**International Journal of Pure and Applied Mathematics**
**Volume 89 No. 4 2013, 553-558**
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
**url:** http://www.ijpam.eu
**doi:** http://dx.doi.org/10.12732/ijpam.v89i4.9

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# CO-IDEALS IN TERNARY SEMIGROUPS

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**Abstract:** In this page we study properties of co-ideal in the ternary semigroup $\langle \mathbb{Z}_0^+, \cdot \rangle$.

**Key Words:** ternary semigroup, ternary semiring, co-ideal, subtractive co-ideal

## 1. Introduction

In [1]. Cayley and Sylvester along with several other mathematicians, in the 19th century considered ternary algebraic structures and cubic relations. The n-ary structures, which are the generalizations of ternary structures create hopes because of their possible applications in Physics. A few important physical applications have been recorded in Ternary semigroups exhibit natural examples of ternary algebras. Also S. Kar [7] studied the ideal theory in the ternary semiring $\langle \mathbb{Z}_0^-, \cdot \rangle$.

In this paper, we introduce the concept of a co-ideal and study properties of a co-ideals of ternary semigroups $\langle \mathbb{Z}_0^+, \cdot \rangle$.

Received: August 25, 2013

## 2. Preliminaries

**Definition 2.1.** [7, p.192] A non-empty set $S$ together with a ternary operation, called ternary multiplication, denoted by juxtaposition, is said to be a *ternary semigroup* if $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in S$.

**Definition 2.2.** [7, p.192] An element $e$ of a ternary semigroup $S$ is called

(1) a left identity (or left unital element) if $eex = x$ for all $x \in S$,

(2) a right identity (or right unital element) if $xee = x$ for all $x \in S$,

(3) a lateral identity (or lateral unital element) if $exe = x$ for all $x \in S$,

(4) a two-sided identity (or bi-unital element) if $eex = xee = x$ for all $x \in S$,

(5) an identity (or unital element) if $eex = exe = xee = x$ for all $x \in S$.

**Example 2.3.** Let $\langle \mathbb{Z}_0^+, \cdot \rangle$ be the set of all non-positive integers. Then with the usual ternary multiplication. $\langle \mathbb{Z}_0^+, \cdot \rangle$ forms a ternary semigroup with zero element 0 and identity element 1

**Definition 2.4.** [7, p.192] A non-empty subset $T$ of a ternary semigroup $S$ is called a *ternary subsemigroup* if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

**Definition 2.5.** [7, p.192] A ternary subsemigroup $I$ of a ternary semigroup $S$ is called

(1) a left ideal of $S$ if $SSI \subseteq I$,

(2) a right ideal of $S$ if $ISS \subseteq I$,

(3) a lateral ideal of $S$ if $SIS \subseteq I$,

(4) a two-sided ideal of $S$ if $I$ is both left and right ideal of $S$,

(5) an ideal of $S$ if $I$ is a left, a right, a lateral ideal of $S$.

An ideal $I$ of a ternary semigroup $S$ is called a proper ideal if $I \neq S$.

## 3. Co-Ideals in Ternary Semigroup

We define co-ideal of a ternary semigroup, and investigate it properties.

**Definition 3.1.** A non-empty subset $I$ of a ternary semigroup $S$ is called a *co-ideal* if

(1) $a, b, c \in I$ implies $abc \in I$,

(2) $a \in I, s \in S$ implies $as \in I$.

**Definition 3.2.** A co-ideal $I$ of a ternary semigroup $S$ is called *subtractive* if $a, b, abc \in I, c \in S$, then $c \in I$

**Theorem 3.3.** *Let $I, J$ be a co-ideal of a ternary semigroup $S$. Then*

*(1) $I \cap J$ is a co-ideal of $S$,*

*(2) $IJ$ is a co-ideal of $S$,*

*(3) $IJ \subseteq I \cap J$.*

*Proof.* (1) Let $a, b, c \in I \cap J$ then $abc \in I$, since $I, J$ is a co-ideal and so $abc \in J$. Hence $abc \in I \cap J$.

Let $a \in I \cap J$ and $r \in S$ then $as \in I$ and $as \in J$, since $I, J$ is a co-ideal of $S$. Thus $as \in I \cap J$. Hence $I \cap J$ is a co-ideal of $S$.

(2) Let $x = ab, y = a'b', z = a^*b^* \in IJ$. Then

$$\begin{aligned}
xyz &= (ab)(a'b')(a^*b^*) \\
&= [(aa')(ab')(ba')(bb')](a^*b^*) \\
&= ([(aa')(a^*b^*)][(ab')(a^*b^*)][(ba')(a^*b^*)][(bb')(a^*b^*)]) \\
&= [(aa^*)(ab^*)(a'a^*)(a'b^*)][(aa^*)(ab^*)(b'a^*)(b'b^*)] \\
&\quad [(ba^*)(bb^*)(a'a^*)(a'b^*)][(ba^*)(bb^*)(b'a^*)(b'b^*)] \\
&= [(aa^*)(ab^*)(a'a^*)(a'b^*)(aa^*)(ab^*)(b'a^*)(b'b^*) \\
&\quad (ba^*)(bb^*)(a'a^*)(a'b^*)(ba^*)(bb^*)(b'a^*)(b'b^*)] \\
&= (aa^*)^2(ab^*)^2a'(a^*b^*)b'(a^*b^*)b(a^*b^*)a'(a^*b^*)b(a^*b^*)b'(a^*b^*) \\
&= (aa^*)^2(ab^*)^2a'b'ba'bb'(a^*b^*) \\
&= (aa^*)^2(ab^*)^2a'^2b^2b'^2(a^*b^*).
\end{aligned}$$

Since $(a^*b^*) \in IJ$ we have $xyz \in IJ$.

Let $r \in S$ then $xr = (ab)r \in I$, since $I$ is a co-ideal of $S$. Hence $IJ$ is a co-ideal of $S$.

(3) Let $x = ab \in IJ$ where $a \in I, b \in J$. Now $a \in I, b \in S$ and $I$ is a co-ideal implies $x = ab \in I$. Similarly $x \in J$. Hence $x \in I \cap J$ so $IJ \subseteq I \cap J$.

□

**Corollary 3.4.** *Every ideal is a co-ideal.*

*Proof.* Let $I$ is an ideal of ternary semigroup of $S$ and $a, b, c \in I$. Then $abc \in I$. Let $a \in I$ and $s \in S$ then $as \in I$, since $I$ is an ideal of $S$. Thus $I$ is a co-ideal of $S$. □

**Theorem 3.5.** *Let $I, J$ be a subtractive co-ideal of a ternary semigroup $S$. Then $I \cap J$ is a subtractive co-ideal of $S$.*

*Proof.* By Theorem 3.3(1) we have $I \cap J$ is a co-ideal of $S$. If $a, b, abc \in I \cap J$, $c \in S$ then $c \in I \cap J$, since $I, J$ is a subtractive co-ideal of $S$. □

**Lemma 3.6.** *Let $I, J$ be a subtractive co-ideal of a ternary semigroup $S$ and $a \in I, b \in J, c \in I \cap J$. Then the following conditions are equivalent*

*(1) $abc \in I \cap J$,*

*(2) $a \in I \cap J$ or $b \in I \cap J$,*

*(3) $a, b \in I$ or $a, b \in J$.*

*Proof.* (1) $\Rightarrow$ (2) Let $abc \in I \cap J$. Without loss of generality assume that $abc \in I$. Since $I$ is a subtractive co-ideal we have $b \in I$. Hence $b \in I \cap J$.

(2) $\Rightarrow$ (3) Let $a \in I \cap J$ or $b \in I \cap J$. We must to show that $a, b \in I$ or $a, b \in J$

Case 1. If $a \in I \cap J$ then $a \in I$ and $a \in J$

Case 2. If $b \in I \cap J$. then $b \in I$ and $b \in J$

From case1.,2. we have $a, b \in I$ or $a, b \in J$.

(3) $\Rightarrow$ (1) Let $a, b \in I$ or $a, b \in J$ then $ab \in I$ or $ab \in J$. Since $I, J$ is a subtractive co-ideal and $c \in I \cap J$ we have $abc \in I \cap J$. □

**Theorem 3.7.** *Let $I, J$ be subtractive co-ideals of a ternary semigroup $S$ and $A$ be ideal of $S$ such that $A \cap I \cap J \neq$. Then $A \subseteq I \cap J$ if and only if $A \subseteq I$ or $A \subseteq J$*

*Proof.* Let $c \in A \cap i \cap J$. Let $A \subseteq I \cap J$ and $A \nsubseteq I$. Choose $a \in A$ such that $a \notin I$. Then $a \in J$. We claim that $A \cap I \subseteq J$. Let $b \in A \cap I$. Now $a, b, c \in A$ and $A$ is a co-ideal implies that $abc \in A \subseteq I \cap J$. By Lemma 3.6 $a, b, \in I$ or $a, b \in J$. Then $a, b \in J$, since $a \notin I$. Hence $A \cap I \subseteq J$. Now $A = A \cap (I \cap J) = (A \cap I) \cap (A \cap J) \subseteq J \cap J = J$. Converse is trivial. □

**Corollary 3.8.** *Let $I, J$ be subtractive co-ideals of a ternary semigroup $S$ such that $I \cap J \neq$. Then $I \cap J$ is co-ideal of $S$ if and only if $I \subseteq J$ or $J \subseteq I$. if and only if $A \subseteq I$ or $A \subseteq J$*

*Proof.* Let $I \cap J$ is a co-ideal of $S$. Now $I \cap J \subseteq I \cap J$ and hence by Theorem 3.7, $I \cap J \subseteq J$ or $I \cap J \subseteq J$. Then $A \subseteq I$ or $A \subseteq J$.

Conversely let $A \subseteq I$ or $A \subseteq J$ to show that $I \cap J$ is a co-ideal. Let $a, b, c \in I \cap J$ then $abc \in I \cap J$, since $I, J$ is a subtractive co-ideals. Let $a \in I \cap J$ and $s \in S$ by Theorem 3.5, $I \cap J$ is a subtractive. Thus $ar \in I \cap J$. Hence $I \cap J$ is a co-ideal. □

## 4. Co-Ideals in Ternary Semigroup $\langle \mathbb{Z}_0^+, \cdot \rangle$

We study properties of co-ideal in ternary semigroup $\mathbb{Z}_0^+$

**Lemma 4.1.** $I_n = \{a \in \mathbb{Z}_0^+ : a \leq n\}$ *is a co-ideal.*

*Proof.* Let $a, b, c \in I_n$ then $a, b, c \in \mathbb{Z}_0^+$. Thus $abc \in \mathbb{Z}_0^+$ so $abc \in I_n$.

Let $a \in I_n$ and $r \in S$ then $ar \in \mathbb{Z}_0^+$. Thus $ar \in I_n$. Hence $I_n$ is a co-ideal of $\mathbb{Z}_0^+$

□

**Theorem 4.2.** *A non-empty subset $I$ of the ternary semigroup $\langle \mathbb{Z}_0^+, \cdot \rangle$ is a co-ideal if and only if $I = I_n$*

*Proof.* ($\Rightarrow$) Assume that $I$ is a co-ideal of $\langle \mathbb{Z}_0^+, \cdot \rangle$. We must show that $I = I_n$ Since $I$ is a non-empty, $I$ has the largest element say $n$. Let $x \in I_n$ then $x \leq n$ and $x = ny$ for some $y \in \mathbb{Z}^+$. Now $x \in I$, since $I$ is a co-ideal. Hence $I_n \subseteq I$. But $I \subseteq I_n$. So $I = I_n$.

($\Leftarrow$) Assume that $I = I_n$ by Lemma4.1 then $I_n$ is a co-ideal of $\mathbb{Z}_0^+$. Thus $I$ is a co-ideal of $\langle \mathbb{Z}_0^+, \cdot \rangle$. □

## Acknowledgments

The author is very grateful to the anonymous referee for stimulating comments and improving presentation of the paper.

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