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NEW CONSTRUCTIONS OF FUNCTION AND SET CHAINS

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Abstract: Function and set chains can be constructed using ordinal notation systems. Here, earlier results are improved, and longer chains constructed. In particular, a simpler method is given for constructing set chains.

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1. Function Chains for Schemes

Function chains for schemes have been given in [4]. These will be reviewed here, with some changes. Recall from [6] that a scheme over κ where $\kappa \in \mathrm{Inac}$ is a pair $\boldsymbol{\sigma} = \langle \sigma, \phi \rangle$ where $\sigma < \kappa^+$ and ϕ is a function whose domain is the set of limit ordinals $\alpha \leq \sigma$. For $\alpha \in \mathrm{Dom}(\phi)$, $\phi(\alpha)$ is an increasing function with domain an ordinal $\eta \leq \kappa$, and whose range is an unbounded subset of α. If $\mathrm{Cf}(\alpha) < \kappa$ then $\eta < \kappa$, and if $\mathrm{Cf}(\alpha) = \kappa$ then $\eta = \kappa$.

Recursions and inductions on schemes are broken into four cases, as follows.

Case 0: $\sigma = 0$.

Case 1: $\sigma = \tau + 1$. $\boldsymbol{\tau}$ denotes $\boldsymbol{\sigma}_{\leq \tau}$.

Case 2: $\sigma \in \mathrm{Lim}$, $\mathrm{Cf}(\sigma) < \kappa$. For $\xi < \eta$ σ_ξ denotes $\phi(\sigma)(\xi)$; and $\boldsymbol{\sigma}_\xi$ denotes $\boldsymbol{\sigma}_{\leq \sigma_\xi}$.

Case 3: $\sigma \in \mathrm{Lim}$, $\mathrm{Cf}(\sigma) = \kappa$. σ_ξ and $\boldsymbol{\sigma}_\xi$ are as in case 2, for $\xi < \kappa$.

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The thin subset $T_{\boldsymbol{\sigma}} \subseteq \kappa$ is defined recursively as follows.
0: $\emptyset$.
1: $T_{\boldsymbol{\tau}}$.
2: $(\eta+1) \cup \cup_{\xi<\eta} T_{\boldsymbol{\sigma}_\xi}$.
3: $\triangledown_{\xi<\kappa} T_{\boldsymbol{\sigma}_\xi}$.
A function $f_{\boldsymbol{\sigma}} : \mathrm{In}_\kappa \mapsto \kappa$ is defined recursively as follows.
0: The identically 0 function.
1: $f_{\boldsymbol{\tau}} + 1$.
2: $\sup_{\xi<\eta} f_{\boldsymbol{\sigma}_\xi}$, with $f_{\boldsymbol{\sigma}}(\lambda)$ set to 0 if $\lambda \leq \theta$.
3: $\mathrm{dsup}_{\xi<\kappa} f_{\boldsymbol{\sigma}_\xi}$.

This definition was given in [4], for domain κ, and it was observed that these functions are the well-known canonical functions, which are unique mod the thin ideal. As noted in [7], the domain In_κ is convenient for defining f_{κ^+}. This results in the triviality that when λ is the smallest inaccessible, $f_{\boldsymbol{\sigma}}$ is the empty function for all $\boldsymbol{\sigma}$. The proviso in case 2 ensures that $f_\sigma(\lambda) < \lambda^+$, as a simple induction shows; this does not seem to be necessary, though.

Lemma 1. *Suppose $\boldsymbol{\sigma}', \boldsymbol{\sigma}$ are schemes, and $\lambda \notin T_{\boldsymbol{\sigma}'} \cup T_{\boldsymbol{\sigma}}$.*
a. If $\sigma' \leq \sigma$ then $f_{\boldsymbol{\sigma}'}(\lambda) \leq f_{\boldsymbol{\sigma}}(\lambda)$.
b. If $\sigma' = \sigma$ then $f_{\boldsymbol{\sigma}'}(\lambda) = f_{\boldsymbol{\sigma}}(\lambda)$.
c. If $\sigma = \tau + 1$ then $f_{\boldsymbol{\sigma}}(\lambda) = f_{\boldsymbol{\tau}}(\lambda) + 1$.
d. If $\sigma' < \sigma$ then $f_{\boldsymbol{\sigma}'}(\lambda) < f_{\boldsymbol{\sigma}}(\lambda)$.

Proof. Part a follows by induction on $\boldsymbol{\sigma}$. In case 0, $\boldsymbol{\sigma}$ is 0 also and the claim is trivial. In case 1, if $\sigma' \leq \tau$ then inductively $f_{\boldsymbol{\sigma}'}(\lambda) \leq f_{\boldsymbol{\tau}}(\lambda)$, and clearly $f_{\boldsymbol{\tau}}(\lambda) < f_\sigma(\lambda)$. If $\sigma' = \sigma$ then inductively $f_{\boldsymbol{\tau}'}(\lambda) = f_{\boldsymbol{\tau}}(\lambda)$, and $f_{\boldsymbol{\sigma}'}(\lambda) = f_{\boldsymbol{\sigma}}(\lambda)$ follows. In case 2, if $\sigma' < \sigma$ then $\sigma' < \sigma_\xi$ for some $\xi < \eta$. Inductively, $f_{\boldsymbol{\sigma}'}(\lambda) \leq f_{\boldsymbol{\sigma}_\xi}(\lambda)$, and clearly $f_{\boldsymbol{\sigma}_\xi}(\lambda) \leq f_{\boldsymbol{\sigma}}(\lambda)$. If $\sigma' = \sigma$, by the argument just given, $f_{\boldsymbol{\sigma}'_\xi}(\lambda) \leq f_{\boldsymbol{\sigma}}(\lambda)$ for all $\xi < \eta'$, whence $f_{\boldsymbol{\sigma}'}(\lambda) \leq f_{\boldsymbol{\sigma}}(\lambda)$. The argument for case 3 is similar to that for case 2. Part b is immediate from part a. For part c let $\boldsymbol{\tau}_1 = \boldsymbol{\sigma}_{\leq\tau}$ and use part b. Part d follows by parts a and c. □

A scheme $\boldsymbol{\sigma} \downarrow \lambda$ over λ, for $\lambda \notin T_{\boldsymbol{\sigma}}$, may be defined recursively as follows. Write $\boldsymbol{\sigma} \downarrow \lambda$ as $\langle \sigma', \phi' \rangle$.
0: The scheme with $\sigma' = 0$.
1: $\boldsymbol{\tau} \downarrow \lambda$ with τ' replaced by $\tau' + 1$.
2: $\sqcup_{\xi<\eta} \boldsymbol{\sigma}_\xi \downarrow \lambda$, with $\phi'(\sigma')(\xi)$ set to $(\phi(\sigma)(\xi))'$.
3: $\sqcup_{\xi<\lambda} \boldsymbol{\sigma}_\xi \downarrow \lambda$, with $\phi'(\sigma')(\xi)$ set to $(\phi(\sigma)(\xi))'$.

Lemma 2. *Suppose $\boldsymbol{\sigma}$ is a scheme, $\lambda \in \mathrm{In}_\kappa$, and $\lambda \notin T_{\boldsymbol{\sigma}}$.*
a, $f_{\boldsymbol{\sigma}} \restriction \lambda = f_{\boldsymbol{\sigma}\downarrow\lambda}$.

b, $f_{\boldsymbol{\sigma}}(\lambda) = \sigma \downarrow \lambda$.

Proof. This is lemma 5 of [5]; for convenience the details are given. For part a, the cases of the induction are as follows, where $\mu \in \mathrm{In}_\lambda$.
0: $f_0(\mu) = 0 = f_{0\downarrow\lambda}(\mu)$.
1: $f_{\boldsymbol{\sigma}}(\mu) = f_{\boldsymbol{\tau}}(\mu) + 1 = f_{\boldsymbol{\tau}\downarrow\lambda}(\mu) + 1 = f_{\boldsymbol{\sigma}\downarrow\lambda}(\mu)$.
2: $f_{\boldsymbol{\sigma}}(\mu) = \sup_{\xi<\eta} f_{\boldsymbol{\sigma}_\xi}(\mu) = \sup_{\xi<\eta} f_{\boldsymbol{\sigma}_\xi\downarrow\lambda}(\mu) = f_{\boldsymbol{\sigma}\downarrow\lambda}(\mu)$.
3: $f_{\boldsymbol{\sigma}}(\mu) = \mathrm{dsup}_{\xi<\kappa} f_{\boldsymbol{\sigma}_\xi}(\mu) = \mathrm{dsup}_{\xi<\lambda} f_{\boldsymbol{\sigma}_\xi\downarrow\lambda}(\mu) = f_{\boldsymbol{\sigma}\downarrow\lambda}(\mu)$.
For part b, the cases of the induction are as follows.
0: $f_0(\lambda) = 0 = 0 \downarrow \lambda$.
1: $f_{\boldsymbol{\sigma}}(\lambda) = f_{\boldsymbol{\tau}}(\lambda) + 1 = \tau \downarrow \lambda + 1 = \sigma \downarrow \lambda$.
2: $f_{\boldsymbol{\sigma}}(\lambda) = \sup_{\xi<\eta} f_{\boldsymbol{\sigma}_\xi}(\lambda) = \sup_{\xi<\eta} \sigma_\xi \downarrow \lambda = \sigma \downarrow \lambda$.
3: $f_{\boldsymbol{\sigma}}(\lambda) = \mathrm{dsup}_{\xi<\kappa} f_{\boldsymbol{\sigma}_\xi}(\lambda) = \mathrm{dsup}_{\xi<\lambda} \boldsymbol{\sigma}_\xi \downarrow \lambda = \sigma \downarrow \lambda$.

□

2. Infinitary Veblen Functions

The infinitary Veblen functions were defined in [11]. Further work on them includes [9] and [10]. A certain set of symbols for them are called Klammersymbols, or Schutte brackets. For convenience a self-contained treatment will be given here, following [11].

For basic facts about normal functions $f : \mathrm{Ord} \mapsto \mathrm{Ord}$, and club subclasses $R \subseteq \mathrm{Ord}$, see [8], section 4.2. In particular, the following hold.

- If f is a normal function then $\mathrm{Ran}(f)$ is a club subclass.
- If R is a club subclass then the function $\mathrm{Enum}(R)$ enumerating R in natural order is normal.
- If f is normal then the function $\mathrm{Fix}(f)$ enumerating the subclass $\{\alpha : f(\alpha) = \alpha\}$ of fixed points of f is normal.
- The intersection of a set of club class is club.
- Given a normal function f, suppose $f(\alpha) > \alpha$. Let $\alpha_0 = \alpha$, and for $n \geq 0$ let $\alpha_{n+1} = f(\alpha_n)$. Then $\sup_n \alpha_n$ is the smallest fixed point of f which is greater than α.
- If $f(0) > 0$ then $\mathrm{Fix}(f)(0) > f(0)$.

Let $\mathcal{A}$ be the set of ordinal valued sequences $\alpha_0, \ldots, \alpha_\mu$ for some μ, where only finitely many α_ξ are nonzero, and $\alpha_\mu > 0$ if $\mu > 0$. $\mathcal{A}$ may be ordered in reverse lexicographic order, where if $\bar{\alpha} = \alpha_0, \ldots, \alpha_\mu$ and $\bar{\alpha}' = \alpha'_0, \ldots, \alpha'_{\mu'}$, then $\bar{\alpha}' <_{\mathrm{rl}} \bar{\alpha}$ iff $\mu' < \mu$, or $\mu' = \mu$, and for some $\gamma \leq \mu$, $\alpha'_\gamma < \alpha_\gamma$ and $\alpha'_\xi = \alpha_\xi$ for $\gamma < \xi \leq \mu$. This relation is well-known to be a well-order (see [11] for example).

Let $\mathcal{A}_*$ be the sequences obtained from the sequences of $\mathcal{A}$, by replacing some α_γ by $*$, provided $\alpha_\delta = 0$ for $\delta < \gamma$. The order $<_{\rm rl}$ may be extended to $\mathcal{A} \cup \mathcal{A}_*$, by taking $\alpha < *$ for $\alpha \in$ Ord.

Let θ be a cardinal. A function $\phi : \mathcal{A} \mapsto$ Ord will be defined, together with functions $\phi_{\tilde\alpha}$: Ord $\mapsto$ Ord for each $\tilde\alpha \in \mathcal{A}_*$. The cases of the recursion are as follows.

1. $\phi_{\tilde\alpha}(\zeta) = \phi(S_1(\tilde\alpha, \zeta))$, where in $S_1(\tilde\alpha, \zeta)$, $*$ is replaced by ζ (and trailing 0's removed if $\tilde\alpha_\mu = *$ and $\zeta = 0$).
2. For $\alpha \in$ Ord, $\phi(\alpha) = \theta^\alpha$.
3. For $\tilde\alpha \in \mathcal{A}_*$ with $\mu > 0$, $\tilde\alpha_0 = *$, $\tilde\alpha_\gamma = 0$ for $0 < \gamma < \nu$, and $\tilde\alpha_\nu > 0$, $\phi_{\tilde\alpha} = \mathrm{Enum}(\cap_{\beta<\alpha_\nu,\gamma<\nu}\mathrm{Ran}(\mathrm{Fix}(\phi_{S_2(\alpha,\beta,\gamma)})))$, where in $S_2(\alpha, \beta, \gamma)$, α_0 is replaced by 0 and α_γ by $*$ (this is a null operation if $\gamma = 0$), and α_ν is replaced by β (and trailing 0's removed if $\nu = \mu$ and $\beta = 0$).

Lemma 3. *For $\tilde a \in \mathcal{A}_*$, $\phi_{\tilde\alpha}$ is normal.*

Proof. The proof is by induction on $<_{\rm rl}$. For the basis, $\tilde\alpha = *$; the lemma follows since $\zeta \mapsto \theta^\zeta$ is normal. For the induction step, if $\tilde\alpha_0 = *$ the lemma follows by the induction hypothesis and the definition of ϕ. Otherwise, using obvious notation write $\tilde\alpha$ as $0\bar0*\bar\eta$; then $\phi_{0\bar0*\bar\eta}(\zeta) = \phi_{*\bar0\zeta\bar\eta}(0)$. Suppose $\xi < \zeta$. Directly from the definition of ϕ, $\mathrm{Ran}(\phi_{*\bar0\xi\bar\eta}) \supseteq \mathrm{Ran}(\phi_{*\bar0\zeta\bar\eta})$ and $\phi_{*\bar0\xi\bar\eta}(0) < \phi_{*\bar0\zeta\bar\eta}(0)$. In particular, $\phi_{0\bar0*\bar\eta}(\xi) < \phi_{0\bar0*\bar\eta}(\zeta)$; that is, $\phi_{\tilde\alpha}$ is increasing.

For $\zeta \in$ Lim let $\chi = \sup_{\xi<\zeta} \phi_{\tilde\alpha}(\xi)$. By what was just proved, $\chi \le \phi_{\tilde\alpha}(\zeta)$, whence $\chi \le \phi_{*\bar0\zeta\bar\eta}(0)$. Thus, to show that $\phi_{\tilde\alpha}$ is continuous it suffices to show that $\chi \in \mathrm{Ran}(\phi_{*\bar0\zeta\bar\eta})$, since then $\chi = \phi_{*\bar0\zeta\bar\eta}(0) = \phi_{\tilde\alpha}(\zeta)$. For this in turn it suffices to show that $\phi_{S_2(*\bar0\zeta\bar\eta,\beta,\gamma)}(\chi) = \chi$ for $\beta < \zeta$, $\gamma < \nu$.

Suppose $\beta < \xi$, and for ease of notation let

$$\tilde\alpha_1 \text{ denote } S_2(*\bar0\zeta\bar\eta, \beta, \gamma), \text{ and } \tilde\alpha_2 \text{ denote } *\bar0\zeta\bar\eta.$$

From the definition of ϕ, $\mathrm{Ran}(\phi_{\tilde\alpha_2}) \subseteq \mathrm{Ran}(\mathrm{Fix}(\phi_{\tilde\alpha_1}))$, and so $\phi_{\tilde\alpha_2}(0) \in \mathrm{Fix}(\phi_{\tilde\alpha_1})$, whence $\phi_{\tilde\alpha_1}(\phi_{\tilde\alpha_2}(0)) = \phi_{\tilde\alpha_2}(0)$. Using this fact,

$$\begin{aligned}&\phi_{S_2(\phi_{*\bar0\zeta\bar\eta},\beta,\gamma)}(\chi) = \\ &\quad \phi_{S_2(\phi_{*\bar0\zeta\bar\eta},\beta,\gamma)}(\sup_{\xi<\zeta} \phi_{\tilde\alpha}(\xi)) = \sup_{\xi<\zeta} \phi_{S_2(\phi_{*\bar0\zeta\bar\eta},\beta,\gamma)}(\phi_{\tilde\alpha}(\xi)) = \\ &\quad \sup_{\beta<\xi<\zeta} \phi_{S_2(\phi_{*\bar0\zeta\bar\eta},\beta,\gamma)}(\phi_{\tilde\alpha}(\xi)) = \sup_{\beta<\xi<\zeta} \phi_{\tilde\alpha}(\xi) = \chi.\end{aligned}$$

□

Lemma 4. *If $\tilde\beta <_{rl} \tilde\alpha$ then Ran(Fix($\phi_{\tilde\beta}$)) $\cup \{1\} \supseteq$ Ran($\phi_{\tilde\alpha}$).*

Proof. Writing $\tilde\alpha$ as $0\bar0*\bar\alpha$., $\phi_{0\bar0*\bar\alpha}(\zeta) = \phi_{*\bar0\zeta\bar\alpha}(0)$, provided $\zeta \ne 0$ if $\bar\alpha$ is null. By the definition of ϕ and induction on $<_{\rm rl}$, $\phi_{\tilde\alpha}(\zeta) \in \mathrm{Ran}(\mathrm{Fix}(\phi_{\tilde\beta})) \cup \{1\}$, proving the lemma. □

Let $\mathcal{A}^0_* = \{\tilde\alpha \in \mathcal{A}_* : \tilde\alpha_0 = *\}$. The following are readily seen, for $\tilde\beta <_{\rm rl} \tilde\alpha$.

- For $\tilde{\alpha} \in \mathcal{A}_*^0$, $1 \in \text{Ran}(\phi_{\tilde{\alpha}})$ iff $\tilde{\alpha} = *$.
- For $\tilde{\alpha}, \tilde{\beta} \in \mathcal{A}_*^0$, If $\tilde{\beta} <_{\text{rl}} \tilde{\alpha}$ then $\text{Ran}(\text{Fix}(\phi_{\tilde{\beta}})) \supseteq \text{Ran}(\phi_{\tilde{\alpha}})$.
- For $\tilde{\alpha}, \tilde{\beta} \in \mathcal{A}_*$, $\text{Ran}(\text{Fix}(\phi_{\tilde{\beta}})) \supseteq \text{Ran}(\text{Fix}(\phi_{\tilde{\alpha}}))$.

For $\mu \in \text{Ord}$ let $\tilde{\alpha}_\mu$ be the element of $\mathcal{A}_*$ where $\tilde{\alpha}_{\mu 0} = *$, $\tilde{\alpha}_{\mu\mu} = 1$, and $\tilde{\alpha}_{\mu\xi} = 0$ for $0 < \xi < \mu$. Let ψ be the function $\mu \mapsto \tilde{\alpha}_\mu(0)$.

Lemma 5. *The function ψ is normal.*

Proof. For $\mu \in \text{Ord}$ let $\tilde{\alpha}_\mu^*$ be the element of $\mathcal{A}_*$ where $\tilde{\alpha}_{\mu\mu} = *$ and $\tilde{\alpha}_{\mu\xi} = 0$ for $0 \le \xi < \mu$. From the definition of ϕ, $\phi_{\tilde{\alpha}_\mu} = \text{Enum}(\cap_{\gamma<\mu}\text{Ran}(\text{Fix}(\phi_{\tilde{\alpha}_\gamma^*})))$. It follows from this and lemma 4 that $\mu \mapsto \tilde{\alpha}_\mu(0)$ is increasing. For $\mu \in \text{Lim}$ let $\chi = \sup_{\gamma<\mu} \phi_{\tilde{\alpha}_\gamma}(0)$. From the definition of ϕ, for $\gamma < \delta$ $\text{Ran}(\phi_{\tilde{\alpha}_\delta}) \subseteq \text{Ran}(\text{Fix}(\phi_{\tilde{\alpha}_\gamma^*}))$. Using this fact, it follows by arguments as given above that $\phi_{\tilde{\alpha}_\mu}(0) = \chi$. □

The least fixed point of ψ is called the "large Veblen ordinal". Notation varies; Λ_0 will be used here. Clearly, $\Lambda_0 = \phi_{\tilde{\alpha}_{\Lambda_0}}(0) = \sup_{\mu<\Lambda_0} \phi_{\tilde{\alpha}_\mu}(0)$. It follows that for $\mu < \Lambda_0$, Λ_0 is a fixed point of $\phi_{\tilde{\alpha}_\mu}$. Using lemma 4, if $\tilde{\alpha} < \tilde{\alpha}_{\Lambda_0}$ then Λ_0 is a fixed point of $\phi_{\tilde{\alpha}}$, whence if $\zeta < \Lambda_0$ then $\phi_{\tilde{\alpha}}(\zeta) < \Lambda_0$.

Say that $\bar{\alpha} \in \mathcal{A}$ is maximal if $\phi(\bar{\beta}) \neq \phi(\bar{\alpha})$ whenever $\beta >_{\text{rl}} \alpha$.

Lemma 6. *For $\bar{\alpha} \in \mathcal{A}$, let $\nu_1 > \cdots > \nu_k$ be the indices such that $\alpha_\nu > 0$.*

a. *For $i > 1$, $\phi(\bar{\alpha}) > \alpha_{\nu_i}$.*
b. *$\phi(\bar{\alpha}) > \alpha_{\nu_1}$ iff $\bar{\alpha}$ is maximal.*
c. *If $\phi(\bar{\alpha}) \notin \text{Ran}(\psi)$ then $\nu_k < \phi(\bar{\alpha})$.*
d. *If $\tau \in \text{Ran}(\phi_*)$ then $\tau = \phi(\bar{\alpha})$ for a unique maximal $\bar{\alpha}$.*

Proof. For $1 \le j < i$ let $\bar{\alpha}_i$ be $\bar{\alpha}$ with α_{ν_j} replaced by 0. It is easily seen that $\phi(\bar{\alpha}_{i-1}) > \phi(\bar{\alpha}_i) \ge \alpha_{\nu_i}$ for $i > 1$, and part a follows. Write $\bar{\alpha}$ as $\bar{0}\alpha_\nu\alpha_{\nu+1}\bar{\alpha}'$, where for ease of notation $\mu > \nu$ is assumed. Let $\tilde{\alpha} = \bar{0}*\alpha_{\nu+1}\bar{\alpha}'$. Since $\phi_{\tilde{\alpha}}$ is normal, $\phi(\alpha) \ge \alpha_\nu$. Let $\tilde{\beta} = *\bar{0}0, \alpha_{\nu+1}+1, \bar{\alpha}'$. Then $\text{Ran}(\phi_{\tilde{\beta}}) = \text{Ran}(\text{Fix}(\phi_{\tilde{\alpha}}))$, and so $\phi(\bar{\alpha}) = \alpha_\nu$ iff $\alpha_\nu \in \text{Ran}(\phi_{\tilde{\beta}})$ iff $\bar{\alpha}$ is not maximal. The case $\nu = \mu$ is similar, proving part b. For part c, let $\gamma \in \text{Ran}(\psi)$ be least such that $\phi(\bar{\alpha}) < \gamma$. Then $\phi(\bar{\alpha}_{\nu_k}) \le \phi(\bar{\alpha}) < \gamma$, whence $\nu_k < \phi(\bar{\alpha}_{\nu_k}) \le \phi(\bar{\alpha})$. For part d, there is a μ such that $\tau < \phi_{\tilde{\alpha}_\mu}(0) \le \phi_{\tilde{\alpha}_\mu}(\tau)$, and so there is a lexicographically least $\tilde{\alpha}$ such that $\tau < \phi_{\tilde{\alpha}}(\tau)$.

If $\tilde{\alpha} = *$ then $\tau = \phi(\zeta)$ for some ζ by hypothesis, and $\zeta < \tau$ since $\zeta = \tau$ is impossible. Now, $\tau = \phi_{\tilde{\beta}}(\tau)$ for $\tilde{\beta} <_{\text{rl}} \tilde{\alpha}$. If $\tilde{\alpha}_0 = *$ then by the definition of ϕ $\tau = \phi_{\tilde{\alpha}}(\zeta)$ for some ζ, and $\zeta = \tau$ is impossible. Otherwise $\tilde{\alpha} = 0\bar{0} * \bar{\alpha}'$, and $\tau < \phi_{0\bar{0}*\bar{\alpha}'}(\tau) = \phi_{*\bar{0}\tau\bar{\alpha}'}(0)$; but then by the definition of ϕ $\tau = \phi_{*\bar{0}\tau\bar{\alpha}'}(\zeta)$ for some ζ, which is impossible. □

An infinitary Veblen function is a function $\phi_k(\zeta, \xi_1, \alpha_1, \ldots, \xi_k, \alpha_k)$ where $k \geq 1$. If $\xi_1 < \cdots < \xi_k$ and $\alpha_i > 0$ the arguments are said to be proper; the value of ϕ_k is $\phi(\bar{\alpha})$ where $\bar{\alpha}$ is the sequence $\zeta, \ldots \alpha_1 \ldots \alpha_2 \ldots \alpha_k$, where α_i occurs in position $1 + \zeta_i$, and all other elements of $\bar{\alpha}$ are 0. If the arguments are improper the value is 0.

Corollary 7. *For $\tau \in Ran(\phi_*)$ with $\tau < \Lambda_0$ there are a unique k and values $\zeta, \xi_1, \alpha_1, \ldots, \xi_k, \alpha_k < \tau$ such that $\tau = \phi_k(\zeta, \xi_1, \alpha_1, \ldots, \xi_k, \alpha_k)$.*

Proof. Immediate. □

Let O denote the function with ordinal arguments, where $\mathrm{O}(\alpha_1, \alpha_2)$ equals -1 if $\alpha_1 < \alpha_2$, 0 if $\alpha_1 = \alpha_2$, and $+1$ if $\alpha_1 > \alpha_2$,

Lemma 8. *Suppose $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{A}^0_*$ and $\tilde{\alpha}_1 <_{rl} \tilde{\alpha}_2$. Then $O(\phi_{\tilde{\alpha}_1}(\zeta_1), \phi_{\tilde{\alpha}_2}(\zeta_2)) = O(\zeta_1, \phi_{\tilde{\alpha}_2}(\zeta_2))$.*

Proof. $\mathrm{Ran}(\phi_{\tilde{\alpha}_2}) \subseteq \mathrm{Ran}(\mathrm{Fix}(\phi_{\tilde{\alpha}_1}))$, so $\phi_{\tilde{\alpha}_1}(\phi_{\tilde{\alpha}_2}(\zeta_2)) = \phi_{\tilde{\alpha}_2}(\zeta_2)$; the lemma follows. □

3. IV$_o$ Terms

As in [7], a CNF function is a function $C_k(\eta_k, \sigma_k, \ldots, \eta_1, \sigma_1)$, where $k \geq 1$. If $\eta_k > \cdots > \eta_1$ and $0 < \sigma_i < \theta$ the arguments are said to be proper, and the value of C_k is $\theta^{\eta_k} \cdot \sigma_k + \cdots + \theta^{\eta_1} \cdot \sigma_1$. If the arguments are improper, the value is defined to be 0.

Let IV$_o$ be the system of terms, whose leaves are ordinals σ, and whose interior nodes are either a CNF function C_k or an infinitary Veblen function ϕ_k. A proper term is one where the arguments are proper at each interior node. A normal form term is a proper term where at each interior note, the value of the function is greater than the value of any of its arguments.

Theorem 9. *If $\alpha < \Lambda_0$ then there is a unique normal form IV$_o$ term whose value is α.*

Proof. Let $\kappa^{+\eta_k} \cdot \sigma_k + \cdots + \kappa^{+\eta_1} \cdot \sigma_1$ be the Cantor normal form for α. The proof is by induction on α, with cases as follows, where t denotes the term.

Case 0: $\alpha = 0$. Then $t = 0$.

Case 1: $k = 1$ and $\eta_1 = 0$. Then $t = \sigma_1$.

Case 2: $k > 1$ or $\sigma_1 > 1$ or $\eta_1 > 0$ and $\eta_1 < \kappa^{+\eta}$. Then $t = C_k(t_{\eta_k}, t_{\sigma_k}, \ldots, t_{\eta_1}, t_{\sigma_1})$.

Case 3: $k = 1$ and $\sigma_1 = 1$ and $\eta_1 = \kappa^{+\eta}$. In this case, $t = \phi_k(t_\zeta, t_{\xi_1}, t_{\alpha_1}, \ldots, t_{\xi_k}, t_{\alpha_k})$ where $\zeta, \xi_1, \alpha_1, \ldots, \xi_k, \alpha_k < \alpha$ are the unique values as in lemma 7. ☐

If t_i is a term with value α_i for $i = 1, 2$, write $\mathrm{O}(t_1, t_2)$ for $\mathrm{O}(\alpha_1, \alpha_2)$. Let $\mathrm{P}(t)$ equal 1 if t is proper, else 0 Let $\mathrm{N}(t)$ equal 1 if t is normal, else 0.

Lemma 10. *a. $O(t_1, t_2)$ depends only on the values $O(\sigma_1, \sigma_2)$ and $P(\sigma)$ for the leaves of t_1, t_2.*
b. $P(t)$ depends only on the values $O(\sigma_1, \sigma_2)$ and $P(\sigma)$ for the leaves of t.
c. $N(t)$ depends only on the values $O(\sigma_1, \sigma_2)$ and $P(\sigma)$ for the leaves of t.

Proof. Parts a and b are proved by induction on the total number of nodes. For part b, if t is a leaf then $\mathrm{P}(t) = 1$. If t is a CNF function, $\mathrm{P}(t)$ iff $\mathrm{P}(s)$ holds for all subterms s, and certain inequalities hold among the subterms. If t is a IV function the argument is similar.

Part a falls into the following cases. If t_1 a leaf and t_2 a leaf then $\mathrm{O}(t_1, t_2)$ is known by hypothesis. Suppose t_1 an ordinal and t_2 is a CNF function. If t_2 is improper then $O(t_1, t_2)$ equals 0 if $t_1 = 0$, else 1. If t_2 is proper then $O(t_1, t_2)$ equals -1, unless $t_2 = \kappa^{+0} \cdot \sigma_2$, in which case it equals $O(t_1, \sigma_2)$. Suppose t_1 an ordinal and t_2 is an IV function. If t_2 is improper then $O(t_1, t_2)$ equals 0 if $t_1 = 0$, else 1. If t_2 is proper then $O(t_1, t_2) = -1$. In the remaining cases, if t_1 and t_2 are both improper then $O(t_1, t_2) = 0$, if t_1 is improper and t_2 is proper then $O(t_1, t_2) = -1$, and if t_1 is proper and t_2 is improper then $O(t_1, t_2) = 1$; so it may be assumed that t_1 and t_2 are proper. If t_1 and t_2 are both CNF functions, $O(t_1, t_2)$ is determined by the lexicographic order, using the induction hypothesis. Suppose t_1 is a CNF function and t_2 is an IV functions. If $\eta_k < \alpha_2$ (where η_k) is the leading exponent in t_1) then $\alpha_1 < \alpha_2$. If $\eta_k > \alpha_2$ then $\alpha_1 > \alpha_2$. If $\eta_k = \alpha_2$ then $\alpha_1 > \alpha_2$, unless $k = 1$ and $\sigma_k = 1$, in which case $\alpha_1 = \alpha_2$. If t_1 is an IV function and t_2 is a CNF functions, reverse the roles of t_1 and t_2 in the preceding argument. Suppose t_1 and t_2 are IV functions, say $\phi_{\tilde{\alpha}_1}(\zeta_1)$ and $\phi_{\tilde{\alpha}_2}(\zeta_2)$, If $\tilde{\alpha}_1 <_{\mathrm{rl}} \tilde{\alpha}_2$ (which may be determined from the arguments using the induction hypothesis) then $\mathrm{O}(t_1, t_2) = \mathrm{O}(\zeta_1, \alpha_2)$. If $\tilde{\alpha}_1 = \tilde{\alpha}_2$ then $\mathrm{O}(t_1, t_2) = \mathrm{O}(\zeta_1, \zeta_2)$. If $\tilde{\alpha}_2 <_{\mathrm{rl}} \tilde{\alpha}_1$ then $\mathrm{O}(t_1, t_2) = \mathrm{O}(\alpha_1, \zeta_2)$.

Part c now follows inductively. The claim is trivial if t is a leaf. If t is a CNF function then t is normal if its subterms are normal, except if $k = 1$ and $\eta_k = 0$. If t is an IV function then t is normal if its subterms are normal, except if $t = \phi_{\tilde{\alpha}_1}(\phi_{\tilde{\alpha}_2}(\zeta_2))$ where $\tilde{\alpha}_1 <_{\mathrm{rl}} \tilde{\alpha}_2$. ☐

4. IV Scheme Terms

From hereon the cardinal θ will be κ^+ for some $\kappa \in \text{Inac}$. A scheme term over κ is defined to be a term, whose leaves are 0 or a scheme over κ with $\sigma > 0$; and whose interior nodes are either a CNF function C_k or an infinitary Veblen function ϕ_k where $k \geq 1$. The value α of a term $\boldsymbol{\alpha}$ is the value of the IV$_o$ term obtained by replacing each scheme $\boldsymbol{\sigma}$ by its value σ. Clearly α depends only on the values of the schemes.

It follows from theorem 9 that for each $\alpha < \Lambda_0$ there is a normal form term $\boldsymbol{\alpha}$, whose ordinal is α. $\boldsymbol{\alpha}$ is unique, up to the choice of schemes $\boldsymbol{\sigma}$ for the leaf ordinals σ. From hereon, unless specifically stated otherwise, IV scheme terms will be assumed to be in normal form.

For a term $\boldsymbol{\alpha}$ let $T_{\boldsymbol{\alpha}}$ be the union over the $\boldsymbol{\sigma}$ occurring at leaves of $\boldsymbol{\alpha}$ of the $T_{\boldsymbol{\sigma}}$.

Let $\boldsymbol{\alpha} \downarrow \lambda$ be $\boldsymbol{\alpha}$, with each leaf $\boldsymbol{\sigma}$ replaced by $\boldsymbol{\sigma} \downarrow \lambda$, each node C_k replaced by $C_{\lambda k}$, and each node ϕ_k replaced by $\phi_{\lambda k}$, Given $\boldsymbol{\alpha}$, the notation $\alpha \downarrow \lambda$ will be used to denote the ordinal specified by $\boldsymbol{\alpha} \downarrow \lambda$.

Lemma 11. *Suppose $\boldsymbol{\alpha}$ is a scheme, $\lambda \in In_\kappa$, and $\lambda \notin T_{\boldsymbol{\alpha}}$.*
a, $f_{\boldsymbol{\alpha}} \upharpoonright \lambda = f_{\boldsymbol{\alpha} \downarrow \lambda}$.
b, $f_{\boldsymbol{\alpha}}(\lambda) = \alpha \downarrow \lambda$.

Proof. For schemes this is lemma 2. Let the argument list of a function be denoted $A_1, \ldots, A_l$. For part a, suppose $\mu < \lambda$. For CNF functions, $f_{\boldsymbol{\alpha} \downarrow \lambda}(\mu) = f_{C_k(A_1 \downarrow \lambda, \ldots, A_l \downarrow \lambda)}(\mu) = C_{\mu k}(f_{A_1 \downarrow \lambda}(\mu), \ldots, f_{A_l \downarrow \lambda}(\mu)) = C_{\mu k}(f_{A_1}(\mu), \ldots, f_{A_l}(\mu)) = f_{C_k(A_1, \ldots, A_l)}(\mu) = f_{\boldsymbol{\alpha}}(\mu)$. The proof for IV functions is similar. For part b, for CNF functions, $f_{\boldsymbol{\alpha}}(\lambda) = f_{C_k(A_1, \ldots, A_l)}(\lambda) = C_{\lambda k}(f_{A_1}(\lambda), \ldots, f_{A_l}(\lambda)) = C_{\lambda k}(A_1 \downarrow \lambda, \ldots, A_l \downarrow \lambda) = C_k(A_1, \ldots, A_l) \downarrow \lambda = \boldsymbol{\alpha} \downarrow \lambda$. The proof for IV functions is similar. □

5. Function Chains for IV Terms

For each IV term $\boldsymbol{\alpha}$ a function $f_{\boldsymbol{\alpha}} : \text{In}_\kappa \mapsto \kappa$ may be defined by recursion on $\boldsymbol{\alpha}$. The cases of the recursion are numbered as in theorem 9, and are as follows. In cases 2 and 3, $C_{\lambda k}$ and $\phi_{\lambda k}$ is written for the function over the cardinal λ^+.

0. $f_0 = 0$
1. $f_{\boldsymbol{\sigma}}$ where $\boldsymbol{\alpha}$ is the scheme $\boldsymbol{\sigma}$.
2. $f_{\boldsymbol{\alpha}}(\lambda) = C_{\lambda k}(f_{\boldsymbol{\eta}_k}(\lambda), f_{\boldsymbol{\sigma}_k}(\lambda), \ldots, f_{\boldsymbol{\eta}_1}(\lambda), f_{\boldsymbol{\sigma}_1}(\lambda))$.
3. $f_{\boldsymbol{\alpha}}(\lambda) = \phi_{\lambda k}(f_{\boldsymbol{\zeta}}(\lambda), f_{\boldsymbol{\xi}_1}(\lambda), f_{\boldsymbol{\alpha}_1}(\lambda), \ldots, f_{\boldsymbol{\xi}_k}(\lambda), f_{\boldsymbol{\alpha}_k}(\lambda))$.

Theorem 12. *Suppose* $\boldsymbol{\alpha}$*, etc. are proper terms, and* $\lambda \in In_\kappa$*.*
a. *If* $\lambda \notin T_{\boldsymbol{\alpha}_1} \cup T_{\boldsymbol{\alpha}_2}$ *then* $O(f_{\boldsymbol{\alpha}_1}(\lambda), f_{\boldsymbol{\alpha}_2}(\lambda)) = O(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$.
b. *If* $\lambda \notin T_{\boldsymbol{\alpha}}$ *then* $P(f_{\boldsymbol{\alpha}}(\lambda)) = P(\boldsymbol{\alpha})$.
c. *If* $\lambda \notin T_{\boldsymbol{\alpha}}$ *then* $N(f_{\boldsymbol{\alpha}}(\lambda)) = N(\boldsymbol{\alpha})$.

Proof. This follows by theorem 10 and lemma 1. □

Theorem 13. *Suppose* $\boldsymbol{\alpha}$ *and* β *are IV terms,* $\alpha = \beta + 1$*,* $\lambda \in In_\kappa$*, and* $\lambda \notin T_\alpha \cup T_\beta$*. Then* $f_\alpha(\lambda) = f_\beta(\lambda) + 1$.

Proof. In the CNF for α, $\eta_1 = 0$; let $\boldsymbol{\alpha}_p$ be $\boldsymbol{\alpha}$ with the last term $\boldsymbol{\sigma}_1$ replaced by $\boldsymbol{\tau}_1$, or removed entirely if $\sigma_1 = 1$. By theorem 12, $f_{\boldsymbol{\beta}}(\lambda) = Tf_{\boldsymbol{\alpha}_p}(\lambda)$. It follows that $f_{\boldsymbol{\beta}}(\lambda) + 1 = Tf_{\boldsymbol{\alpha}_p}(\lambda) + 1 = f_{\boldsymbol{\alpha}}(\lambda)$. □

6. Set Chains for IV Terms

The reader is assumed to be familiar with [7]. A definition of $\mathrm{H}_{\boldsymbol{\alpha}}$ for an IV term $\boldsymbol{\alpha}$ will be given. A definition of $\mathrm{H}_{1\boldsymbol{\alpha}}$ was given in [7] only for a subset of the BV terms. H is a map from subsets of In_κ to subsets of In_κ; when it is necessary to specify κ, $\mathrm{H}_{\boldsymbol{\alpha}}$ will be denoted $\mathrm{H}_{\kappa\boldsymbol{\alpha}}$.

For an IV term $\boldsymbol{\alpha}$ and $X \subseteq \mathrm{In}_\kappa$, say that $\lambda \in \mathrm{H}_{\boldsymbol{\alpha}}(X)$ iff $\lambda \in X$ and $\mathrm{H}_{\boldsymbol{\beta}}(X \cap \lambda)$ is a stationary subset of λ for all IV terms $\boldsymbol{\beta}$ over λ where $\beta < f_{\boldsymbol{\alpha}}(\lambda)$.

For $X, Y \subseteq \mathrm{In}_\kappa$ say that $X \subseteq_t Y$ if $\{\lambda \in \mathrm{In}_\kappa : \lambda \in X \text{ and } \lambda \notin Y\}$ is thin; and similarly for $\equiv_t$.

Theorem 14. *If* $\alpha' \leq \alpha$ *then for any* $X \subseteq In_\kappa$*,* $H_{\boldsymbol{\alpha}'}(X) \supseteq_t H_{\boldsymbol{\alpha}}(X)$*.*

Proof. The proof is by induction on κ. For the basis, κ is the smallest inaccessible cardinal, X is always empty, and the claim is trivial. For arbitrary κ, there is a thin set T such that if $\lambda \in \mathrm{In}_\kappa$ and $\lambda \notin T$ then $f_{\boldsymbol{\alpha}'}(\lambda) \leq f_{\boldsymbol{\alpha}}(\lambda)$. For such λ, if $\mathrm{H}_{\boldsymbol{\beta}}(X \cap \lambda)$ is stationary for $\beta < f_{\boldsymbol{\alpha}}(\lambda)$, then by the induction hypothesis $\mathrm{H}_{\boldsymbol{\beta}}(X \cap \lambda)$ is stationary for $\beta < f_{\boldsymbol{\alpha}'}(\lambda)$. □

Lemma 15. *Suppose* $\boldsymbol{\alpha}$ *is an IV term,* $X \subseteq In_\kappa$*,* $\lambda \in In_\kappa$*, and* $\lambda \notin T_{\boldsymbol{\alpha}}$*. Then* $H_{\boldsymbol{\alpha}\downarrow\lambda}(X \cap \lambda) = H_{\boldsymbol{\alpha}}(X) \cap \lambda$*.*

Proof. Suppose $\mu < \lambda$. Then $\mu \in \mathrm{H}_{\boldsymbol{\alpha}\downarrow\lambda}(X \cap \lambda)$ iff $\mathrm{H}_{\boldsymbol{\beta}}(X \cap \mu)$ is stationary for $\beta < f_{\boldsymbol{\alpha}\downarrow\lambda}(\mu)$ iff (by lemma 11) $\mathrm{H}_{\boldsymbol{\beta}}(X \cap \mu)$ is stationary for $\beta < f_{\boldsymbol{\alpha}}(\mu)$ iff $\mu \in \mathrm{H}_{\boldsymbol{\alpha}}(X)$. □

Theorem 16. *Suppose* $\alpha = \beta + 1$, $\lambda \in \mathrm{In}_\kappa$, *and* $\lambda \notin T_{\boldsymbol{\alpha}}$. *Then* $\lambda \in H_{\boldsymbol{\alpha}}(X)$ *iff* $\lambda \in H(H_{\boldsymbol{\beta}}(X))$.

Proof. $\lambda \in \mathrm{H}_{\boldsymbol{\alpha}}(X)$ iff $\lambda \in X$ and $\mathrm{H}_{\boldsymbol{\gamma}}(X \cap \lambda)$ is stationary for $\boldsymbol{\gamma}$ with $\gamma < f_{\boldsymbol{\alpha}}(\lambda)$ iff $\lambda \in \mathrm{H}_{\boldsymbol{\beta}}(X)$ and (A) $\mathrm{H}_{\boldsymbol{\gamma}}(X \cap \lambda)$ is stationary for $\boldsymbol{\gamma}$ with $\gamma = f_{\boldsymbol{\alpha}}(\lambda)$. Using lemma 11.b, (A) holds iff $\mathrm{H}_{\boldsymbol{\beta}\downarrow\lambda}(X \cap \lambda)$ is stationary, and by lemma 15 this holds iff $\mathrm{H}_{\boldsymbol{\beta}}(X) \cap \lambda$ is stationary. □

Recall from [7] the definition of ρ_R. By theorems 14 and 16, it follows that if $\mathrm{H}_{\boldsymbol{\alpha}}(\mathrm{In}_\kappa)$ is stationary then $\rho_R(\mathrm{H}_{\boldsymbol{\alpha}}(\mathrm{In}_\kappa) \geq \alpha$. This was proved in [7] only for a subset of the BV terms (with a different definition of $\mathrm{H}_{\boldsymbol{\alpha}}$; but see below).

7. A Continuity Property

As noted in [11], explicit ascending sequences can be given for limit ordinals less than Λ_0, using IV terms. A sequences $\boldsymbol{\alpha}_\xi$ for $\boldsymbol{\alpha}$ may be defined by recursion on $\boldsymbol{\alpha}$. Case 0 is irrelevant. In case 1, let $\boldsymbol{\sigma}_\xi$ be as determined by the last node of $\boldsymbol{\sigma}$.

Case 2 is divided into subcases as follows. In subcase 2.1, in the CNF for α, $k > 1$. Write $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$. Let $\boldsymbol{\alpha}_\xi$ equal $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_{2\xi}$.

In subcase 2.2 $\boldsymbol{\alpha} = \kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}$ where $\sigma \in \mathrm{Lim}$ and $\sigma > 1$. Let $\boldsymbol{\alpha}_\xi$ equal $\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}_\xi$.

In subcase 2.3 $\boldsymbol{\alpha} = \kappa^{+\boldsymbol{\eta}}$ where $\eta \in \mathrm{Lim}$ and $\eta < \kappa^{+\eta}$. Let $\boldsymbol{\alpha}_\xi$ equal $\kappa^{+\boldsymbol{\eta}_\xi}$.

In subcase 2.4 $\boldsymbol{\alpha} = \kappa^{+\boldsymbol{\eta}}$ where $\eta = \eta^- + 1$. Let $\boldsymbol{\alpha}_\xi$ equal $\kappa^{+\boldsymbol{\eta}^-} \cdot \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is some particular scheme with ordinal ξ.

To divide case 3 into subcases, write the IV function as $\phi_k(\zeta, \pi_1, \alpha_1, \ldots)$. Cases will be denoted XYZ where X is the type of ζ (L for limit, 0, or S for successor), Y is the type of α_1 (L or S), and Z is the type of ξ_1 (L, 0, or S); in some cases Y or Z may be *, denoting any possibility. If α is a successor ordinal write α^- for its predecessor. Sequences $\boldsymbol{\alpha}_\xi$ in each case are as follows.

Case L**: $\boldsymbol{\alpha}_\xi = \phi_k(\boldsymbol{\zeta}_\xi, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_1, \ldots)$.

Case 0L*: $\boldsymbol{\alpha}_\xi = \phi_k(0, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_{1\xi}, \ldots)$.

Case SL*: $\boldsymbol{\gamma} = \phi_k(\boldsymbol{\zeta}^-, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_1, \ldots) + 1$, $\boldsymbol{\alpha}_\xi = \phi_k(\boldsymbol{\gamma}, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_{1\xi} \ldots)$.

Case 0SL: $\boldsymbol{\alpha}_\xi = \phi_k(0, \boldsymbol{\pi}_{1\xi}, \boldsymbol{\alpha}_1^-, \ldots)$.

Case SSL: $\boldsymbol{\gamma} = \phi_k(\boldsymbol{\zeta}^-, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_1, \ldots) + 1$, $\boldsymbol{\alpha}_\xi = \phi_k(\boldsymbol{\gamma}, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_{1\xi}, \ldots)$.

Case 0S0: $\boldsymbol{\alpha}_0 = \phi_k(0, 0, \boldsymbol{\alpha}_1^-, \ldots)$, $\boldsymbol{\alpha}_{n+1} = \phi_k(\boldsymbol{\alpha}_n, 0, \boldsymbol{\alpha}_1^-, \ldots)$.

Case SS0: $\boldsymbol{\alpha}_0 = \phi_k(\boldsymbol{\zeta}^-, 0, \boldsymbol{\alpha}_1, \ldots) + 1$, $\boldsymbol{\alpha}_{n+1} = \phi_k(\boldsymbol{\alpha}_n, 0, \boldsymbol{\alpha}_1^-, \ldots)$.

Case 0SS: $\boldsymbol{\alpha}_0 = \phi_{k+1}(0, \boldsymbol{\pi}_1^-, 1, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_1^-, \dots)$, $\boldsymbol{\alpha}_{n+1} = \phi_{k+1}(0, \boldsymbol{\pi}_1^-, \boldsymbol{\alpha}_n, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_1^-, \dots)$.

Case SSS: $\boldsymbol{\alpha}_0 = \phi_k(\boldsymbol{\zeta}^-, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_1, \dots)$, $\boldsymbol{\alpha}_{n+1} = \phi_{k+1}(0, \boldsymbol{\pi}_1^-, \boldsymbol{\alpha}_n, \boldsymbol{\pi}_1, \boldsymbol{\alpha}_1^-, \dots)$.

Lemma 17. *Suppose $\boldsymbol{\alpha}$ is a term, $\lambda \in In_\kappa$, and $\lambda \notin T_{\boldsymbol{\alpha}}$. If α is a successor ordinal then $f_{\boldsymbol{\alpha}}(\lambda)$ is a successor ordinal. If Cf$(\alpha) = \eta$ where $\eta < \kappa$ (whence $\eta < \lambda$) then Cf$(f_{\boldsymbol{\alpha}}(\lambda)) = \eta$. If Cf$(\alpha) = \kappa$ then Cf$(f_{\boldsymbol{\alpha}}(\lambda)) = \kappa$. If Cf$(\alpha) = \kappa^+$ then Cf$(f_{\boldsymbol{\alpha}}(\lambda)) = \lambda^+$.*

Proof. This follows by induction on α, using the ascending sequences given above, and the definition of $f_{\boldsymbol{\alpha}}(\lambda)$. □

The notation $f \le_t g$ will be used to denote that $\{\lambda : f(\lambda) > g(\lambda\}$ is thin; and similarly for $=_t$. It is readily verified that sup and dsup produce least upper bounds in the order $\le_t$. See also lemma 1.2 of [2]

Theorem 18. *If $\eta < \kappa$ and α_ξ for $\xi < \eta$ is an ascending sequence with $\alpha = \sup_{\xi<\eta} \alpha_\xi$ then $f_{\boldsymbol{\alpha}} =_t \sup_{\xi<\eta} f_{\boldsymbol{\alpha}_\xi}$.*

Proof. The proof is by induction on $\boldsymbol{\alpha}$. By remarks preceding the theorem it suffices to prove the claim for a particular sequence α_ξ. The sequence used will be that given above. Case 0 is irrelevant. Case 1 follows directly. The remaining cases follow inductively, using properties of sup, with subcase 2.4 being irrelevant. □

The theorem can be improved slightly. The set of λ where the equation might not hold is contained in $T_{\boldsymbol{\alpha}} \cup (\eta + 1) \cup \cup_{\xi<\eta} T_{\boldsymbol{\alpha}_\xi}$.

Theorem 19. *If α_ξ for $\xi < \kappa$ is an ascending sequence with $\alpha = \sup_{\xi<\kappa} \alpha_\xi$ then $f_{\boldsymbol{\alpha}} =_t$ dsup$_{\xi<\kappa} f_{\boldsymbol{\alpha}_\xi}$.*

Proof. The proof is by induction on $\boldsymbol{\alpha}$. As in the preceding theorem it suffices to prove the claim for the sequence given above. Case 0 is irrelevant. Case 1 follows directly. The remaining cases follow inductively, using the definition of dsup and properties of sup, with subcase 2.4 and the subcases of case 3 not involving an L being irrelevant. □

Theorem 20. a. *Suppose $\eta < \kappa$, α_ξ is ascending, and $\alpha = \sup_{\xi<\eta} \alpha_\xi$. Then $H_{\boldsymbol{\alpha}} \equiv_t \cap_{\xi<\eta} H_{\boldsymbol{\alpha}_\xi}$.*

b. *Suppose α_ξ is ascending and $\alpha = \sup_{\xi<\kappa} \alpha_\xi$. Then $H_{\boldsymbol{\alpha}} \equiv_t$ $Bt_{\xi<\kappa} H_{\boldsymbol{\alpha}_\xi}$.*

Proof. For part a, $\lambda \in \mathrm{H}_{\boldsymbol{\alpha}}(X)$ iff $\mathrm{H}_{\boldsymbol{\beta}}(X \cap \lambda)$ is stationary whenever $\beta < f_{\boldsymbol{\alpha}}(\lambda)$. By theorem 18, except for a thin set of λ, this holds iff, for all $\xi < \eta$, $\mathrm{H}_{\boldsymbol{\beta}}(X \cap \lambda)$ is stationary whenever $\beta < f_{\boldsymbol{\alpha}_\xi}(\lambda)$, iff $\lambda \in \mathrm{H}_{\boldsymbol{\alpha}_\xi}(X)$. Part b follows similarly, with "$\xi < \eta$" replaced by "$\xi < \lambda$". □

Readily from the definitions, $\mathrm{H}_0 = \mathrm{Id}$ (the identity function). By theorem 16, 18, and 19, $\mathrm{H}_{\boldsymbol{\sigma}}$ as defined here agrees with $\mathrm{H}^{\boldsymbol{\sigma}}$ as defined in [7]. It may be seen that this continues to hold at κ^+; further discussion is omitted here.

8. Enforceability

Let Φ be the sentence $\forall A, C, X, Y, Z($"A is an IV scheme term" $\wedge$ "C is club" $\wedge\, X = \mathrm{Inac} \wedge Y = \mathrm{H}_A(X) \wedge Z = Y \cap C \Rightarrow Z \neq \emptyset)$. Most of the subformulas of the matrix are readily seen to be Δ^1_0; discussions will be given for the remaining ones.

There is a Δ^1_0 predicate stating that the class X represents a well-order on a subset of κ. This states that X is a class of ordered pairs, which as a binary relation is transitive and reflexive, and has no descending chains of length ω.

There is a Δ^1_0 predicate stating that the class X represents a scheme for κ. Namely, it represents a pair $\langle \sigma, \phi \rangle$ where σ is represented as above, and ϕ is a function whose domain is the limit points $\alpha < \sigma$, where $\phi(\alpha)$ is a function with domain either an ordinal, or all ordinals, etc.

An IV term may be given as a class coding the sequence of classes $\langle t, \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k \rangle$ where t is a hereditarily finite set coding the tree of the term, and $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k$ are (codes for) the schemes at the leaves in order. It follows that the formula "A is an IV scheme term" is Δ^1_0.

A function $F : \mathrm{Ord} \mapsto \mathrm{Ord}$ is a class (namely its ordered pairs). Let f_x denote the value of $f_{\boldsymbol{\sigma}}$ at stage x of the iteration. The class of triples $\langle x, \lambda, f_x(\lambda) \rangle$ may serve as a witness that $F = f_A$ for a scheme A. The predicate "W is the witness to $F = f_A$" is Δ^1_0.

The predicate $\alpha = C_{\lambda k}(\eta_1, \sigma_1, \ldots, \eta_k, \sigma_k)$, in the values $\lambda, k, \langle \eta_1, \sigma_1, \ldots, \eta_k, \sigma_k \rangle$, is first-order definable, indeed no doubt Δ^0_1. The same is true of the predicate $\alpha = \phi_{\lambda k}(\zeta, \xi_1, \alpha_1, \ldots, \xi_k, \alpha_k)$, Further discussion is omitted here.

The witness to $F = f_A$ where A is any term is the sequence of classes $\langle F_1, \ldots, F_r, W_1, \ldots, W_s \rangle$ where r is the number of nodes in the tree of A, s is the number of leaves, for each leaf index i W_i witnesses that $F_{l_i} = f_{\boldsymbol{\sigma}_i}$, and for each interior node index i W_i witnesses that F_{n_i} is obtained by applying the appropriate function per λ to the F_j of the sons. The predicate "W is the witness to $F = f_A$" is Δ^1_0.

From the foregoing the predicate $F = f_A$ is Σ^1_1, whence Δ^1_1.

A term $\boldsymbol{\beta}$ over $\lambda \in \mathrm{In}_\kappa$ is a set. The predicates "β is the ordinal of $\boldsymbol{\beta}$" and "$\gamma = f_{\boldsymbol{\beta}}(\mu)$" are first order definable. The function $\lambda \mapsto S_\lambda$ where $S_\lambda = \{\langle \boldsymbol{\beta}, x, y\rangle : y = \mathrm{H}_{\boldsymbol{\beta}}(x)\}$ may be defined by recursion on λ; Indeed, $\langle \boldsymbol{\beta}, x, y\rangle \in S_\lambda$ iff $\forall \mu(\mu \in y$ iff $\lambda \in x \wedge \forall \boldsymbol{\beta}($"$\boldsymbol{\beta}$ is term over μ" $\wedge\ \gamma < f_{\boldsymbol{\beta}}(\mu) \Rightarrow$ "$\mathrm{H}_{\boldsymbol{\gamma}}(X \cap \mu)$ is stationary")).

Now, $Y = \mathrm{H}_A(X)$ may be expressed as $\forall \lambda(\lambda \in Y$ iff $\lambda \in X \wedge \forall \boldsymbol{\beta}($"$\boldsymbol{\beta}$ is term over λ" $\wedge\ \exists \gamma(\langle \lambda, \gamma\rangle \in F \wedge \beta < \gamma \Rightarrow$ "$\mathrm{H}_{\boldsymbol{\beta}}(X \cap \lambda)$ is stationary")). From the foregoing, "$Y = \mathrm{H}_A(X)$" is Δ^1_1.

Say that an inaccessible cardinal κ is Λ_0-Mahlo iff $\models_{V_\kappa} \Phi$.

Theorem 21. *There is a Π^1_1 sentence Φ such that κ is Λ_0-Mahlo iff $\models_{V_\kappa} \Phi$.*

Proof. Immediate by remarks above. □

Lemma 22. *Suppose κ is weakly compact. For each scheme $\boldsymbol{\sigma}$ there is a Π^1_1 formula $\Phi_{\boldsymbol{\sigma}}$ such that $\models_{V_\kappa} \Phi_{\boldsymbol{\sigma}}$, and for $\lambda \in \mathit{In}_\kappa$, if $\models_{V_\lambda} \Phi_{\boldsymbol{\sigma}}$ then $\boldsymbol{\sigma} \cap V_\lambda$ equals $\boldsymbol{\sigma} \downarrow \lambda$.*

Proof. $\Phi_{\boldsymbol{\sigma}}$ is constructed by recursion on $\boldsymbol{\sigma}$. Various details are omitted, and may be found in [3]. Cases of the recursion are as follows, where Φ has second order parameters $\boldsymbol{\alpha}, P$. In case 0, Φ is a sentence enforcing that κ is inaccessible. In case 1, $\Phi_{\boldsymbol{\sigma}}$ is "$\boldsymbol{\sigma}$ is a scheme" $\wedge$ "σ is a successor ordinal" $\wedge$ $\forall \boldsymbol{\tau}(\tau = \boldsymbol{\sigma}^- \Rightarrow \Phi_{\boldsymbol{\sigma}^-}(\boldsymbol{\tau}, P))$. In case 2, $\Phi_{\boldsymbol{\sigma}}$ is "$\boldsymbol{\sigma}$ is a scheme" $\wedge\ \sigma \in \mathrm{Lim} \wedge \exists \eta($"$\eta$ is the domain of the ascending sequence of the last node" $\wedge\ \forall \xi < \eta \forall \boldsymbol{\tau}(\tau = \boldsymbol{\sigma}_\xi \Rightarrow \Phi_{\boldsymbol{\sigma}_\xi}(\boldsymbol{\tau}, P)))$. The formulas $\Phi_{\boldsymbol{\sigma}_\xi}$ may be combined into a single formula in a well-known manner. Case 3 is similar. □

Corollary 23. *Suppose κ is weakly compact. For each IV term $\boldsymbol{\alpha}$ there is a Π^1_1 formula $\Phi_{\boldsymbol{\alpha}}$ such that $\models_{V_\kappa} \Phi_{\boldsymbol{\alpha}}$, and for $\lambda \in \mathit{In}_\kappa$, if $\models_{V_\lambda} \Phi_{\boldsymbol{\alpha}}$ then $\boldsymbol{\alpha} \cap V_\lambda$ equals $\boldsymbol{\alpha} \downarrow \lambda$.*

Proof. This follows readily from the lemma and the definition of $\boldsymbol{\alpha} \downarrow \lambda$. □

Theorem 24. *Suppose κ is weakly compact. Then $\models_{V_\kappa} \Phi$.*

Proof. It suffices to show by induction on $\boldsymbol{\alpha}$ that for all $\boldsymbol{\alpha}$, $\models_{V_\kappa}$"$\mathrm{H}_{\boldsymbol{\alpha}}(\mathrm{Inac})$ is stationary". Inductively, $\models_{V_\kappa} \forall \boldsymbol{\beta} < \boldsymbol{\alpha}$"$\mathrm{H}_{\boldsymbol{\beta}}(\mathrm{Inac})$ is stationary". By lemma 11 and corollary 23, for a stationary set of λ, $\models_{V_\lambda} \forall \boldsymbol{\beta} < f_{\boldsymbol{\alpha}}(\lambda)$"$\mathrm{H}_{\boldsymbol{\beta}}(\mathrm{Inac})$ is stationary". $\models_{V_\kappa}$"$\mathrm{H}_{\boldsymbol{\alpha}}(\mathrm{Inac})$ is stationary" follows. □

9. A New Axiom

It seems helpful to define axiom M to be the second order axiom, to be added to NBG, which states that Ord is "…-Mahlo" where "…" is a specification of the length of stationary set chains which have been constructed. As of this paper, axiom M states that "Ord is Λ_0-Mahlo". The sentence Φ of the previous section gives a more detailed statement.

Only some brief remarks on justifying this axiom will be given here. The reader is assumed to be familiar with section 7 of [6]. Inductively, it may be assumed that $\forall \boldsymbol{\beta} < \boldsymbol{\alpha}$"$\mathrm{H}_{\boldsymbol{\beta}}$(Inac). Indeed, letting $\boldsymbol{\alpha}$ be a term in some sufficiently large outer universe, successive λ such that $\models_{V_\lambda} \forall \boldsymbol{\beta} < f_{\boldsymbol{\alpha}}(\lambda)$"$\mathrm{H}_{\boldsymbol{\beta}}$(Inac) is stationary" may be collected. The general fact that something that may be repeated may be repeated stationarily often results in a universe in which "$\mathrm{H}_{\boldsymbol{\alpha}}$(Inac) is stationary" holds.

As usual, more details of this justification would be desirable, but we omit this subject here.

10. Normal Ultrafilters

An application of function chains was given in [5], with further discussion given in [7]. An essential fact is the following.

Theorem 25. *For any measurable cardinal κ, any normal ultrafilter U on κ, and any IV scheme term α over κ, $f_{\boldsymbol{\alpha}}$ represents α in the ultrapower $Ult_U(V)$.*

Proof. The proof is by induction on $\boldsymbol{\alpha}$, using Los' theorem. First the claim is proved for schemes. For case 0 the claim is immediate. For case 1, inductively $f_{\boldsymbol{\tau}}$ represents τ, whence $f_{\boldsymbol{\tau}} + 1$ represents $\tau + 1$. For case 2, inductively $f_{\boldsymbol{\sigma}_\xi}$ represents σ_ξ, whence $\sup_\xi f_{\boldsymbol{\sigma}_\xi}$ represents $\sup_\xi \sigma_\xi$. Case 3 is similar. For the induction on scheme terms, cases 0 and 1 have already been proved. Cases 2 and 3 follow readily by the definability of the functions C_k and ϕ_k. □

It is independent whether there are canonical functions of rank κ^+ (see [1]). The foregoing shows that the functions $f_{\boldsymbol{\alpha}}$ have a weaker property of interest. It would be of interest to obtain a characterization of the property which made no use of ultrafilters.

Suppose κ is a measurable cardinal. Let $o(\kappa)$ denote its Mitchell order. For $\boldsymbol{\alpha}$ an IV scheme term let $S_{\boldsymbol{\alpha}}$ denote $\{\lambda \in \mathrm{In}_\kappa : o(\lambda) \geq f_{\boldsymbol{\alpha}}(\lambda)\}$. Recall the definition of $<_R$ from [7].

Theorem 26. *Suppose $\alpha < o(\kappa)$. Then $S_{\boldsymbol{\alpha}}$ is stationary, and if $\beta < \alpha$ then $S_{\boldsymbol{\beta}} <_R S_{\boldsymbol{\alpha}}$.*

Proof. Let U_1 be a normal ultrafilter on κ with $O(U_1) = \alpha$. By lemma 19.34 of [Jech], o represents $o(U_1)$ in $\mathrm{Ult}_{U_1}(V)$. Since by theorem 25 $f_{\boldsymbol{\alpha}}$ also represents $\alpha = o(U_1)$, $\{\lambda \in \mathrm{In}_\kappa : o(\lambda) = f_{\boldsymbol{\alpha}}(\lambda)\} \in U_1$. It follows that $S_{\boldsymbol{\alpha}}$ is stationary.

Suppose $\lambda \in S_{\boldsymbol{\beta}+1}$. Then $o(\lambda) \geq f_{\boldsymbol{\beta}+1}(\lambda)$, so except for a thin set of λ there is a normal ultrafilter U' on λ with $o(U') = f_{\boldsymbol{\beta}}(\lambda)$. By lemma 11 and theorem 25, except for a thin set of λ, $f_{\boldsymbol{\beta}} \restriction \lambda$ represents $f_{\boldsymbol{\beta}}(\lambda)$ in $\mathrm{Ult}_{U'}(V)$, whence it represents $o(U')$. By an argument just given, $\{\mu < \lambda : o(\mu) = f_{\boldsymbol{\alpha}}(\mu)\} \in U'$, whence $S_{\boldsymbol{\beta}} \cap \lambda \in U'$. This shows that $S_{\boldsymbol{\beta}+1} \subseteq_t \mathrm{H}(S_{\boldsymbol{\beta}})$, completing the proof of the theorem. □

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