International Journal of Pure and Applied Mathematics
Volume 89 No. 5 2013, 659-665
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v89i5.2

ijpam.eu

CUBIC KU-SUBALGEBRAS

Muhammad Akram[1 §], Naveed Yaqoob[2], Muhammad Gulistan[3]

[1]Department of Mathematics
University of Gujrat
Gujrat, PAKISTAN

[2]Department of Mathematics
COMSATS Institute of Information Technology
Abbottabad, PAKISTAN

[3]Department of Mathematics
Hazara University
Mansehra, PAKISTAN

Abstract: In this paper, we study some new properties of cubic KU-subalgebras.

AMS Subject Classification: 06F35, 03G25, 08A72
Key Words: KU-algebras, cubic sets, cubic KU-subalgebras

1. Introduction

Prabpayak and Leerawat [3] introduced a new algebraic structure which is called KU-algebra. They introduced the concept of homomorphisms of KU-algebras and investigated some related properties in [4]. Mostafa et al. [2] introduced the notion of fuzzy KU-ideals of KU-algebras.

Jun et al. [1] introduced the notion of cubic sub-algebras and ideals in BCK/BCI-algebras. They discussed relationship between a cubic subalgebra and a cubic ideal.

Received: June 27, 2013

§Correspondence author

In this paper, we provided some new properties of cubic KU-subalgebras.

2. Preliminaries and Basic Definitions

In this section, we will recall some definitions of KU-algebra and cubic sets.

Definition 1. By a KU-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ with a single binary operation $*$ that satisfies the following identities: for any $x, y, z \in X$,

(ku1) $(x * y) * [(y * z) * (x * z)] = 0$,

(ku2) $x * 0 = 0$,

(ku3) $0 * x = x$,

(ku4) $x * y = 0 = y * x$ implies $x = y$.

In a KU-algebra, the identity $z * z = 0$ is true [2].

Definition 2. [4] A subset S of KU-algebra X is called a KU-subalgebra of X if $x * y \in S$, whenever $x, y \in S$.

An interval number is $\widetilde{a} = [a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$, i.e.,

$$D[0,1] = \{\widetilde{a} = [a^-, a^+] : a^- \leq a^+, \text{ for } a^-, a^+ \in I\}.$$

We define the operations "$\succeq$", "$\preceq$", "$=$", "rmin" and "rmax" in case of two elements in $D[0,1]$. We consider two elements $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ in $D[0,1]$. Then:

(1) $\widetilde{a} \succeq \widetilde{b}$ if and only if $a^- \geq b^-$ and $a^+ \geq b^+$,

(2) $\widetilde{a} \preceq \widetilde{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,

(3) $\text{rmin}\{\widetilde{a}, \widetilde{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$.

Definition 3. [1] Cubic sets defined on a non-empty set X as object having the form $\Xi = \left\{\left\langle x, \widetilde{\delta}_\Xi(x), \theta_\Xi(x)\right\rangle : x \in X\right\}$, which is briefly denoted by $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$, where the functions $\widetilde{\delta}_\Xi : X \longrightarrow D[0,1]$ and $\theta_\Xi : X \rightarrow [0,1]$.

3. Cubic KU-Subalgebras

In this section, we will discuss some new properties of cubic KU-subalgebras.

Definition 4. Let X be a KU-algebra. A cubic subset $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ in X is called a cubic KU-subalgebra of X if

$$\widetilde{\delta}_\Xi(x * y) \succeq r\min\{\widetilde{\delta}_\Xi(x), \widetilde{\delta}_\Xi(y)\}$$

and

$$\theta_\Xi(x * y) \leq \max\{\theta_\Xi(x), \theta_\Xi(y)\},$$

for all $x, y \in X$.

Proposition 5. *If $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ is a cubic KU-subalgebra of X, then for all $x \in X$, $\widetilde{\delta}_\Xi(0) \succeq \widetilde{\delta}_\Xi(x)$ and $\theta_\Xi(0) \leq \theta_\Xi(x)$.*

Proof. The proof is straightforward. □

Theorem 6. *Let $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ be a cubic KU-subalgebra of X. If there exists a sequence x_n in X such that $\lim_{n\to\infty} \widetilde{\delta}_\Xi(x_n) = [1,1]$ and $\lim_{n\to\infty} \theta_\Xi(x_n) = 0$, then $\widetilde{\delta}_\Xi(0) = [1,1]$ and $\theta_\Xi(0) = 0$.*

Proof. Let $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ be a cubic KU-subalgebra of X. By Proposition 5, $\widetilde{\delta}_\Xi(0) \succeq \widetilde{\delta}_\Xi(x)$ for all $x \in X$, therefore $\widetilde{\delta}_\Xi(0) \succeq \widetilde{\delta}_\Xi(x_n)$ for positive integer n. Consider, $[1,1] \succeq \widetilde{\delta}_\Xi(0) \succeq \lim_{n\to\infty} \widetilde{\delta}_\Xi(x_n) = [1,1]$. Hence, $\widetilde{\delta}_\Xi(0) = [1,1]$.

Also, by Proposition 5, $\theta_\Xi(0) \leq \theta_\Xi(x)$ for all $x \in X$, thus $\theta_\Xi(0) \leq \theta_\Xi(x_n)$ for positive integer n. Now, $0 \leq \theta_\Xi(0) \leq \lim_{n\to\infty} \theta_\Xi(x_n) = 0$. Hence, $\theta_\Xi(0) = 0$. □

For any elements x and y of X, let us write

$$\prod^m x * y \quad \text{for} \quad (\ldots((x * x) * x)\ldots) * y,$$

where x appears m times. Here m is a natural number.

Theorem 7. *Let $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ be a cubic KU-subalgebra of X and let $m \in N$ (N be the set of natural numbers). Then*

(i) $\widetilde{\delta}_\Xi\left(\prod^m x * x\right) \succeq \widetilde{\delta}_\Xi(x)$ and $\theta_\Xi\left(\prod^m x * x\right) \leq \theta_\Xi(x)$, for any odd number m,

(ii) $\widetilde{\delta}_\Xi\left(\prod^m x * x\right) = \widetilde{\delta}_\Xi(x)$ and $\theta_\Xi\left(\prod^m x * x\right) = \theta_\Xi(x)$, for any even number m.

Proof. (i) Let $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ be a cubic KU-subalgebra of X and let $x \in X$. Assume that m is odd. Then $m = 2q-1$ for some positive integer q. Now we prove the theorem by induction. Now by Proposition 5, $\widetilde{\delta}_\Xi(x * x) = \widetilde{\delta}_\Xi(0) \succeq \widetilde{\delta}_\Xi(x)$ and $\theta_\Xi(x * x) = \theta_\Xi(0) \leq \theta_\Xi(x)$. Suppose that $\widetilde{\delta}_\Xi\left(\prod^{2q-1} x * x\right) \succeq \widetilde{\delta}_\Xi(x)$ and $\theta_\Xi\left(\prod^{2q-1} x * x\right) \leq \theta_\Xi(x)$. Then by assumption,

$$\begin{aligned}
\widetilde{\delta}_\Xi\left(\prod^{2(q+1)-1} x * x\right) &= \widetilde{\delta}_\Xi\left(\prod^{2q+1} x * x\right) \\
&= \widetilde{\delta}_\Xi\left(\prod^{2q-1} x * ((x * x) * x)\right) \\
&= \widetilde{\delta}_\Xi\left(\prod^{2q-1} x * (0 * x)\right) \\
&= \widetilde{\delta}_\Xi\left(\prod^{2q-1} x * x\right) \succeq \widetilde{\delta}_\Xi(x),
\end{aligned}$$

and

$$\begin{aligned}
\theta_\Xi\left(\prod^{2(q+1)-1} x * x\right) &= \theta_\Xi\left(\prod^{2q+1} x * x\right) \\
&= \theta_\Xi\left(\prod^{2q-1} x * ((x * x) * x)\right) \\
&= \theta_\Xi\left(\prod^{2q-1} x * (0 * x)\right) \\
&= \theta_\Xi\left(\prod^{2q-1} x * x\right) \leq \theta_\Xi(x).
\end{aligned}$$

This completes the proof of (i). The proof of (ii) is similar to (i). □

Theorem 8. *Let $\mathcal{A}$ be a non-empty subset of X and $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ be a cubic subset in X defined by*

$$\widetilde{\delta}_\Xi(x) = \begin{cases} [\beta_1, \beta_2] & \text{if } x \in \mathcal{A} \\ [\gamma_1, \gamma_2] & \textit{otherwise} \end{cases} \quad \text{and} \quad \theta_\Xi(x) = \begin{cases} \vartheta & \text{if } x \in \mathcal{A} \\ \eta & \textit{otherwise} \end{cases}$$

for all $[\beta_1, \beta_2], [\gamma_1, \gamma_2] \in D[0,1]$ *and* $\vartheta, \eta \in [0,1]$ *with* $[\beta_1, \beta_2] \succeq [\gamma_1, \gamma_2]$ *and* $\vartheta \leq \eta$. *Then* $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ *is a cubic KU-subalgebra of* X *if and only if* $\mathcal{A}$ *is a KU-subalgebra of* X.

Proof. Let $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ be a cubic KU-subalgebra of X. Let $x, y \in X$ be such that $x, y \in \mathcal{A}$. Then

$$\begin{aligned} \widetilde{\delta}_\Xi(x * y) &\succeq r\min\{\widetilde{\delta}_\Xi(x), \widetilde{\delta}_\Xi(y)\} \\ &= r\min\{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = [\beta_1, \beta_2] \end{aligned}$$

and

$$\theta_\Xi(x * y) \leq \max\{\theta_\Xi(x), \theta_\Xi(y)\} = \max\{\vartheta, \vartheta\} = \vartheta.$$

So $x * y \in \mathcal{A}$. Hence $\mathcal{A}$ is a KU-subalgebra of X.

Conversely, suppose that $\mathcal{A}$ is a KU-subalgebra of X. Let $x, y \in X$. We have two cases here:

Case (i): If $x, y \in \mathcal{A}$ then $x * y \in \mathcal{A}$, thus

$$\widetilde{\delta}_\Xi(x * y) = [\beta_1, \beta_2] = r\min\{\widetilde{\delta}_\Xi(x), \widetilde{\delta}_\Xi(y)\}$$

and

$$\theta_\Xi(x * y) = \vartheta = \max\{\theta_\Xi(x), \theta_\Xi(y)\}.$$

Case (ii): If $x \notin \mathcal{A}$ or $y \notin \mathcal{A}$, then

$$\widetilde{\delta}_\Xi(x * y) \succeq [\gamma_1, \gamma_2] = r\min\{\widetilde{\delta}_\Xi(x), \widetilde{\delta}_\Xi(y)\}$$

and

$$\theta_\Xi(x * y) \leq \eta = \max\{\theta_\Xi(x), \theta_\Xi(y)\}.$$

Hence, $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ is a cubic KU-subalgebra of X. □

Theorem 9. *If every cubic KU-subalgebra* $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ *of* X *has the finite image, then every descending chain of KU-subalgebras of* X *terminates at finite step.*

Proof. Consider a strictly descending chain $H_0 \supset H_1 \supset H_2 \cdots$ of KU-subalgebras of X which does not terminate at finite step. Define a cubic set $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ in X by

$$\widetilde{\delta}_\Xi(x) = \begin{cases} \left[\frac{n+1}{n+2}, \frac{n+3}{n+4}\right] & \text{if } x \in H_n \setminus H_{n+1} \\ [1,1] & \text{if } x \in \cap_{n=0}^{\infty} H_n \end{cases}$$

$$\text{and} \quad \theta_\Xi(x) \quad = \quad \begin{cases} \frac{n+2}{n+3} & \text{if } x \in H_n \setminus H_{n+1} \\ 0 & \text{if } x \in \cap_{n=0}^{\infty} H_n \end{cases}$$

where $n = 0, 1, 2, \ldots$ and H_0 stands for X. Let $x, y \in X$. Assume that $x \in H_n \setminus H_{n+1}$ and $y \in H_k \setminus H_{k+1}$ for $n = 0, 1, 2, \ldots$; $k = 0, 1, 2, \ldots$. We may assume that $n \leq k$. Then obviously x and $y \in H_n$, so $x * y \in H_n$ because H_n is a KU-subalgebra of X. Hence,

$$\begin{aligned} \widetilde{\delta}_\Xi(x * y) &\succeq \left[\frac{n+1}{n+2}, \frac{n+3}{n+4}\right] = \mathrm{r}\min\{\widetilde{\delta}_\Xi(x), \widetilde{\delta}_\Xi(y)\} \\ \theta_\Xi(x * y) &\leq \frac{n+2}{n+3} = \max\{\theta_\Xi(x), \theta_\Xi(y)\}. \end{aligned}$$

If $x, y \in \cap_{n=0}^{\infty} H_n$, then $x * y \in \cap_{n=0}^{\infty} H_n$. Thus

$$\widetilde{\delta}_\Xi(x * y) = [1, 1] = r\min\{\widetilde{\delta}_\Xi(x), \widetilde{\delta}_\Xi(y)\}$$

and

$$\theta_\Xi(x * y) = 0 = \max\{\theta_\Xi(x), \theta_\Xi(y)\}.$$

If $x \notin \cap_{n=0}^{\infty} H_n$ and $y \in \cap_{n=0}^{\infty} H_n$, then there exists a positive integer p such that $x \in H_p \setminus H_{p+1}$. It follows that $x * y \in H_p$ so that

$$\begin{aligned} \widetilde{\delta}_\Xi(x * y) &\succeq \left[\frac{p+1}{p+2}, \frac{p+3}{p+4}\right] = \mathrm{r}\min\{\widetilde{\delta}_\Xi(x), \widetilde{\delta}_\Xi(y)\} \\ \theta_\Xi(x * y) &\leq \frac{p+2}{p+3} = \max\{\theta_\Xi(x), \theta_\Xi(y)\}. \end{aligned}$$

Finally suppose that $x \in \cap_{n=0}^{\infty} H_n$ and $y \notin \cap_{n=0}^{\infty} H_n$. Then $y \in H_q \setminus H_{q+1}$ for some positive integer q. It follows that $x * y \in H_q$, and hence

$$\begin{aligned} \widetilde{\delta}_\Xi(x * y) &\succeq \left[\frac{q+1}{q+2}, \frac{q+3}{q+4}\right] = \mathrm{r}\min\{\widetilde{\delta}_\Xi(x), \widetilde{\delta}_\Xi(y)\} \\ \theta_\Xi(x * y) &\leq \frac{q+2}{q+3} = \max\{\theta_\Xi(x), \theta_\Xi(y)\}. \end{aligned}$$

This proves that $\Xi = \left\langle \widetilde{\delta}_\Xi, \theta_\Xi \right\rangle$ is a cubic KU-subalgebra with an infinite number of different values, which is a contradiction. This completes the proof. $\square$

References

[1] Y.B. Jun, C.S. Kim, M.S. Kang, Cubic subalgebras and ideals of BCK/BCI-algebras, *Far East. J. Math. Sci.*, **44** (2010), 239-250.

[2] S.M. Mostafa, M.A. Abd-Elnaby, M.M. Yousef, Fuzzy ideals of KU-algebras, *Int. Math. Forum*, **6**, No. 63 (2011), 3139-3149.

[3] C. Prabpayak, U. Leerawat, On ideas and congurences in KU-algebras, *Scientia Magna J.*, **5**, No. 1 (2009), 54-57.

[4] C. Prabpayak, U. Leerawat, On isomorphisms of KU-algebras, *Scientia Magna J.*, **5**, No. 3 (2009), 25-31.

[5] N. Yaqoob, S.M. Mostafa, M.A. Ansari, On cubic KU-ideals of KU-algebras, *ISRN Algebra*, Article ID 935905 (2013), 10 pages.