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LYAPUNOV CONVERGENCE ANALYSIS FOR ASYMPTOTIC TRACKING USING FORWARD AND BACKWARD EULER APPROXIMATION OF DISCRETE DIFFERENTIAL EQUATIONS

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Abstract: This paper proposes an analysis of the convergence of discrete differential equations obtained by Euler approximation methods. Backward and Feed-forward Euler approximations are considered. These kinds of methods are very often used in discretisation of continuous models because of their straightforward structure which allows an easy implementation in microprocessor applications. These two kinds of discretisations are very important in the representation of controllers in which the use of a fast algorithm of its discrete representation is a basic condition for the whole stability of the closed loop control structure.

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1. Introduction

The topic of tracking trajectories is known in the literature. Works as [1] and

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[2] define a perfect tracking problem. Using the assumption that the system is flat proposed possible solutions. In [3] an algorithm for tracking trajectory is developed basically using a flatness approach. In the presented work, no structural assumptions on the system are used except the affinity of the nonlinear system to obtain an asymptotic tracking. In practical application the affinity hypothesis is a no conservative one. In fact, most of the applications present an affine structure. In [4] a practical application shows the importance of the asymptotic tracking problem. This contribution, starting from the contribution in [5] analyzes Backward and Feed-forward Euler approximations. These kinds of methods are very often used in discretisation of continuous models because of their straightforward structure which allows an easy implementation in microprocessor applications. These two kinds of discretisations are very important in the representation of controllers in which the use of a fast algorithm of its discrete representation is a basic condition for the whole stability of the closed loop control structure. The paper is organized in the following way. Section 1.1 presents the problem formulation together with a possible solution using Lyapunov approach as in [5]. The main contribution in this paper is represented by Section 2 in which an analysis of the Backward and Feed-forward Euler approximations is shown. Possible solutions in the case of Backward and Feed-forward Euler approximations are also shown. The Section of the conclusion closes the paper.

1.1. Problem Formulation

Problem 1. Let define the following affine nonlinear system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}), \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, and $\mathbf{y}(t) \in \mathbb{R}^p$, with $n, m, p \in \mathbb{N}$. If a state vector trajectory $\mathbf{x}_d(t)$ is given to be tracked, find a possible general $\mathbf{u}(t)$ which realizes the tracking for the system described by (1) and (2). □

In order to propose a possible solution of Problem 1, the following field is defined:

$$\mathbf{K}(t) = \mathbf{G}\big(\mathbf{x}_d(t) - \mathbf{x}(t)\big), \tag{3}$$

where $\mathbf{G} = \begin{bmatrix} \lambda_{11} & 0 & ...0_{1n} \\ 0 & \lambda_{2n} & 0 \\ ... & ... & ... \\ ... & ... & ... \\ ... & ... & ... \\ 0 & 0 & \lambda_{mn} \end{bmatrix}$ is a diagonal matrix with $\lambda_{i,j} \in \mathbb{R}$ and $i = 1, 2, ..., p,\ j, = 1, 2, ..., n$; variable $\mathbf{x}_d(t)$ represents the vector of the desired trajectories. If the following Lyapunov function is defined:

$$\mathbf{V}(\mathbf{K}_i) = \frac{\mathbf{K}_i^2(t)}{2}, \tag{4}$$

then it follows that:

$$\dot{\mathbf{V}}(\mathbf{K}_i) = \mathbf{K}_i(t)\dot{\mathbf{K}}_i(t). \tag{5}$$

In order to find the stability of the solution, it is possible to choose the following functions:

$$\dot{\mathbf{V}}(\mathbf{K}_i) = -\mathbf{M}_d\mathbf{K}_i{}^2(t), \tag{6}$$

where $\mathbf{M}_d$ indicates a diagonal positive definite matrix consisting of parameter $\eta_i$which is responsible for the velocity of the tracking convergence. $\mathbf{K}_i$ represents each component of the vector field $\mathbf{K}$, with $i = 1, 2, ..., p$. As above mentioned, p is the number of outputs of the system defined in (1). Comparing (5) with (6), the following relation is obtained:

$$\mathbf{K}_i(t)\dot{\mathbf{K}}_i(t) = -\mathbf{M}_d\mathbf{K}_i^2(t), \tag{7}$$

and finally

$$\mathbf{K}_i(t)\big(\dot{\mathbf{K}}_i(t) + \mathbf{M}_d\mathbf{K}_i(t)\big) = 0. \tag{8}$$

The no trivial solution follows from the condition

$$\dot{\mathbf{K}}_i(t) + \mathbf{M}_d\mathbf{K}_i(t) = 0. \tag{9}$$

From (3) it follows:

$$\dot{\mathbf{K}}_i(t) = \mathbf{G}\big(\dot{\mathbf{x}}_d(t) - \dot{\mathbf{x}}(t)\big) = \mathbf{G}\dot{\mathbf{x}}_d(t) - \mathbf{G}\dot{\mathbf{x}}(t). \tag{10}$$

The main idea is to find a $\mathbf{u}_{eq}(t)$, an equivalent input, and after that a $\mathbf{u}(t)$, such that $\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_d(t)$. For that, from (1) it follows that:

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_d(t) = \mathbf{f}(\mathbf{x}_d(t)) + \mathbf{g}(\mathbf{x}_d(t))\mathbf{u}(t), \tag{11}$$

and from (10) the following relation is obtained:

$$\dot{\mathbf{K}}_i(t) = \mathbf{G}\dot{\mathbf{x}}_d(t) - \mathbf{G}\mathbf{f}(x_d(t)) - \mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\mathbf{u}(t) = \mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big(\mathbf{u}_{eq}(t) - \mathbf{u}(t)\big), \quad (12)$$

where $\mathbf{u}_{eq}(t)$ is the equivalent input which, in our case, assumes the following expression:

$$\mathbf{u}_{eq}(t) = \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big)^{-1}\mathbf{G}\Big(\dot{\mathbf{x}}_d(t) - \mathbf{f}(x_d(t))\Big). \quad (13)$$

After inserting (12) into (9), the following relation is obtained:

$$\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big(\mathbf{u}_{eq}(t) - \mathbf{u}(t)\big) + \mathbf{M}_d\mathbf{K}_i(t) = 0, \quad (14)$$

and in particular:

$$\mathbf{u}(t) = \mathbf{u}_{eq}(t) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(t))\big)^{-1}\mathbf{M}_d\mathbf{K}_i(t). \quad (15)$$

Normally, it is a difficult job to calculate $\mathbf{u}_{eq}(t)$.

2. Forward and Backward Euler Approximations

If equation (12) is rewritten in a discrete form using Forward Euler approximation, then it follows:

$$\frac{\mathbf{K}_i((k+1)T_s) - \mathbf{K}_i(kT_s)}{T_s} = \mathbf{G}\mathbf{g}(\mathbf{x}_d(k))\big(\mathbf{u}_{eq}(kT_s) - \mathbf{u}(kT_s)\big). \quad (16)$$

Parameter $T_s \in \mathbb{N}$ and represents the sampling time of the discretisation. If equation (15) is also rewritten in a discrete form, then:

$$\mathbf{u}(kT_s) = \mathbf{u}_{eq}(kT_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(k))\big)^{-1}\mathbf{M}_d\mathbf{K}_i(kT_s). \quad (17)$$

Equation (16) can be also rewritten as:

$$\mathbf{u}_{eq}(kT_s) = \mathbf{u}(kT_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(k))\big)^{-1}\frac{\mathbf{K}_i((k+1)T_s) - \mathbf{K}_i(kT_s)}{T_s}. \quad (18)$$

Equation (18) can be estimated to one-step backward in the following way:

$$\begin{aligned}\mathbf{u}_{eq}((k-1)T_s) &= \mathbf{u}((k-1)T_s) \\ &\quad + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(k-1))\big)^{-1}\frac{\mathbf{K}_i(kT_s) - \mathbf{K}_i((k-1)T_s)}{T_s}. \quad (19)\end{aligned}$$

Because of function $\mathbf{u}_{eq}(t)$ being a continuous one, we can write:

$$\mathbf{u}_{eq}(kT_s) \approx \mathbf{u}_{eq}((k-1)T_s). \tag{20}$$

Considering equation (20), then equation (19) becomes:

$$\mathbf{u}_{eq}(kT_s) = \mathbf{u}((k-1)T_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(k-1))\big)^{-1} \frac{\mathbf{K}_i(kT_s) - \mathbf{K}_i((k-1)T_s)}{T_s}. \tag{21}$$

Inserting (21) into (17) it follows:

$$\mathbf{u}(kT_s) = \mathbf{u}((k-1)T_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(k-1))\big)^{-1} \Big(\mathbf{M}_d\mathbf{K}_i(kT_s) + \frac{\mathbf{K}_i(kT_s) - \mathbf{K}_i((k-1)T_s)}{T_s}\Big), \tag{22}$$

and finally the Forward Euler solution is the following:

$$\mathbf{u}(kT_s) = \mathbf{u}((k-1)T_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(k-1))T_s\big)^{-1} \Big(\mathbf{M}_d T_s\mathbf{K}_i(kT_s) + \mathbf{K}_i(kT_s) - \mathbf{K}_i((k-1)T_s)\Big). \tag{23}$$

If equation (12) is rewritten in a discrete form using Backward Euler approximation, then it follows:

$$\frac{\mathbf{K}_i((k+1)T_s) - \mathbf{K}_i(kT_s)}{T_s} = \mathbf{G}\mathbf{g}(\mathbf{x}_d(k+1))\big(\mathbf{u}_{eq}((k+1)T_s) - \mathbf{u}((k+1)T_s)\big). \tag{24}$$

If equation (24) is estimated to one-steps backward, the Backward Euler solution assumes the following structure:

$$\mathbf{u}(kT_s) = \mathbf{u}((k)T_s) + \big(\mathbf{G}\mathbf{g}(\mathbf{x}_d(k))T_s\big)^{-1} \Big(\mathbf{M}_d T_s\mathbf{K}_i(kT_s) + \mathbf{K}_i(kT_s) - \mathbf{K}_i((k-1)T_s)\Big). \tag{25}$$

Remark 1. Equation (25) states the solution of the proposed problem. This solution depends on different aspects: parameters T_s, λ_{ij} and matrix $\mathbf{M}_d$. Parameter T_s is responsible for the validity of assumption stated in (20) which the proposed solution is based on. In fact, the convergence of the result is shown just for the continuous case mathematically and through the assumption of equation (20) the result is extended to the discrete case. If the dynamics of the considered system are very high (very fast dynamics), then parameter T_s must be chosen very small in order to guarantee assumption (20) and thus the convergence of the proposed solution. □

Considering equations (8) and (9) the following relation can be written:

$$\mathbf{G}(\dot{\mathbf{x}}_d(t) - \dot{\mathbf{x}}(t)) + \mathbf{M}_d\mathbf{G}(\mathbf{x}_d(t) - \mathbf{x}(t)) = 0. \tag{26}$$

Considering the Forward Euler sampling approximation, equation (26) becomes:

$$\begin{aligned}\mathbf{K}(kT_s) - \mathbf{K}((k-1)T_s) + T_s\mathbf{M}_b\mathbf{K}((k-1)T_s)\\ = T_s\mathbf{\Delta}(\mathbf{x}_d((k-1)T_s), \mathbf{x}((k-1)T_s)),\end{aligned} \tag{27}$$

where T_s equals the sampling time and term $\Delta(x_d((k-1)T_s), x((k-1)T_s))$ represents a residual term due to the discretisation approximation and to assumption in (20). It is well known that in order to obtain the asymptotic stability, diagonal matrix $\mathbf{M}_d$ must be such that $\eta_i < 2/T_s$ $\forall i$. So parameters η_i cannot be used to minimize the approximation error. In fact, we can write the following relation:

$$\mathbf{K}(kT_s) = (\mathbf{I} - T_s\mathbf{M}_d)\mathbf{K}((k-1)T_s) + T_s\mathbf{\Delta}(\mathbf{x}_d(k-1), \mathbf{x}(k-1)). \tag{28}$$

If Backward Euler sampling approximation is considered, then equation (26) becomes:

$$\mathbf{K}(kT_s) - \mathbf{K}((k-1)T_s) + T_s\mathbf{M}_d\mathbf{K}(kT_s) = T_s\mathbf{\Delta}(\mathbf{x}_d(kT_s), \mathbf{x}(kT_s)), \tag{29}$$

and in case of no exact cancelation, the bigger parameters η_i are, the smaller the error becomes. In fact, equation (29) can be writen in the following way:

$$\begin{aligned}\mathbf{K}(kT_s) = (\mathbf{I} + T_s\mathbf{M}_d)^{-1}\mathbf{K}((k-1T_s))+\\ (\mathbf{I} + T_s\mathbf{M}_d)^{-1}T_s\mathbf{\Delta}(\mathbf{x}_d((k-1)T_s), \mathbf{x}((k-1)T_s).\end{aligned} \tag{30}$$

3. Conclusions

This paper deals with a general tracking problem for affine nonlinear systems in discrete domain. Forward and Backward Euler discretisation methods are analyzed and their possible solutions are calculated. These approaches and their proposed solutions are quite general.

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