WHEEL AS AN EDGE-MAGIC GRAPH

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Abstract: An image of a plane graph, $G = (V, E)$ of order $n$ and size $m$, is said to be an edge-magic plane graph if there is a bijection $f : E \rightarrow \{1, 2, ..., m\}$ such that for all $s$-side faces of $G$, except the infinite face, the sum of the labels of its edges is a constant $k(s)$. Such a bijection will be called an edge-magic plane labeling of $G$. In case that all the finite sides of a graph $G$ having the same size we will be interested in determining the minimum and the maximum number, $k$, such that there exists an edge-magic plane labeling of $G$, in which $k$ is the sum of the edge labeling of each face. In this paper we find such a minimum and maximum numbers for a wheel with even order. Furthermore we conjecture that the same formula is valid for the odd case.

Key Words: magic graph, plane graph, wheel, edge magic, minimal magic graph, maximal magic graph

1. Introduction

We study undirected graphs without loops or multiple edges. Given a graph $G$; $V(G)$, $E(G)$, $v(G)$ and $e(G)$ stands for the set of vertices, the set of edges, the order (number of vertices) and the size (number of edges) of $G$. $K_n$, and $C_n$ stand for the complete graph and the cycle of order $n$. For two graphs $G$ and $H$ we denote by $G + H$ the graph obtained from the disjoint union $G \cup H$ by adding all edges between $G$ and $H$. 
A wheel, $W_n$, is a graph of order $n + 1$ composed of a vertex, which will be called the hub, adjacent to all vertices of a cycle of order $n$. The cycle will be called the rim of the wheel, and the edges connecting the hub to the vertices of the rim will be called the spokes. i.e., $W_n = C_n + K_1$.

1.1. Total Magic Graphs

There are quite a lot of different definitions for a magic graph. We will point out two of the most popular.

Definition. Let $G = (V, E)$ be a graph of order $n$ and size $m$. A bijection $f : V \cup E \rightarrow \{1, 2, \ldots, n + m\}$ is called a vertex-magic total labeling of $G$, if there exist a constant $k$, such that,

$$\forall x \in V, \ f(x) + \sum_{xy \in E} f(xy) = k.$$ 

A graph $G$ is called a total vertex-magic graph if there exist a vertex-magic total labeling of $G$.

Definition. Let $G = (V, E)$ be a graph of order $n$ and size $m$. A bijection $f : V \cup E \rightarrow \{1, 2, \ldots, n + m\}$ is called an edge-magic total labeling of $G$, if there exist a constant $k$, such that,

$$\forall xy \in E, \ f(x) + f(y) + f(xy) = k.$$ 

A graph $G$ is called a total edge-magic graph if there exist an edge-magic total labeling of $G$.

It was proven in [5] that:

**Theorem 1.1.1.** $W_n$ has vertex-magic total labeling iff $n \leq 11$.

It was proven in [3] that:

**Theorem 1.1.2.** $W_n$ is not total edge-magic $\forall n \equiv 3 \text{mod} 4$.

It was conjectured at [3], but not yet proven, that $W_n$ is total edge-magic whenever $n \equiv 3/\text{mod} 4$.

1.2. Plane Magic Graphs

Koh wei lih defined in [4] the notions of magic labellings of a plane graph. In this paper, we will use the term edge-magic plane graph for what was defined as edge-magic graph in [4], to differ it from other definitions of edge-magic graph.
**Definition.** Let $G$ be a plane graph of size $m$. A bijection $f : E(G) \rightarrow \{1, 2, \ldots, m\}$ is called **edge-magic plane labeling** if the sum of the edge labels surrounding each $s$-sided face of $G$ is a constant.

**Definition.** A plane graph $G$ is called **edge-magic-plane graph** if there exist an edge-magic plane labeling of $G$.

**Definition.** Let $G$ be a plane graph such that all its bounded faces having the same size. $G$ will be called **$k$-edge-magic plane graph** if there exist an edge-magic plane labeling of $G$, such that the sum of labels surrounding each face of $G$ is $k$.

**Notation.** For a plane graph $G$, such that all its bounded faces having the same size, we denote by $EMP(G)$ the set of natural numbers, $k$, such that $G$ is a $k$-edge-magic plane graph.

On this paper we will find $\min(EMP(W_n))$ and $\max(EMP((W_n)))$ for all odd natural number $n$.

### 2. Labeling of wheels

Let $(a_1, \ldots, a_n)$ be the labeling of the spokes and $(b_1, \ldots, b_n)$ the labeling of the rim edges of $W_n$, such that the sum of the labels on each face of the wheel is $k$. Since each spoke belongs to two faces and each rim edge belongs to only one face, we conclude that:

$$2 \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = nk.$$  

Therefore:

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{2n} i = \sum_{i=1}^{n} a_i + (2n + 1)n = nk \quad (1)$$

Furthermore, from (1) and the following inequality

$$\frac{(n+1)n}{2} = \sum_{i=1}^{n} i \leq \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} n + i = \frac{(3n+1)n}{2}$$

we can derive that

$$\left\lfloor \frac{n+1}{2} + 2n + 1 \right\rfloor \leq k \leq \left\lceil \frac{3n+1}{2} + 2n + 1 \right\rceil \quad (2)$$
**Theorem 2.1.** For any odd natural number \( n \geq 3 \),

\[
\min(\text{EMP}(W_n)) = \frac{n + 1}{2} + 2n + 1.
\]

**Proof.** Let \( m \) be the natural number, such that \( n = 2m + 1 \).
Let \( v_0 \) be the hub vertex and \((v_1, \ldots, v_n)\) be the rim vertices, ordered counter clockwise.

Set \( f : E(W_n) \to \{1, \ldots, 2n\} \) be the function which admits the rule:

\[
f(v_i, v_j) = n + i \quad \forall i; \ 1 \leq i \leq n - 1
\]

\[
f(v_n, v_1) = 2n
\]

\[
f(v_0, v_{n-2k}) = k + 1 \quad \forall k; \ 0 \leq k \leq m
\]

\[
f(v_0, v_{2k}) = n - k + 1 \quad \forall k; \ 1 \leq k \leq m
\]

Then:

a. \( \forall k; \ 1 \leq k \leq m; \ f(v_0, v_{n-2k}) + f(v_0, v_{n-2k+1}) + f(v_{n-2k}, v_{n-2k+1}) = (k + 1) + [n - (m - k + 1) + 1] + (2n - 2k) = 2n + \frac{n + 1}{2} + 1. \)

b. \( \forall k; \ 0 \leq k \leq m - 1; \ f(v_0, v_{n-2k-1}) + f(v_0, v_{n-2k}) + f(v_{n-2k-1}, v_{n-2k}) = [n - (m - k) + 1] + (k + 1) + (2n - 2k - 1) = 2n + \frac{n + 1}{2} + 1 \)

c. \( f(v_0, v_n) + f(v_0, v_1) + f(v_n, v_1) = 1 + (m + 1) + 2n = 2n + \frac{n + 1}{2} + 1. \)

Thus, the assertion is derived from (2).
Such a labeling is demonstrated on \( W_7 \) at Figure 1.

![Figure 1. min. labeling of \( W_7 \)](image-url)
**Remark.** The above labeling of $W_n$ can be described as followed. We label the rim edges by $(n+1, n+2, \ldots, 2n)$ counter clockwise. Then we label the spoke between the rim edges labeled by $2n-1$ and $2n$ by 1 and we label all other spokes edges by $2, 3, \ldots, n$, clockwise skipping one edge every time.

**Theorem 2.2.** For any odd natural number $n \geq 3$,

$$\max(EMP(W_n)) = \frac{3n+1}{2} + 2n + 1.$$  

**Proof.** Let $m$ be the natural number such that $n = 2m + 1$. Let $v_0$ be the hub vertex and $(v_1, \ldots, v_n)$ be the rim vertices, ordered counter clockwise.

Set $f : E(W_n) \rightarrow \{1, \ldots, 2n\}$ be the function which admits the rule:

$$f(v_i, v_j) = i \ ; \ \forall i : 1 \leq i \leq n - 1$$

$$f(v_n, v_1) = n$$

$$f(v_0, v_{2k}) = n + k + 1 \ ; \ \forall k : 0 \leq k \leq m$$

$$f(v_0, v_{n-2k}) = 2n - k + 1 \ ; \ \forall k : 1 \leq k \leq m$$

Then:

a. $\forall k : 1 \leq k \leq m$ ; $f(v_0, v_{n-2k}) + f(v_0, v_{n-2k+1}) + f(v_{n-2k}, v_{n-2k+1})$

$$= (n + k + 1) + [2n - (m - k + 1) + 1] + (n - 2k) = \frac{3n+1}{2} + 2n + 1.$$  

b. $\forall k : 0 \leq k \leq m - 1$ ; $f(v_0, v_{n-2k-1}) + f(v_0, v_{n-2k}) + f(v_{n-2k-1}, v_{n-2k})$

$$= [2n - (m - k) + 1] + (n + k + 1) + (n - 2k - 1) = \frac{3n+1}{2} + 2n + 1.$$  

c. $f(v_0, v_n) + f(v_0, v_1) + f(v_n, v_1) = (n + 1) + (n + m + 1) + n$

$$= \frac{3n+1}{2} + 2n + 1.$$  

Thus, the assertion is derived from (2).

Such a labeling is demonstrated on $W_7$ at Figure 2.
Remark. The above labeling of $W_n$ can be obtained from the minimum labeling of $W_n$, described at theorem 2.1., by adding $n$ to the label of every rim edge and subtracting $n$ from every label of a spoke edge.

3. Discussion

We saw that for any odd natural number $n \geq 3$,

$$\min(EMP(W_n)) = \left\lceil \frac{n + 1}{2} + 2n + 1 \right\rceil, \quad \max(EMP(W_n)) = \left\lfloor \frac{3n + 1}{2} + 2n + 1 \right\rfloor$$

The question is whether these formulas are valid also in the case of even numbers. The following figures shows that it is valid at least for all $n \leq 8$. 

Figure 2. max. labeling of $W_7$

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Figure 4. min. labeling of $W_6$ max. labeling of $W_6$

Figure 5. min. labeling of $W_8$ max. labeling of $W_8$

References


