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WHEEL AS AN EDGE-MAGIC GRAPH

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Abstract: An image of a plane graph, $G = (V, E)$ of order n and size m, is said to be an *edge-magic plane graph* if there is a bijection $f : E \rightarrow \{1, 2, .., m\}$ such that for all $s-side$ faces of G, except the infinite face, the sum of the labels of its edges is a constant $k(s)$. Such a bijection will be called an edge-magic plane labeling of G. In case that all the finite sides of a graph G having the same size we will be interested in determining the minimum and the maximum number, k, such that there exists an edge-magic plane labeling of G, in which k is the sum of the edge labeling of each face. In this paper we find such a minimum and maximum numbers for a wheel with even order. Furthermore we conjecture that the same formula is valid for the odd case.

Key Words: magic graph, plane graph, wheel, edge magic, minimal magic graph, maximal magic graph

1. Introduction

We study undirected graphs without loops or multiple edges. Given a graph G; $V(G), E(G), v(G)$ and $e(G)$ stands for the set of vertices, the set of edges, the order (number of vertices) and the size (number of edges) of G. K_n, and C_n stand for the complete graph and the cycle of order n. For two graphs G and H we denote by $G + H$ the graph obtained from the disjoint union $G \cup H$ by adding all edges between G and H.

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A wheel, W_n, is a graph of order $n+1$ composed of a vertex, which will be called the *hub*, adjacent to all vertices of a cycle of order n. The cycle will be called the *rim* of the wheel, and the edges connecting the hub to the vertices of the rim will be called the *spokes*. i.e., $W_n = C_n + K_1$.

1.1. Total Magic Graphs

There are quite a lot of different definitions for a magic graph. We will point out two of the most popular.

Definition. Let $G = (V, E)$ be a graph of order n and size m. A bijection $f : V \cup E \to \{1, 2, .., n + m\}$ is called a *vertex-magic total labeling* of G, if there exist a constant k, such that,

$$\forall x \in V, \;\; f(x) + \sum_{xy \in E} f(xy) = k.$$

A graph G is called a *total vertex-magic graph* if there exist a vertex-magic total labeling of G.

Definition. Let $G = (V, E)$ be a graph of order n and size m. A bijection $f : V \cup E \to \{1, 2, .., n + m\}$ is called an *edge-magic total labeling* of G, if there exist a constant k, such that,

$$\forall xy \in E, \;\; f(x) + f(y) + f(xy) = k.$$

A graph G is called a *total edge-magic graph* if there exist an edge-magic total labeling of G.

It was proven in [5] that:

Theorem 1.1.1. *W_n has vertex-magic total labeling iff $n \leq 11$.*

It was proven in [3] that:

Theorem 1.1.2. *W_n is not total edge-magic $\forall n \equiv 3mod4$.*

It was conjectured at [3], but not yet proven, that W_n is total edge-magic whenever $n \equiv 3/mod4$.

1.2. Plane Magic Graphs

Koh wei lih defined in [4] the notions of magic labellings of a plane graph. In this paper, we will use the term *edge-magic plane graph* for what was defined as edge-magic graph in [4], to differ it from other definitions of edge-magic graph.

Definition. Let G be a plane graph of size m. A bijection $f : E(G) \to \{1, 2, ., m\}$ is called *edge-magic plane labeling* if the sum of the edge labels surrounding each s-sided face of G is a constant.

Definition. A plane graph G is called *edge-magic-plane graph* if there exist an edge-magic plane labeling of G.

Definition. Let G be a plane graph such that all its bounded faces having the same size. G will be called *k-edge-magic plane graph* if there exist an edge-magic plane labeling of G, such that the sum of labels surrounding each face of G is k.

Notation. For a plane graph G, such that all its bounded faces having the same size, we denote by $EMP(G)$ the set of natural numbers, k, such that G is *k-edge-magic plane graph.*

On this paper we will find $min(EMP(W_n))$ and $max(EMP((W_n))$ for all odd natural number n.

2. Labeling of wheels

Let $(a_1, \ldots, a_n)$ be the labeling of the spokes and $(b_1, \ldots, b_n)$ the labeling of the rim edges of W_n, such that the sum of the labels on each face of the wheel is k. Since each spoke belongs to two faces and each rim edge belongs to only one face, we conclude that:

$$2\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = nk.$$

Therefore:

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{2n} i = \sum_{i=1}^{n} a_i + (2n+1)n = nk \tag{1}$$

Furthermore, from (1) and the following inequality

$$\frac{(n+1)n}{2} = \sum_{i=1}^{n} i \leq \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} n + i = \frac{(3n+1)n}{2}$$

we can derive that

$$\lceil \frac{n+1}{2} + 2n + 1 \rceil \leq k \leq \lfloor \frac{3n+1}{2} + 2n + 1 \rfloor \tag{2}$$

Theorem 2.1. *For any odd natural number $n \geq 3$,*

$$min(EMP(W_n)) = \frac{n+1}{2} + 2n + 1.$$

Proof. Let m be the natural number, such that $n = 2m + 1$.

Let v_0 be the hub vertex and $(v_1, \ldots, v_n)$ be the rim vertices, ordered counter clockwise.

Set $f : E(W_n) \to \{1, \ldots, 2n\}$ be the function which admits the rule:

$$f(v_i, v_j) = n + i \;\; ; \;\; \forall i \, ; \; 1 \leq i \leq n - 1$$

$$f(v_n, v_1) = 2n$$

$$f(v_0, v_{n-2k}) = k + 1 \;\; ; \;\; \forall k \, ; \; 0 \leq k \leq m$$

$$f(v_0, v_{2k}) = n - k + 1 \;\; ; \;\; \forall k \, ; \; 1 \leq k \leq m$$

Then:

a. $\forall k \, ; \; 1 \leq k \leq m \, ; \;\; f(v_0, v_{n-2k}) + f(v_0, v_{n-2k+1}) + f(v_{n-2k}, v_{n-2k+1})$

$$= (k+1) + [n - (m - k + 1) + 1] + (2n - 2k) = 2n + \frac{n+1}{2} + 1.$$

b. $\forall k \, ; \; 0 \leq k \leq m - 1 \, ; \;\; f(v_0, v_{n-2k-1}) + f(v_0, v_{n-2k}) + f(v_{n-2k-1}, v_{n-2k})$

$$= [n - (m - k) + 1] + (k + 1) + (2n - 2k - 1) = 2n + \frac{n+1}{2} + 1$$

c. $f(v_0, v_n) + f(v_0, v_1) + f(v_n, v_1) = 1 + (m + 1) + 2n = 2n + \frac{n+1}{2} + 1.$

Thus, the assertion is derived from (2).

Such a labeling is demonstrated on W_7 at Figure 1.

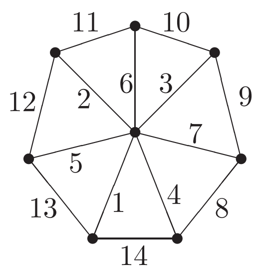

Figure 1. min. labeling of W_7

Remark. The above labeling of W_n can be described as followed. We label the rim edges by $(n+1, n+2, \ldots, 2n)$ counter clockwise. Then we label the spoke between the rim edges labeled by $2n-1$ and $2n$ by 1 and we label all other spokes edges by $2, 3, \ldots, n$, clockwise skipping one edge every time.

Theorem 2.2. *For any odd natural number $n \geq 3$,*

$$max(EMP(W_n)) = \frac{3n+1}{2} + 2n + 1.$$

Proof. Let m be the natural number such that $n = 2m+1$.

Let v_0 be the hub vertex and $(v_1, \ldots, v_n)$ be the rim vertices, ordered counter clockwise.

Set $f : E(W_n) \to \{1, \ldots, 2n\}$ be the function which admits the rule:

$$f(v_i, v_j) = i \ ; \ \forall i \, ; \ 1 \leq i \leq n-1$$

$$f(v_n, v_1) = n$$

$$f(v_0, v_{n-2k}) = n + k + 1 \ ; \ \forall k \, ; \ 0 \leq k \leq m$$

$$f(v_0, v_{2k}) = 2n - k + 1 \ ; \ \forall k \, ; \ 1 \leq k \leq m$$

Then:

a. $\forall k \, ; \ 1 \leq k \leq m \, ; \ f(v_0, v_{n-2k}) + f(v_0, v_{n-2k+1}) + f(v_{n-2k}, v_{n-2k+1})$

$$= (n+k+1) + [2n - (m-k+1) + 1] + (n-2k) = \frac{3n+1}{2} + 2n + 1.$$

b. $\forall k \, ; \ 0 \leq k \leq m-1 \, ; \ f(v_0, v_{n-2k-1}) + f(v_0, v_{n-2k}) + f(v_{n-2k-1}, v_{n-2k})$

$$= [2n - (m-k) + 1] + (n+k+1) + (n-2k-1) = \frac{3n+1}{2} + 2n + 1.$$

c. $f(v_0, v_n) + f(v_0, v_1) + f(v_n, v_1) = (n+1) + (n+m+1) + n$

$$= \frac{3n+1}{2} + 2n + 1.$$

Thus, the assertion is derived from (2).

Such a labeling is demonstrated on W_7 at Figure 2.

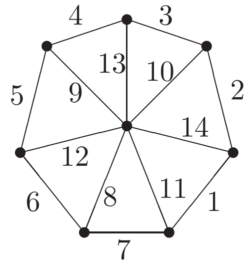

Figure 2. max. labeling of W_7

Remark. The above labeling of W_n can be obtained from the minimum labeling of W_n, described at theorem 2.1., by adding n to the label of every rim edge and subtracting n from every label of a spoke edge.

3. Discussion

We saw that for any odd natural number $n \geq 3$,

$$min(EMP(W_n)) = \lceil \frac{n+1}{2} + 2n + 1 \rceil \text{ , } max(EMP(W_n)) = \lfloor \frac{3n+1}{2} + 2n + 1 \rfloor$$

The question is whether these formulas are valid also in the case of even numbers. the following figures shows that it is valid at least for all $n \leq 8$.

Figure 3. min. labeling of W_4 max. labeling of W_4

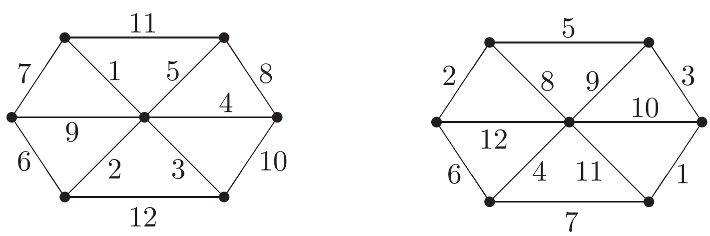

Figure 4. min. labeling of W_6 max. labeling of W_6

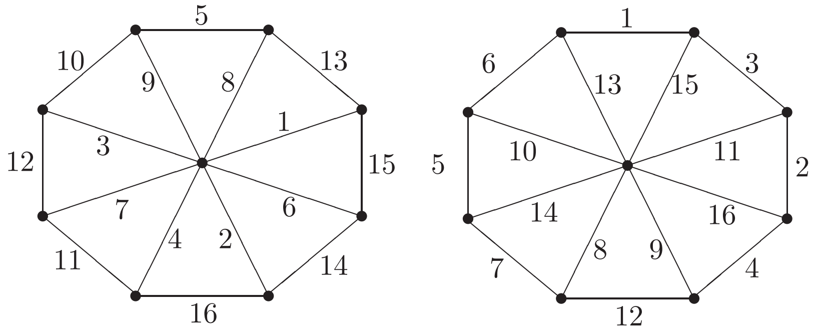

Figure 5. min. labeling of W_8 max. labeling of W_8

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