International Journal of Pure and Applied Mathematics
Volume 89 No. 4 2013, 559-564
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v89i4.10
ijpam.eu

SOME EXTENSION AND GENERALIZATION OF THE BOUNDS FOR THE ZEROS OF A POLYNOMIAL WITH RESTRICTED COEFFICIENTS

Mohammad Al-Hawari[1 §], Farah M. AL-Askar[2]
[1]College of Applie Medical Sciences
Majmaah University
P.O. Box 1405, Almajmaah, 11952, KINGDOM OF SAUDI ARABIA
[2]College of Education Department of mathematics
Majmaah University
P.O. Box 1405, Almajmaah, 11952, KINGDOM OF SAUDI ARABIA

Abstract: Let $P(z)$ be a polynomial of degree n with decreasing coefficients. Then all its zeros lie in $|z| \leq 1$. In this paper we present some generalizations of this result and a refinement of a classical bounds.

AMS Subject Classification: 0C10, 30C15
Key Words: polynomial, bounds, zeros, Enestrom-Kakeya theorem

1. Introduction

The location of the zeros of a polynomial with restricted coefficients have been invoked by Enestrom and Kakeya [1] and Abdul Aziz, Bashir Ahmad Zargar [3], also see M.H. Gulzar [5]. It has been shown in [1] that if

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0$$

be a polynomial of degree n, such that $a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0$, then $p(z)$ does not vanish in

$$|z| > 1. \tag{1}$$

Received: August 26, 2013

§Correspondence author

Also, it has been shown in [6] that if

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0$$

be a polynomial of degree n, then all the zeros of $p(z)$ lie in the closed disk

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \{2a_\lambda - a_{n-1} + |a_0| (2-\rho) - \rho a_0\}, \tag{2}$$

for some positive numbers ρ, with $0 < \rho \leq 1$, and $\lambda, 0 \leq \lambda \leq n-1$, $a_n \leq a_{n-1} \leq ... \leq a_\lambda \geq a_{\lambda-1} \geq ... \geq a_1 \geq \rho a_0$.

The aim of this paper is to present some generalizations of the result "all its zeros lie in" $|z| \leq 1$ and a refinement of a classical bounds.

2. Main Results

Theorem 2.1. *Let*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0$$

be a polynomial of degree n, then all the zeros of $p(z)$ lie in the closed disk

$$|z + k_1 - 1| \leq k_1 + k_2 \frac{a_\lambda}{a_n} + \frac{2a_0}{a_n}(1-\rho), \tag{3}$$

for some positive numbers k_1, k_2, ρ, and λ, with $k_1, k_2 \geq 1$, $0 < \rho \leq 1$, $0 < \lambda \leq n-1$, and $k_1 a_n \geq a_{n-1} \geq ... \geq k_2 a_\lambda \geq a_{\lambda-1} \geq ... \geq \rho a_0 \geq 0$.

Proof. Consider

$$\begin{aligned} F(z) = (1-z)\, p(z) &= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + ... + (a_1 - a_0) z + a_0 \\ &= -a_n z^{n+1} + a_n z^n - k_1 a_n z^n + (k_1 a_n - a_{n-1}) z^n + ... + a_\lambda z^\lambda \\ &\quad - k_2 a_\lambda z^\lambda + (k_2 a_\lambda - a_{\lambda-1}) z^\lambda + ... + (a_1 - \rho a_0) z + (\rho - 1)\, a_0 z + a_0. \end{aligned}$$

Then, if $|z| > 1$, we obtain

$$\begin{aligned} |F(z)| = \Big| &-a_n z^{n+1} + a_n z^n - k_1 a_n z^n + (k_1 a_n - a_{n-1}) z^n + ... + a_\lambda z^\lambda \\ &- k_2 a_\lambda z^\lambda + (k_2 a_\lambda - a_{\lambda-1}) z^\lambda + ... + (a_1 - \rho a_0) z + (\rho - 1)\, a_0 z + a_0 \Big| \\ \geq |a_n|\, |z|^n \Bigg[&|z + k_1 - 1| - \frac{1}{|a_n|} \Big\{ \Big| (k_1 a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) \frac{1}{z} + ... \end{aligned}$$

$$-k_2 a_\lambda z^\lambda + (k_2 a_\lambda - a_{\lambda-1}) z^\lambda + ... + (a_0 - \rho a_0) z + (\rho - 1)\frac{a_0}{z^n} + \frac{a_0}{z^n}\Big|\}\Big]$$
$$\geq |a_n|\,|z|^n\left[|z+k_1-1| - \frac{1}{|a_n|}\{k_1 a_n + k_2 a_\lambda - \rho a_0 + (1-\rho)\,|a_0| + |a_0|\}\right]$$
$$= |a_n|\,|z|^n\left[|z+k_1-1| - \frac{1}{a_n}\{k_1 a_n + k_2 a_\lambda - \rho a_0 + (1-\rho)\,|a_0| + a_0\}\right] > 0,$$

if

$$|z+k_1-1| > \frac{k_1 a_n + k_2 a_\lambda - \rho a_0 + (1-\rho)\,|a_0| + a_0}{a_n},$$

then we shows that

$$|z| > 1 \rightarrow |F(z)| > 0.$$

Also, if

$$|z+k_1-1| > k_1 + k_2\frac{a_\lambda}{a_n} + \frac{2a_0}{a_n}(1-\rho),$$

then all the zeros of $p(z)$ lie in the closed disk

$$|z+k_1-1| \leq k_1 + k_2\frac{a_\lambda}{a_n} + \frac{2a_0}{a_n}(1-\rho).$$

If we take $k_1 = \frac{a_{n-1}}{a_n} \geq 1$, $k_2 = \frac{a_n}{a_\lambda} \geq 1$, in Theorem 2.1, then we obtain the following corollary.

Corollary 2.1. *Let*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0$$

be a polynomial of degree n. Then all the zeros of $p(z)$ lie in the closed disk

$$\left|z + \frac{a_{n-1}}{a_n} - 1\right| \leq \frac{a_{n-1}}{a_n} + 1 + \frac{2a_0}{a_n}(1-\rho), \tag{4}$$

, for some positive numbers ρ and λ, with $0 < \rho \leq 1$, $0 < \lambda \leq n-1$, and $a_{n-1} \geq a_{n-2} \geq ... \geq a_n \geq a_{\lambda-1} \geq ... \geq \rho a_0 \geq 0$.

If we take $k_1 = k_2 = \rho = 1$, in Theorem 2.1, then we obtain the following corollary.

Corollary 2.2. sl Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0$$

be a polynomial of degree n, then all the zeros of $p(z)$ lie in the closed disk

$$|z| \leq 1 + \frac{a_\lambda}{a_n} \tag{5}$$

and $a_n \geq a_{n-1} \geq ... \geq a_\lambda \geq a_{\lambda-1} \geq ... \geq a_0 \geq 0$.

If we take $k_2 = 1$, in Theorem 2.1, then we obtain the following corollary.

Corollary 2.3. *Let*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0$$

be a polynomial of degree n. Then all the zeros of $p(z)$ lie in the closed disk

$$|z + k_1 - 1| \leq k_1 + \frac{a_\lambda}{a_n} + \frac{2a_0}{a_n}(1 - \rho), \tag{6}$$

for some positive numbers k_1, ρ, λ with $k_1 \geq 1$, $0 < \rho \leq 1$, $0 < \lambda \leq n - 1$, and $k_1 a_n \geq a_{n-1} \geq ... \geq a_\lambda \geq a_{\lambda-1} \geq ... \geq \rho a_0 \geq 0$.

If we put $k_1 = k_2 = a_n$ and $\rho = 1$, in Theorem 2.1, then we obtain the following corollary.

Corollary 2.4. *Let*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0$$

be a polynomial of degree n. Then all the zeros of $p(z)$ lie in the closed disk

$$|z + a_n - 1| \leq a_n + a_\lambda. \tag{7}$$

Here $a_n^2 \geq a_{n-1} \geq ... \geq a_n a_\lambda \geq a_{\lambda-1} \geq ... \geq a_0 \geq 0$.

Theorem 2.2. *Let*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0$$

be a polynomial of degree n. Then all the zeros of $p(z)$ lie in the closed disk

$$|z + k - 1| \leq \frac{1}{a_n}\{2a_\lambda - k a_n + 2|a_0| - \rho(|a_0| - a_0)\}, \tag{8}$$

for some positive numbers k, ρ, and λ, with $0 < k \leq 1$, $0 < \rho \leq 1$, and $0 < \lambda \leq n - 1$, and $k a_n \leq a_{n-1} \leq ... \leq a_\lambda \geq a_{\lambda-1} \geq .. > a_1 \geq \rho a_0 \geq 0$.

Proof. Consider

$$F(z) = (1 - z)\, p(z) = -a_n z^{n+1} + (a_n - a_{n-1}) z^n + ...$$
$$+ (a_\lambda - a_{\lambda-1}) z^\lambda + ... + (a_1 - a_0) z + a_0$$
$$= -a_n z^{n+1} + a_n z^n - k a_n z^n + (k a_n - a_{n-1}) z^n + ... + (a_{\lambda+1} - a_\lambda) z^{\lambda+1} + (a_\lambda - a_{\lambda-1}) z^\lambda$$

$$+ \dots + (a_1 - \rho a_0)z + (\rho - 1)\, a_0 z + a_0$$
$$\geq |a_n|\, |z|^n \, [|z + k$$
$$-1| - \frac{1}{|a_n|}\{\left| \begin{array}{c} (ka_n - a_{n-1}) + (a_{n-1} - a_{n-2})\frac{1}{z} + \dots - a_\lambda z^\lambda + \\ (a_\lambda - a_{\lambda-1})z^\lambda + \dots + (a_0 - \rho a_0)z + (\rho - 1)\, \frac{a_0}{z^n} + \frac{a_0}{z^n} \end{array} \right| \}]$$
$$= |a_n|\, |z|^n \left[|z + k - 1| - \frac{1}{a_n}\{-ka_n + 2a_\lambda + 2\,|a_0| - \rho\,(|a_0| - a_0)\} \right] > 0,$$

if

$$|z + k - 1| \leq \frac{1}{a_n}\{2a_\lambda - ka_n + 2\,|a_0| - \rho\,(|a_0| - a_0)\,\}.$$

Corollary 2.5. *If we put $k = 1$, in Theorem 2.2, then we obtain the inequality (2).*

Corollary 2.6. *If we take $\rho = 1$, and $a_i > 0$ for all $i = 0, 1....n$ in Theorem 2.2, then:*

$$|z + k - 1| \leq \frac{1}{a_n}\{2a_\lambda - ka_n\}. \tag{9}$$

Corollary 2.7. *If we take $\rho = 1$, $k = 1$ and $a_i > 0$ for all $i = 0, 1....n$ in Theorem 2.2, then:*

$$|z| \leq \frac{1}{a_n}\{2a_\lambda - 1\}. \tag{10}$$

References

[1] P.V. Krishnalah, On Kakeya theorem, *Journal of London Mathematical Society*, **20**, No. 3 (1955), 314-319, **doi:** 10.1112/jlms/s1-30.3.314.

[2] A. Aziz, Q.G. Mohammad, On the zeros of a certain class of polynomials and related analytic functions, *Journal of Mathematical Analysis and Applications*, **75**, No. 2 (1980), 495-502, **doi:** 10.1016/0022-247X(80)90097-9.

[3] Abdul Aziz, Bashir Ahmad Zargar, Bounds for the zeros of a polynomial with restricted coefficients, *Applied Mathematics*, 3 (2012), 30-33, **doi:** 10.4236/am.2012.31005.

[4] N.K. Govil, Q.I. Rahman, On the Enestrom-Kakeya theorem II, *Tohoku Mathematical Journal*, **20**, No. 2 (1968), 126-136, **doi:** 10.2748/tmj/1178243172.

[5] M.H. Gulzar, On the zeros of a polynomial with restricted coefficients, *Research Journal of Pure Algebra*, **1**, No. 9 (2011), 205-208.

[6] A. Aziz, B.A. Zargar, Some extensions of Enestrom Kakeya theorem, *Glasnick Matematicki*, **31** (1996), 239-244.