CUBIC KU-SUBALGEBRAS

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Abstract: In this paper, we study some new properties of cubic KU-subalgebras.

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1. Introduction

Prabpayak and Leerawat [3] introduced a new algebraic structure which is called KU-algebra. They introduced the concept of homomorphisms of KU-algebras and investigated some related properties in [4]. Mostafa et al. [2] introduced the notion of fuzzy KU-ideals of KU-algebras.

Jun et al. [1] introduced the notion of cubic sub-algebras and ideals in BCK/BCI-algebras. They discussed relationship between a cubic subalgebra and a cubic ideal.

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In this paper, we provided some new properties of cubic KU-subalgebras.

2. Preliminaries and Basic Definitions

In this section, we will recall some definitions of KU-algebra and cubic sets.

**Definition 1.** By a KU-algebra we mean an algebra \((X, *, 0)\) of type \((2, 0)\) with a single binary operation \(*\) that satisfies the following identities: for any \(x, y, z \in X\),

\[
\begin{align*}
(ku1) \quad (x * y) * [(y * z) * (x * z)] &= 0, \\
(ku2) \quad x * 0 &= 0, \\
(ku3) \quad 0 * x &= x, \\
(ku4) \quad x * y &= 0 = y * x \implies x = y.
\end{align*}
\]

In a KU-algebra, the identity \(z * z = 0\) is true \([2]\).

**Definition 2.** \([4]\) A subset \(S\) of KU-algebra \(X\) is called a KU-subalgebra of \(X\) if \(x * y \in S\), whenever \(x, y \in S\).

An interval number is \(\tilde{a} = [a^-, a^+]\), where \(0 \leq a^- \leq a^+ \leq 1\). Let \(D[0, 1]\) denote the family of all closed subintervals of \([0, 1]\), i.e.,

\[
D[0, 1] = \{\tilde{a} = [a^-, a^+] : a^- \leq a^+, \text{ for } a^-, a^+ \in I\}.
\]

We define the operations "\(\succeq\)", "\(\preceq\)", "\(\succ\)", "\(\preccurlyeq\)" and "\(\text{rmin}\)" and "\(\text{rmax}\)" in case of two elements in \(D[0, 1]\). We consider two elements \(\tilde{a} = [a^-, a^+]\) and \(\tilde{b} = [b^-, b^+]\) in \(D[0, 1]\). Then:

(1) \(\tilde{a} \succeq \tilde{b}\) if and only if \(a^- \geq b^-\) and \(a^+ \geq b^+\),

(2) \(\tilde{a} \preceq \tilde{b}\) if and only if \(a^- \leq b^-\) and \(a^+ \leq b^+\),

(3) \(\text{rmin}\{\tilde{a}, \tilde{b}\} = \{\text{min}\{a^-, b^-, a^+, b^+\}\} \).

**Definition 3.** \([1]\) Cubic sets defined on a non-empty set \(X\) as object having the form \(\Xi = \{\left\langle x, \tilde{\delta}_\Xi(x), \theta_\Xi(x) \right\rangle : x \in X\}\), which is briefly denoted by \(\Xi = \left\langle \tilde{\delta}_\Xi, \theta_\Xi \right\rangle\), where the functions \(\tilde{\delta}_\Xi : X \to D[0, 1]\) and \(\theta_\Xi : X \to [0, 1]\).
3. Cubic KU-Subalgebras

In this section, we will discuss some new properties of cubic KU-subalgebras.

**Definition 4.** Let $X$ be a KU-algebra. A cubic subset $\Xi = \langle \tilde{\delta}_\Xi, \theta_\Xi \rangle$ in $X$ is called a cubic KU-subalgebra of $X$ if

$$\tilde{\delta}_\Xi(x \ast y) \succeq r \min\{\tilde{\delta}_\Xi(x), \tilde{\delta}_\Xi(y)\}$$

and

$$\theta_\Xi(x \ast y) \leq \max\{\theta_\Xi(x), \theta_\Xi(y)\},$$

for all $x, y \in X$.

**Proposition 5.** If $\Xi = \langle \tilde{\delta}_\Xi, \theta_\Xi \rangle$ is a cubic KU-subalgebra of $X$, then for all $x \in X$, $\tilde{\delta}_\Xi(0) \succeq \tilde{\delta}_\Xi(x)$ and $\theta_\Xi(0) \leq \theta_\Xi(x)$.

**Proof.** The proof is straightforward. □

**Theorem 6.** Let $\Xi = \langle \tilde{\delta}_\Xi, \theta_\Xi \rangle$ be a cubic KU-subalgebra of $X$. If there exists a sequence $x_n$ in $X$ such that $\lim_{n \to \infty} \tilde{\delta}_\Xi(x_n) = [1, 1]$ and $\lim_{n \to \infty} \theta_\Xi(x_n) = 0$, then $\tilde{\delta}_\Xi(0) = [1, 1]$ and $\theta_\Xi(0) = 0$.

**Proof.** Let $\Xi = \langle \tilde{\delta}_\Xi, \theta_\Xi \rangle$ be a cubic KU-subalgebra of $X$. By Proposition 5, $\tilde{\delta}_\Xi(0) \succeq \tilde{\delta}_\Xi(x)$ for all $x \in X$, therefore $\tilde{\delta}_\Xi(0) \succeq \tilde{\delta}_\Xi(x_n)$ for positive integer $n$. Consider, $[1, 1] \succeq \tilde{\delta}_\Xi(0) \succeq \lim_{n \to \infty} \tilde{\delta}_\Xi(x_n) = [1, 1]$. Hence, $\tilde{\delta}_\Xi(0) = [1, 1]$.

Also, by Proposition 5, $\theta_\Xi(0) \leq \theta_\Xi(x)$ for all $x \in X$, thus $\theta_\Xi(0) \leq \theta_\Xi(x_n)$ for positive integer $n$. Now, $0 \leq \theta_\Xi(0) \leq \lim_{n \to \infty} \theta_\Xi(x_n) = 0$. Hence, $\theta_\Xi(0) = 0$. □

For any elements $x$ and $y$ of $X$, let us write

$$\prod_{i=1}^{m} x \ast y$$

for $((...((x \ast x) \ast x)...) \ast y,$

where $x$ appears $m$ times. Here $m$ is a natural number.

**Theorem 7.** Let $\Xi = \langle \tilde{\delta}_\Xi, \theta_\Xi \rangle$ be a cubic KU-subalgebra of $X$ and let $m \in \mathbb{N}$ (N be the set of natural numbers). Then

(i) $\tilde{\delta}_\Xi\left(\prod_{i=1}^{m} x \ast x\right) \succeq \tilde{\delta}_\Xi(x)$ and $\theta_\Xi\left(\prod_{i=1}^{m} x \ast x\right) \leq \theta_\Xi(x)$, for any odd number $m$,

(ii) $\tilde{\delta}_\Xi\left(\prod_{i=1}^{m} x \ast x\right) = \tilde{\delta}_\Xi(x)$ and $\theta_\Xi\left(\prod_{i=1}^{m} x \ast x\right) = \theta_\Xi(x)$, for any even number $m$. 

Proof. (i) Let $\Xi = \langle \tilde{\delta}_\Xi, \theta_\Xi \rangle$ be a cubic KU-subalgebra of $X$ and let $x \in X$. Assume that $m$ is odd. Then $m = 2q - 1$ for some positive integer $q$. Now we prove the theorem by induction. Now by Proposition 5, $\tilde{\delta}_\Xi(x \ast x) = \tilde{\delta}_\Xi(0) \geq \tilde{\delta}_\Xi(x)$ and $\theta_\Xi(x \ast x) = \theta_\Xi(0) \leq \theta_\Xi(x)$. Suppose that $\tilde{\delta}_\Xi(\prod x \ast x)^{2q-1} \geq \tilde{\delta}_\Xi(x)$ and $\theta_\Xi(\prod x \ast x)^{2q-1} \leq \theta_\Xi(x)$. Then by assumption,

$$\tilde{\delta}_\Xi(\prod x \ast x) = \tilde{\delta}_\Xi(\prod x \ast x)^{2q+1} = \tilde{\delta}_\Xi(\prod x \ast ((x \ast x) \ast x))^{2q-1} \geq \tilde{\delta}_\Xi(x),$$

and

$$\theta_\Xi(\prod x \ast x) = \theta_\Xi(\prod x \ast x)^{2q+1} = \theta_\Xi(\prod x \ast ((x \ast x) \ast x))^{2q-1} \leq \theta_\Xi(x).$$

This completes the proof of (i). The proof of (ii) is similar to (i).

\[ \square \]

**Theorem 8.** Let $A$ be a non-empty subset of $X$ and $\Xi = \langle \tilde{\delta}_\Xi, \theta_\Xi \rangle$ be a cubic subset in $X$ defined by

$$\tilde{\delta}_\Xi(x) = \begin{cases} [\beta_1, \beta_2] & \text{if } x \in A \\ [\gamma_1, \gamma_2] & \text{otherwise} \end{cases} \quad \text{and} \quad \theta_\Xi(x) = \begin{cases} \vartheta & \text{if } x \in A \\ \eta & \text{otherwise} \end{cases}$$
for all \([\beta_1, \beta_2], [\gamma_1, \gamma_2] \in D[0,1]\) and \(\vartheta, \eta \in [0,1]\) with \([\beta_1, \beta_2] \succeq [\gamma_1, \gamma_2]\) and \(\vartheta \leq \eta\). Then \(\Xi = \langle \widetilde{\delta}_{\Xi}, \theta_{\Xi} \rangle\) is a cubic KU-subalgebra of \(X\) if and only if \(A\) is a KU-subalgebra of \(X\).

Proof. Let \(\Xi = \langle \widetilde{\delta}_{\Xi}, \theta_{\Xi} \rangle\) be a cubic KU-subalgebra of \(X\). Let \(x, y \in X\) be such that \(x, y \in A\). Then

\[
\widetilde{\delta}_{\Xi}(x \ast y) \geq r \min\{\widetilde{\delta}_{\Xi}(x), \widetilde{\delta}_{\Xi}(y)\} = r \min\{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = [\beta_1, \beta_2]
\]

and

\[
\theta_{\Xi}(x \ast y) \leq \max\{\theta_{\Xi}(x), \theta_{\Xi}(y)\} = \max\{\vartheta, \vartheta\} = \vartheta.
\]

So \(x \ast y \in A\). Hence \(A\) is a KU-subalgebra of \(X\).

Conversely, suppose that \(A\) is a KU-subalgebra of \(X\). Let \(x, y \in X\). We have two cases here:

Case (i): If \(x, y \in A\) then \(x \ast y \in A\), thus

\[
\widetilde{\delta}_{\Xi}(x \ast y) = [\beta_1, \beta_2] = r \min\{\widetilde{\delta}_{\Xi}(x), \widetilde{\delta}_{\Xi}(y)\}
\]

and

\[
\theta_{\Xi}(x \ast y) = \vartheta = \max\{\theta_{\Xi}(x), \theta_{\Xi}(y)\}.
\]

Case (ii): If \(x \notin A\) or \(y \notin A\), then

\[
\widetilde{\delta}_{\Xi}(x \ast y) \succeq [\gamma_1, \gamma_2] = r \min\{\widetilde{\delta}_{\Xi}(x), \widetilde{\delta}_{\Xi}(y)\}
\]

and

\[
\theta_{\Xi}(x \ast y) \leq \eta = \max\{\theta_{\Xi}(x), \theta_{\Xi}(y)\}.
\]

Hence, \(\Xi = \langle \widetilde{\delta}_{\Xi}, \theta_{\Xi} \rangle\) is a cubic KU-subalgebra of \(X\).

Theorem 9. If every cubic KU-subalgebra \(\Xi = \langle \widetilde{\delta}_{\Xi}, \theta_{\Xi} \rangle\) of \(X\) has the finite image, then every descending chain of KU-subalgebras of \(X\) terminates at finite step.

Proof. Consider a strictly descending chain \(H_0 \supset H_1 \supset H_2 \cdots\) of KU-subalgebras of \(X\) which does not terminate at finite step. Define a cubic set \(\Xi = \langle \widetilde{\delta}_{\Xi}, \theta_{\Xi} \rangle\) in \(X\) by

\[
\widetilde{\delta}_{\Xi}(x) = \begin{cases} 
\left[\frac{n+1}{n+2}, \frac{n+3}{n+4}\right] & \text{if } x \in H_n \setminus H_{n+1} \\
[1,1] & \text{if } x \in \cap_{n=0}^{\infty} H_n
\end{cases}
\]
664 M. Akram, N. Yaqoob, M. Gulistan

and \( \theta_{\Xi}(x) = \begin{cases} \frac{n+2}{n+3} & \text{if } x \in H_{n} \setminus H_{n+1} \\ 0 & \text{if } x \in \cap_{n=0}^{\infty} H_{n} \end{cases} \)

where \( n = 0, 1, 2, \ldots \) and \( H_{0} \) stands for \( X \). Let \( x, y \in X \). Assume that \( x \in H_{n} \setminus H_{n+1} \) and \( y \in H_{k} \setminus H_{k+1} \) for \( n = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots \). We may assume that \( n \leq k \). Then obviously \( x \) and \( y \in H_{n} \), so \( x \ast y \in H_{n} \) because \( H_{n} \) is a KU-subalgebra of \( X \). Hence,

\[
\tilde{\delta}_{\Xi}(x \ast y) \geq \left[ \frac{n+1}{n+2}, \frac{n+3}{n+4} \right] = r \min\{\tilde{\delta}_{\Xi}(x), \tilde{\delta}_{\Xi}(y)\}
\]

\[
\theta_{\Xi}(x \ast y) \leq \frac{n+2}{n+3} = \max\{\theta_{\Xi}(x), \theta_{\Xi}(y)\}.
\]

If \( x, y \in \cap_{n=0}^{\infty} H_{n} \), then \( x \ast y \in \cap_{n=0}^{\infty} H_{n} \). Thus

\[
\tilde{\delta}_{\Xi}(x \ast y) = [1, 1] = r \min\{\tilde{\delta}_{\Xi}(x), \tilde{\delta}_{\Xi}(y)\}
\]

and

\[
\theta_{\Xi}(x \ast y) = 0 = \max\{\theta_{\Xi}(x), \theta_{\Xi}(y)\}.
\]

If \( x \notin \cap_{n=0}^{\infty} H_{n} \) and \( y \in \cap_{n=0}^{\infty} H_{n} \), then there exists a positive integer \( p \) such that \( x \in H_{p} \setminus H_{p+1} \). It follows that \( x \ast y \in H_{p} \) so that

\[
\tilde{\delta}_{\Xi}(x \ast y) \geq \left[ \frac{p+1}{p+2}, \frac{p+3}{p+4} \right] = r \min\{\tilde{\delta}_{\Xi}(x), \tilde{\delta}_{\Xi}(y)\}
\]

\[
\theta_{\Xi}(x \ast y) \leq \frac{p+2}{p+3} = \max\{\theta_{\Xi}(x), \theta_{\Xi}(y)\}.
\]

Finally suppose that \( x \in \cap_{n=0}^{\infty} H_{n} \) and \( y \notin \cap_{n=0}^{\infty} H_{n} \). Then \( y \in H_{q} \setminus H_{q+1} \) for some positive integer \( q \). It follows that \( x \ast y \in H_{q} \), and hence

\[
\tilde{\delta}_{\Xi}(x \ast y) \geq \left[ \frac{q+1}{q+2}, \frac{q+3}{q+4} \right] = r \min\{\tilde{\delta}_{\Xi}(x), \tilde{\delta}_{\Xi}(y)\}
\]

\[
\theta_{\Xi}(x \ast y) \leq \frac{q+2}{q+3} = \max\{\theta_{\Xi}(x), \theta_{\Xi}(y)\}.
\]

This proves that \( \Xi = \langle \tilde{\delta}_{\Xi}, \theta_{\Xi} \rangle \) is a cubic KU-subalgebra with an infinite number of different values, which is a contradiction. This completes the proof. \( \Box \)

References


