International Journal of Pure and Applied Mathematics
Volume 89 No. 5 2013, 689-697
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v89i5.5

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ON THE UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract: Sufficient conditions for the univalence of an integral operator is obtained. Relevant connections of the results obtained with the earlier are also pointed out.

AMS Subject Classification: 30C45, 30C80
Key Words: analytic functions, univalent functions, Hadamard product, integral operators, Schwarz lemma

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0), \tag{1.1}$$

which are analytic in the open disc

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$, which are univalent in Δ.

Received: August 31, 2013

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For two functions $f_j(j=1,2)$ given by

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k \qquad (j=1,2),$$

the Hadamard product or convolution, denoted by $(f_1*f_2)(z)$, given by

$$(f_1*f_2)(z)=z+\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k.$$

For complex numbers $\alpha_1,\alpha_2,\ldots,\alpha_q$ and $\beta_1,\beta_2,\ldots,\beta_s$; $(\beta_j\in\mathbb{C}\setminus\mathcal{Z}_0^-;\mathcal{Z}_0^-=\{0,-1,-2,\ldots\}$ for $j=1,2,..,s)$,recently in [11] an operator $\mathcal{D}_{\lambda,\mu}^m(\alpha_1,\beta_1)f(z):\mathcal{A}\to\mathcal{A}$ is defined by

$$\mathcal{D}_{\lambda,\mu}^m(\alpha_1,\beta_1)f(z)=z+\sum_{k=2}^{\infty}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^m\frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1}\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1}\ldots(\beta_s)_{k-1}(k-1)!}a_kz^k \tag{1.2}$$

where $q\le s+1;q,s,m\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};z\in\Delta$ and $\mathbb{N}$ denotes the set of all positive integers and $0\le\mu\le\lambda\le1$. Also $(x)_k$ is the Pochhammer symbol defined in terms of gamma functions, as

$$(x)_k=\frac{\Gamma(x+k)}{\Gamma(x)}=\begin{cases}1 & \text{if} \quad k=0\\ x(x+1)\ldots(x+k-1) & \text{if} \quad k\in\mathbb{N}.\end{cases}$$

For suitable values of $\alpha_{i's},\beta_{j's},q,s,m,\lambda$ and μ we can deduce several operators such as Sălăgean derivative operator [20], Ruscheweyh derivative operator [19], fractional calculus operator [13], Carlson-Shaffer operator [9], Dziok-Srivatsava operator [10] and also the operator introduced by Darus et al [1] and the operator introduced by Selvaraj et al [21].

Using the operator $\mathcal{D}_{\lambda,\mu}^m(\alpha_1,\beta_1)f(z)$ we now introduce the following. For $n\in\mathbb{N}\cup\{0\}$ and $\gamma_1,\gamma_2,\ldots,\gamma_n,\delta\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$ we define the integral operator
$I_{\gamma_i,\delta}(\lambda,\mu,m;\alpha_1,\beta_1;z):\mathcal{A}^n\to\mathcal{A}$ by

$$I_{\gamma_i,\delta}(\lambda,\mu,m;\alpha_1,\beta_1;z)=\left\{\delta\int_0^z t^{\delta-1}\prod_{i=1}^{n}\left(\frac{\mathcal{D}_{\lambda,\mu}^m(\alpha_1,\beta_1)f_i(t)}{t}\right)^{\frac{1}{\gamma_i}}dt\right\}^{\frac{1}{\delta}}. \tag{1.3}$$

This operator $I_{\gamma_i,\delta}(\lambda,\mu,m;\alpha_1,\beta_1;z)$ generalizes many operators which were introduced and studied by many authors. We list some of them.

1. If $\mu = 0$ the above integral operator reduces to

$$F_{\gamma_i,\delta}(\lambda, m; \alpha_1, \beta_1; z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{\mathcal{D}_\lambda^m(\alpha_1, \beta_1) f_i(t)}{t} \right)^{\frac{1}{\gamma_i}} dt \right\}^{\frac{1}{\delta}}.$$

   This operator was introduced and studied by Selvaraj et al [22].

2. Let $\mu = 0, q = 2, s = 1, \alpha_1 = \beta_1, \alpha_2 = 1, \gamma_i = \frac{1}{\alpha_i}$ and $\delta = 1$, this operator reduces to

$$I(f_1, f_2, \ldots, f_n)(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{D_\lambda^m f_i(t)}{t} \right)_i^a dt$$

   where $D_\lambda^m(f(z))$ is the well known Al-oboudi differential operator. This operator $I(f_1, \ldots, f_n)(z)$ was introduced and studied by S. Bulut [8].

3. For $m = 0, q = 2, s = 1, \alpha_1 = \beta_1, \alpha_2 = 1, \gamma_i = \frac{1}{\alpha - 1}$ and $\delta = n(\alpha - 1) + 1$, then this operator reduces to

$$F_{n,\alpha}(z) = [(n(\alpha - 1) + 1) \int_0^z \prod_{i=1}^{n} (f_i(t))^{\alpha-1} dt]^{\frac{1}{n(\alpha-1)+1}}.$$

   Studied by D. Breaz and N. Breaz and H.M. Srivatsava [7].

4. Let $m = 0, q = 2, s = 1, \alpha_1 = \beta_1, \alpha_2 = 1, \gamma_i = \frac{1}{a_i}$ and $\delta = 1$. Then this operator reduces to

$$F_\alpha(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{a_i} dt$$

   which was introduced and studied by D. Breaz and N. Breaz [4].

5. Let $m = 0, q = 2, s = 1, \alpha_1 = 2, \beta_1 = 1, \alpha_2 = 1, \gamma_i = \frac{1}{\alpha_i}$ and $\delta = 1$ then the operator reduces to

$$G_\alpha(z) = \int_0^z \prod_{i=1}^{n} (f_i'(t))^{\alpha_i} dt$$

   which was introduced and studied by D. Breaz et al [5].

Apart from this several well known and new integral operators will follow as a special case of $I_{\gamma_i,\delta}(\lambda, \mu, m; \alpha_1, \beta_1; z)$.

2. Preliminaries

Theorem 2.1. (Schawarz Lemma) *Let $f \in \mathcal{A}$ satisfy the condition $|f(z)| < 1$ for all $z \in \Delta$, then*

$$|f(z)| \leq |z| \qquad (z \in \Delta).$$

Equality holds for the functions $f(z) = cz$ where $|c| = 1$.

Theorem 2.2. (see [15]) *Let δ be the complex number with $\Re\delta > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1 - |z|^{2\delta}}{\Re\delta} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \qquad (z \in \Delta)$$

then the function

$$F_\delta(z) = \left[\delta \int_0^z t^{\delta-1} f'(t) dt \right]^{\frac{1}{\delta}} = z + \dots$$

is analytic and univalent in Δ.

Theorem 2.3. (see [16]) *Let δ be a complex number and $\Re\delta > 0$ and $c \in \mathbb{C}(|c| < 1; c = -1)$. If $f \in \mathcal{A}$ satisfies*

$$\left| c|z|^{2\delta} + (1 - |z|^{2\delta}) \frac{zf''(z)}{\delta f'(z)} \right| \leq 1 \qquad (z \in \Delta),$$

then the function

$$F_\delta(z) = \left[\delta \int_0^z t^{\delta-1} f'(t) dt \right]^{\frac{1}{\delta}} = z + \dots$$

is analytic and univalent in Δ.

Theorem 2.4. (see [18]) *Let $f \in \mathcal{A}$, satisfies the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1, \qquad (z \in \Delta). \tag{2.1}$$

Also let $\alpha \in [1, \frac{3}{2}]$ and $c \in \mathbb{C}$. If $|c| \leq \frac{3-2\alpha}{\alpha}(c = -1)and|f(z)| \leq 1(z \in \Delta)$ then the function

$$H_\alpha(z) = \left(\alpha \int_0^z [g(t)]^{\alpha-1} dt \right)^{\frac{1}{\alpha}}$$

belongs to $\mathcal{S}$.

3. main results

Theorem 3.1. *Let $\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i \in \mathcal{A}$for $i = 1, 2, \dots, n$ and each satisfy (2.1). Also for $M \geq 1$, let γ_i, δ be complex numbers such that*

$$\Re\delta \geq \sum_{i=1}^{n} \frac{2M+1}{|\gamma_i|} \quad and c \in \mathbb{C}.$$

If

$$|c| \leq 1 - \frac{1}{\Re\delta} \sum_{i=1} . \frac{2M+1}{|\gamma_i|} \tag{3.1}$$

and $|\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)| \leq M$ for all $i = 1, 2, \dots, n$ and $z \in \Delta$, then the function $I_{\gamma_i,\delta}(\lambda,\mu,m;\alpha_1,\beta_1;z)$ defined by (1.3) is univalent.

Proof. Let

$$h(z) = \int_0^z \left(\prod_{i=1}^{n} \frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(t)}{t} \right)^{\frac{1}{\gamma_i}} dt,$$

so that we have

$$h'(z) = \left(\prod_{i=1}^{n} \frac{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)}{z} \right)^{\frac{1}{\gamma_i}},$$

and hence

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(\frac{z(\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)'}{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)} - 1 \right). \tag{3.2}$$

Consider

$$\begin{aligned} &\left| c|z|^{2\delta} + (1-|z|^{2\delta}) \frac{zh''(z)}{\delta h'(z)} \right| \\ &\quad = \left| c|z|^{2\delta} + (1-|z|^{2\delta}) \frac{1}{\delta} \sum_{i=1}^{n} \left(\frac{z(\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)'}{\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)} - 1 \right) \right| \\ &\quad \leq |c| + \frac{1}{\delta} \sum_{i=1}^{n} \frac{1}{|\gamma_i|} \left(\left| \frac{z^2(\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z))'}{[\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)]^2} \right| \frac{|\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)|}{|z|} + 1 \right). \end{aligned}$$

Since $|\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)| \leq M$ for $z \in \Delta$ and for $i = 1, 2, \dots, n$, we have

$$|\mathcal{D}^m_{\lambda,\mu}(\alpha_1,\beta_1)f_i(z)| \leq M|z|$$

for $z \in \Delta$ and for $i = 1, 2, \ldots, n$. In view of (2.1) we get

$$\left| c|z|^{2\delta} + (1 - |z|^{2\delta}) \frac{zh''(z)}{\delta h'(z)} \right| \leq |c| + \frac{1}{\Re\delta} \sum_{i=1}^{n} \frac{2M+1}{|\gamma_i|} \qquad (z \in \Delta).$$

Hence by (3.1) we get

$$\left| c|z|^{2\delta} + (1 - |z|^{2\delta}) \frac{zh''(z)}{\delta h'(z)} \right| \leq 1.$$

Finally applying Theorem 2.3 we get the required result. □

We remark here that by specializing the parameters involved in Theorem 3.1 we may arrive at the results in [7, 6, 22, 12, 18].

4. Another criteria for Univalence

In this section we establish another criterion for univalence of the integral operator $I_{\gamma_i,\delta}(\lambda, \mu, m; \alpha_1, \beta_1; z))$ in the unit disk Δ.

Theorem 4.1. *Let the functions $\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f_i(z) \in \mathcal{S}$ for $i = 1, 2, \ldots, n$, and δ be a complex number with $\Re\delta > 0$. If*

$$\frac{1}{|\gamma_1|} + \frac{1}{|\gamma_2|} + \ldots + \frac{1}{|\gamma_n|} \leq \frac{1}{4},$$

then the function $I_{\gamma_i,\delta}(\lambda, \mu, m; \alpha_1, \beta_1; z)$ is univalent.

Proof. Let $h(z)$ be defined as in Theorem 3.1, then

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(\frac{z(\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f_i(z)'}{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f_i(z)} - 1 \right).$$

Consider

$$\begin{aligned}
\frac{1 - |z|^{2\Re\delta}}{\Re\delta} \left| \frac{zh''(z)}{h'(z)} \right| &= \frac{1 - |z|^{2\Re\delta}}{\Re\delta} \sum_{i=1}^{n} \frac{1}{|\gamma_i|} \left| \frac{z(\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f_i(z))'}{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f_i(z)} - 1 \right| \\
&\leq \frac{1 - |z|^{2\Re\delta}}{\Re\delta} \sum_{i=1}^{n} \frac{1}{|\gamma_i|} \left(\left| \frac{z(\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f_i(z))'}{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1) f_i(z)} \right| + 1 \right) \\
&\leq \frac{1 - |z|^{2\Re\delta}}{\Re\delta} \sum_{i=1}^{n} \frac{1}{|\gamma_i|} \left(\frac{1 + |z|}{1 - |z|} + 1 \right)
\end{aligned}$$

$$= \frac{1-|z|^{2\Re\delta}}{1-|z|}\frac{2}{\Re\delta}\sum_{i=1}^{n}\frac{1}{|\gamma_i|}.$$

Noting the fact that

$$\frac{1-|z|^{2\Re\delta}}{1-|z|} \leq \begin{cases} 1 & \text{if} \quad 0 < \Re\delta < \frac{1}{2} \\ 2\Re\delta & \text{if} \quad \frac{1}{2} < \Re\delta < \infty. \end{cases} \tag{4.1}$$

Using the inequality (4.1) and the hypothesis of the theorem we get

$$\frac{1-|z|^{2\Re\delta}}{\Re\delta}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1.$$

Hence the result follows as an application of Theorem 2.2. □

Finally we remark that by specializing the parameters involved in Theorem 4.1, the results in [17, 22] may be obtained as a special case of the Theorem 4.1.

References

[1] Afaf. A. Ali Abubaker, M. Darus, Neighbourhoods of certain classes of analytic functions defined by a generalized differential operator, *Int. J. Math. Anal.*, **4**, No. 28 (2010), 2373-2380.

[2] R. Aghalary, S. B. Joshi, R. N. Mohapatra, V. Ravichandran, Subordinations for analytic functions defined by Dziok-Srivatsava linear operator, *Appl. Math. Comput.*, **187**, No. 1 (2007), 13-19.

[3] F. M. Al-Oboudi, On the univalent functions defined by a generalized Sălăgean operator, *Int. J. Math. Math. Sci*, **25**-28 (2004), 1429-1436.

[4] D. Breaz, N. Breaz, Two integral operators, *Studia. Univ. Babeş-Bolyai. Mat.*, **47**, No. 3 (2002), 13-19.

[5] D. Breaz, H. Ö. Güney, The integral operator on the classes $\mathcal{S}^*_\alpha(b)$ and $\mathcal{C}_\alpha(b)$, *J. Math. Inequal.*, **2**, No. 1 (2008), 97-100.

[6] D. Breaz, H. Ö. Güney, On the univalence criterion of a general integral operator, *J. Inequal. Appl.*, Article ID **702715, 8** (2008).

[7] D. Breaz, N. Breaz, H. M. Srivatsava, An extension of univalent condition for a family of operators, *Appl. Math. Lett.*, **22**, No. 1 (2009), 41-44.

[8] S. Bulut, Sufficient conditions for univalence of an integral operator defined by Al-Oboudi differential operator, *J. Inequal. Appl.*, Article ID. **957042, 5** (2008).

[9] B. C. Carlson, S. B. Shaffer, Starlike and Prestarlike hypergeometric functions, *SIAM. J. Math. Anal.*, 15**(2002)**, 737-745.

[10] J. Dziok, H. M. Srivatsava, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.* **103**, No. 1 (1999), 1-13.

[11] N Marikkannan, A subclass of analytic functions and a generalised differential operator, *communicated*.

[12] G. I. Oros, G. Oros, D. Breaz, Sufficient condition for univalence for an integral operator, *J. Inequal. Appl.*, Article ID 127645, 7 (2008).

[13] S. Owa, Some applications of the fractional calculus, *Fractional Calculus, Proc. Workshop, Ross Priory, Univ. Strathclyde/Engl. 1984 Res. Notes. Math.***138** (1985), 164-175.

[14] S. Ozaki, M. Nunokawa, The Schwarzian derivative and univalent functions, *Proc. Amer. Math. Soc.*, **33** (1972), 392-394.

[15] N. N. Pascu, On a univalence criterion (II), *International seminar on functional equations, approximation and convexity, Cluj-Napoca*, Preprint Nr.**6** (1985), 153-154.

[16] V. Pescar, A new generalization of Ahlfors's and Becker's criterion of univalence, *Bull. Malaysian. Math. Soc*, **19**, No. 2 (1996), 53-54.

[17] V. Pescar, New Univalence criterion for certain integral operator, *Studia. Univ. Babeş-Bolyai. Math. Soc.*, **46**, No. 2 (2001), 107-109.

[18] V. Pescar, On the univalence of some integral operators, *J. Indian. Acad. Math.*, **27**, No. 2 (2005), 239-243.

[19] St. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, **49** (1975), 109-115.

[20] G. Ş. Sălăgean, Subclasses of univalent functions, in *Complex analysis—fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981)*, 362–372, Lecture Notes in Math., 1013 Springer, Berlin.

[21] C. Selvaraj, K. R.Karthikeyan, Certain classes of analytic functions of complex order involving a family of generalized differential operator, *J. Math. Ineq.*, **2**, No. 4 (2008), 449-458.

[22] C. Selvaraj, K. R. Karthikeyan, Univalence of a generalized operator associated with the generalized hypergeometric function. *Tamsuioxford J. Math. Sci.*, **26**, No. 1 (2010), 41-51.