f(t)\varphi_\lambda(t)\Delta_{(\alpha,\beta)}(t)dt$$

The inverse formula [5] is defined by

$$f(t) = \frac{1}{2\pi}\int_0^{\infty} \widehat{f}(\lambda)\varphi_\lambda(t)d\mu(\lambda),$$

where $d\mu(\lambda) = |c(\lambda)|^{-2}d\lambda$.

The Jacobi transform a unitary isomorphism from $\mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$ onto $\mathrm{L}^2(\mathbb{R}^+, \frac{1}{2\pi}d\mu(\lambda))$, i.e.

$$\|f\|_{2,(\alpha,\beta)} = \|\widehat{f}\|_{\mathrm{L}^2(\mathbb{R}^+,\frac{1}{2\pi}d\mu(\lambda))} \tag{1}$$

Recall from [4] the generalized translation operator τ_h of a suitable function f on $\mathbb{R}^+$, is defined by

$$\tau_h f(x) = \int_0^{\infty} f(z)K(x,h,z)\Delta_{(\alpha,\beta)}(z)dz,$$

where K is an explicity known kernel function such that

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^{\infty} \varphi_\lambda(z)K(x,y,z)\Delta_{(\alpha,\beta)}(z)dz,$$

$$\begin{aligned} K(x,y,z) &= \frac{2^{-2\rho}\Gamma(\alpha+1)(\cosh x\cosh y\cosh z)^{-\alpha-\beta-1}}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})(\sinh x\sinh y\sinh z)^{2\alpha}}(1-B^2)^{\alpha-\frac{1}{2}} \\ &\times\ F(\alpha+\beta,\alpha-\beta,\alpha+\frac{1}{2},\frac{1}{2}(1-B)) \end{aligned}$$

for $|x-y| < z < x+y$ and $K(x,y,z) = 0$ elsewhere and

$$B = \frac{\cosh^2 x + \cosh^2 y + \cosh^2 z - 1}{2\cosh x\cosh y\cosh z}$$

and F is the Gauss hypergeometric function

$$F(a,b,c,z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k,\ \ |z| < 1,$$

where $(a)_0 = 1$ and $(a)_k = a(a+1)...(a+k-1)$.

In [2], we have

$$\widehat{(\tau_h f)}(\lambda) = \varphi_\lambda(h)\widehat{f}(\lambda) \tag{2}$$

2. Generalization of Titchmarsh's Theorem

Definition 2.1. *A function $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$ is said to be in the ψ-Jacobi Lipschitz class, denote by $Lip(\psi, 2)$, if*

$$\|\tau_h f(x) - f(x)\|_{2,(\alpha,\beta)} = O(\psi(h)) \text{ as } h \longrightarrow 0,$$

where

1. *$\psi(t)$ is a continuous increasing function on $[0, \infty)$ and $\psi(0) = 0$,*
2. *$\psi(ts) = \psi(t)\psi(s)$, for all $t, s \in [0, \infty)$,*
3. *$h \leq \psi(h)$ and $\int_0^{1/h} s\psi(s^{-2})ds = O(\frac{1}{h^2}\psi(h^2))$ as $h \longrightarrow 0$.*

Theorem 2.2. *Let $f \in \mathrm{L}^2_{(\alpha,\beta)}(\mathbb{R}^+)$. Then the following are equivalents*

1. *$f \in Lip(\psi, 2)$,*
2. *$\int_r^{\infty} |\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(\psi(r^{-2}))$ as $r \longrightarrow +\infty$.*

Proof. $1 \Longrightarrow 2$ Suppose that $f \in Lip(\psi, 2)$. Then

$$\|\tau_h f(x) - f(x)\|_{2,(\alpha,\beta)} = O(\psi(h)) \text{ as } h \longrightarrow 0$$

From formulas (1) and (2), we have

$$\|\tau_h f(x) - f(x)\|_{2,(\alpha,\beta)} = \int_0^{\infty} |1 - \varphi_\lambda(h)|^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda)$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$ then $\lambda h \geq 1$ and (3) of Lemma 1.1 implies that

$$1 \leq \frac{1}{c^2}|1 - \varphi_\lambda(h)|^2.$$

Then

$$\begin{aligned}
\int_{1/h}^{2/h} |\widehat{f}(\lambda)|^2 d\mu(\lambda) &\leq \int_{1/h}^{2/h} |1 - \varphi_\lambda(h)|^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&\leq \int_0^{\infty} |1 - \varphi_\lambda(h)|^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&= O(\psi^2(h)) = O(\psi(h^2)).
\end{aligned}$$

There exists then a positive constant C such that

$$\int_{1/h}^{2/h} |\widehat{f}(\lambda)|^2 d\mu(\lambda) \leq C\psi(h^2).$$

Then

$$\int_{r}^{2r} |\widehat{f}(\lambda)|^2 d\mu(\lambda) \leq C\psi(r^{-2}) \ as\ r \longrightarrow +\infty$$

Furthermore, we have

$$\begin{aligned} \int_{r}^{\infty} |\widehat{f}(\lambda)|^2 d\mu(\lambda) &= \left(\int_{r}^{2r} + \int_{2r}^{4r} + \int_{4r}^{8r} + ...\right) |\widehat{f}(\lambda)|^2 d\mu(\lambda) \\ &\leq C\psi(r^{-2}) + C\psi((2r)^{-2}) + C\psi((4r)^{-2}) + ... \\ &\leq C\psi(r^{-2})(1 + \psi(2^{-2}) + \psi((2^{-2})^2) + \psi((2^{-2})^3) + ...) \\ &\leq C(1-\psi(2^{-2}))^{-1}\psi(r^{-2}), \end{aligned}$$

since $\psi(2^{-2}) < 1$.

This prove

$$\int_{r}^{\infty} |\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(\psi(r^{-2})) \ as\ r \longrightarrow +\infty.$$

$2 \Longrightarrow 1$ Suppose now that

$$\int_{r}^{\infty} |\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(\psi(r^{-2})) \ as\ r \longrightarrow +\infty.$$

We write

$$\|\tau_h f(x) - f(x)\|_{2,(\alpha,\beta)} = \int_{0}^{\infty} |1-\varphi_\lambda(h)|^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda) = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_{0}^{1/h} |1-\varphi_\lambda(h)|^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda)$$

and

$$\mathrm{I}_2 = \int_{1/h}^{\infty} |1-\varphi_\lambda(h)|^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda)$$

Estimate the summands I_1 and I_2 from above. It follows from (1) of Lemma 1.1 that

$$\mathrm{I}_2 = \int_{1/h}^{\infty} |1-\varphi_\lambda(h)|^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda) \leq 4\int_{1/h}^{\infty} |\widehat{f}(\lambda)|^2 d\mu(\lambda) = O(\psi(h^2)).$$

To estimate I_1, we use the inequalities (1) and (2) of Lemma 1.1

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{1/h} |1-\varphi_\lambda(h)|^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&\leq 2\int_0^{1/h} |1-\varphi_\lambda(h)||\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&\leq 2h^2 \int_0^{1/h} (\lambda^2+\rho^2)|\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&\leq 2\rho^2 h^2 \int_0^{1/h} |\widehat{f}(\lambda)|^2 d\mu(\lambda) + 2h^2 \int_0^{1/h} \lambda^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda)
\end{aligned}$$

Note that

$$\begin{aligned}
2\rho^2 h^2 \int_0^{1/h} |\widehat{f}(\lambda)|^2 d\mu(\lambda) &\leq 2\rho^2 h^2 \int_0^{\infty} |\widehat{f}(\lambda)|^2 d\mu(\lambda) \\
&= 2\rho^2 h^2 \|f\|^2_{2,(\alpha,\beta)} \\
&\leq 2\rho^2 \psi(h^2) \|f\|^2_{2,(\alpha,\beta)} \\
&= O(\psi(h^2)).
\end{aligned}$$

We put

$$\phi(s) = \int_s^{\infty} |\widehat{f}(\lambda)|^2 d\mu(\lambda).$$

Integrating by parts, we obtain

$$\begin{aligned}
2h^2 \int_0^{1/h} \lambda^2 |\widehat{f}(\lambda)|^2 d\mu(\lambda) &= 2h^2 \int_0^{1/h} (-s^2 \phi'(s)) ds \\
&= 2h^2 \left(-\frac{1}{h^2}\phi(\frac{1}{h}) + 2\int_0^{1/h} s\phi(s) ds \right) \\
&= -2\phi(\frac{1}{h}) + 4h^2 \int_0^{1/h} s\phi(s) ds \\
&\leq 4Kh^2 \int_0^{1/h} s\psi(s^{-2}) ds \\
&\leq 4Kh^2 \frac{1}{h^2}\psi(h^2) \\
&= O(\psi(h^2)).
\end{aligned}$$

Finally, then

$$\|\tau_h f(x) - f(x)\|_{2,(\alpha,\beta)} = O(\psi(h)) \ as\ h \longrightarrow 0,$$

which complets the proof of theorem. □

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