A POINTWISE POISSON APPROXIMATION FOR RANDOM SUMS OF GEOMETRIC RANDOM VARIABLES

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Abstract: We determine a bound for the point metric between the distribution of random sums of independent geometric random variables and an appropriate Poisson distribution. Three examples have been given to illustrate the results obtained.

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1. Introduction

Let $X_1, \dots, X_n$ be n independently distributed geometric random variables, each with $P(X_i = k) = (1 - p_i)^k p_i$, $k = 0, 1, \dots$, and let $S_n = \sum_{i=1}^n X_i$ and U_λ a Poisson random variable with mean λ . It is well-known that if all $q_i = 1 - p_i$ are small, then the distribution of S_n can be approximated by a Poisson distribution with mean λ . Correspondingly, the probability function of S_n , denoted by $P_{S_n}(x_0)$, can also be approximated by an appropriate Poisson probability function with mean λ , denoted by $\wp_{\lambda n}(x_0)$, where $x_0 \in \mathbb{N} \cup \{0\}$. In this case, Teerapabolarn [1] gave a bound for the point metric between the distributions

of S_n and U_{λ_n} as follows:

$$|P_{S_n}(x_0) - \wp_{\lambda_n}(x_0)| \leq \sum_{i=1}^n \min \left\{ \frac{1 - e^{-\lambda_n}}{\lambda_n}, \frac{1}{x_0}, p_i \right\} \frac{q_i^2}{p_i} \quad (1.1)$$

for $x_0 \in \mathbb{N}$ and $\lambda_n = \sum_{i=1}^n q_i$.

Let us consider the case where n is a random variable (not a fixed number), that is, if N is a non-negative integer valued random variable and independent of the X_i 's, then the sums $S_N = \sum_{i=1}^N X_i$ is called *random sums* of independent geometric random variables, which makes it more interesting. In this study, we are interest to approximate $P_{S_N}(x_0)$ by $\wp_{\lambda}(x_0)$ when $\lambda = E(\lambda_N) = \sum_{i=1}^N q_i$.

2. Result

A bound for the point metric of $P_{S_N}(x_0)$ and $\wp_{\lambda}(x_0)$ is presented as follows.

Theorem 2.1. For $x_0 \in \mathbb{N}$ and $\lambda = E(\lambda_N) = \sum_{i=1}^N q_i$, then we have

$$\begin{aligned} |P_{S_N}(x_0) - \wp_{\lambda}(x_0)| &\leq \min \left\{ \frac{1 - e^{-\lambda}}{\lambda}, \frac{1}{x_0} \right\} E|\lambda_N - \lambda| \\ &+ \min \left\{ E \left(\frac{1 - e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^N \frac{q_i^2}{p_i} \right), \frac{1}{x_0} E \left(\sum_{i=1}^N \frac{q_i^2}{p_i} \right), E \left(\sum_{i=1}^N q_i^2 \right) \right\}, \end{aligned} \quad (2.1)$$

where $P_{S_N}(0) = E(\prod_{i=1}^N p_i)$.

Proof. From the fact that

$$\begin{aligned} |P_{S_N}(x_0) - \wp_{\lambda}(x_0)| &\leq \sum_{n=0}^{\infty} P(N = n) |P_{S_n}(x_0) - \wp_{\lambda_n}(x_0)| \\ &+ |\wp_{\lambda_N}(x_0) - \wp_{\lambda}(x_0)|. \end{aligned} \quad (2.2)$$

Following the results in (1.1) and [2], we have

$$\begin{aligned} |\wp_{\lambda_N}(x_0) - \wp_{\lambda}(x_0)| &\leq \sum_{n=0}^{\infty} P(N = n) \sum_{i=1}^n \min \left\{ \frac{1 - e^{-\lambda_n}}{\lambda_n}, \frac{1}{x_0}, p_i \right\} \frac{q_i^2}{p_i} \\ &+ \min \left\{ \frac{1 - e^{-\lambda}}{\lambda}, \frac{1}{x_0} \right\} E|\lambda_N - \lambda|, \end{aligned} \quad (2.3)$$

which yields (2.1). □

If X_i 's are identically distributed, then the following corollary is an immediate consequence of the Theorem 2.1

Corollary 2.1. *For $x_0 \in \mathbb{N}$, if $p_1 = p_2 = \dots = p$, then we have the following:*

$$|P_{S_N}(x_0) - \wp_\lambda(x_0)| \leq \min \left\{ \frac{1 - e^{-qE(N)}}{qE(N)}, \frac{1}{x_0} \right\} qE|N - E(N)| \\ + \min \left\{ \frac{q}{p} E(1 - e^{-qN}), \frac{q^2}{px_0} E(N), q^2 E(N) \right\}, \quad (2.4)$$

where $P_{S_N}(0) = pE(N)$.

3. Examples

This section, we give three examples to illustrate the result in the case where X_i 's are identically distributed.

Example 3.1. For n ($n \in \mathbb{N}$) is fixed, let N be a positive integer-valued random variable with probability function

$$P(N = k) = \begin{cases} \frac{1}{2} & , k = n, \\ \frac{1}{2} & , k = 2n, \\ 0 & , \text{otherwise.} \end{cases}$$

Therefore $E(N) = \frac{3n}{2}$ and $E|N - E(N)| = \frac{n}{2}$. Let $p_1 = p_2 = \dots = p$, then we have

$$|P_{S_N}(x_0) - \wp_\lambda(x_0)| \leq \min \left\{ \frac{1 - e^{-\frac{3nq}{2}}}{3}, \frac{nq}{2x_0} \right\} + \min \left\{ \frac{q}{p}, \frac{3nq^2}{2px_0}, \frac{3nq^2}{2} \right\},$$

where $x_0 \in \mathbb{N}$ and $P_{S_N}(0) = \frac{3np}{2}$.

Example 3.2. Let N be a positive integer-valued random variable with probability function

$$P(N = n) = \frac{1}{2^n}, \quad n = 1, 2, \dots,$$

then we have $E(N) = 2$ and $E|N - E(N)| = 1$. If $p_1 = p_2 = \dots = p$, then we obtain

$$|P_{S_N}(x_0) - \wp_\lambda(x_0)| \leq \min \left\{ \frac{1 - e^{-2q}}{2}, \frac{q}{x_0} \right\} + \min \left\{ \frac{q}{p}, \frac{2q^2}{px_0}, 2q^2 \right\},$$

where $x_0 \in \mathbb{N}$ and $P_{S_N}(0) = 2p$.

Example 3.3. Let N has a Poisson random variable with probability function

$$P(N = n) = \frac{e^{-\mu} \mu^n}{n!}, \quad n = 0, 1, \dots,$$

then we have $E(N) = \mu$ and $E|N - E(N)| = 2\mu e^{-\mu}$. If $p_1 = p_2 = \dots = p$, then we obtain

$$|P_{S_N}(x_0) - \wp_\lambda(x_0)| \leq \min \left\{ 1 - e^{-\mu q}, \frac{\mu q}{x_0} \right\} 2e^{-\mu} + \min \left\{ \frac{q}{p}, \frac{\mu q^2}{p x_0}, \mu q^2 \right\},$$

where $x_0 \in \mathbb{N}$ and $P_{S_N}(0) = \mu p$.

References

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