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ENTROPY BOUNDS USING ARITHMETIC-GEOMETRIC-HARMONIC MEAN INEQUALITY

Om Parkash[1 §], Priyanka Kakkar[2]
[1,2]Department of Mathematics
Guru Nanak Dev University
Amritsar, 143005, INDIA

Abstract: In the present communication, we have developed new inequalities using arithmetic-geometric-harmonic mean inequality and consequently, applied our findings to the field of entropy theory. It is observed that our results provide the stronger upper bounds to Shannon entropy as found by Simic[10] and Tapus and Popescu[7]using Jensen's inequality. Moreover, we have extended our findings to the field of coding theory by providing the comparisons of various codeword lengths.

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Key Words: entropy, arithmetic-geometric-harmonic mean inequality, decreasing function, Jensen's inequality, mean codeword length, probability distribution

1. Introduction

In today's world, many developments in computational mathematics have made it possible to compute a much larger number of mathematical entities than could be calculated earlier. However, most of these entities can only be expressed in terms of inequalities and as such the importance of the inequalities has increased

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§Correspondence author

significantly large enough. Moreover, a variety of researchers including physical scientists, engineers, information theoreticians, statisticians and computer scientists have developed inequalities in order to solve various optimizational problems in their respective disciplines.

It is well known that the Arithmetic-Geometric-Harmonic (AGH) mean inequality is a fundamental relationship and a useful tool for problem solving and developing relations with other mathematical disciplines. The importance of this inequality may be significant because of its applications towards the solutions of many optimization problems including utility theory, financial mathematics etc.

The AGH mean inequality is defined as follows:

Let $X = (x_1, x_2, ..., x_n)$ be a sequence of positive numbers and let $P = (p_1, p_2, ..., p_n)$ be the probability distribution associated with X such that $\sum_{i=1}^{n} p_i = 1$. Then

$$A_n = \sum_{i=1}^{n} p_i x_i \geq \prod_{i=1}^{n} x_i^{p_i} = G_n \geq \left(\sum_{i=1}^{n} \frac{p_i}{x_i} \right)^{-1} = H_n \tag{1.1}$$

where A_n is the arithmetic mean, G_n is the geometric mean and H_n is the harmonic mean (Refer Kapur [3]). Authors like Simic [10], [11], [12], Tapus and Popescu [7], Dragmoir [9] worked on Jensen's inequality, established new lower and upper bounds for it and applied their findings in information theory.

The objective of the present paper is to introduce new inequalities and to extend their applications to the field of entropy and coding theory. The organization of the paper is as follows: In Section 2, we use the monotonicity of a function and AGH mean inequality to derive new inequalities. Section 3 provides the sharper upper bounds to Shannon's [2] entropy using the new inequalities. These upper bounds are similar to results achieved by Simic [10] and Tapus and Popescu [7] using Jensen's inequality. As a consequence of the development of new inequalities, we have provided the comparisons of different mean codeword lengths.

2. New Inequalities using Arithmetic-Geometric-Harmonic Mean Inequality

Theorem 2.1. Let f be a monotonically decreasing function on interval $I = [a, b]$, then

$$f(A_n) = f\left(\sum_{i=1}^{n} p_i x_i \right) \leq f\left(\prod_{i=1}^{n} x_i^{p_i} \right) = f(G_n) \tag{2.1}$$

with equality sign if and only if all the members of $X = (x_1, x_2, ..., x_n)$ are equal.

Proof. Using inequality (1.1), the proof directly follows from the definition of monotonically decreasing function. □

Remark. The inequality (2.1) reverses if the function f is monotonically increasing.

Theorem 2.1 may be refined in the form of following inequality:

Theorem 2.2. For a monotonically decreasing function f on interval I, we have

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \min_{1\leq r<s\leq n} f\left(\prod_{i=1, i\neq r,s}^{n} x_i^{p_i} . \left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right) \tag{2.2}$$

where $1 \leq r < s \leq n$.

Proof. Let $x_r, x_s \in X$ where $1 \leq r < s \leq n$ and let p_r, p_s be their corresponding probabilities.

Since $x_r, x_s \in I$, therefore $\frac{p_r x_r + p_s x_s}{p_r + p_s} \in I$. So, Using theorem (2.1), we have

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &= f\left(\sum_{i=1, i\neq r,s}^{n} p_i x_i + (p_r + p_s)\left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)\right) \\ &\leq f\left(\prod_{i=1, i\neq r,s}^{n} x_i^{p_i} . \left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right) \end{aligned}$$

Since $x_r, x_s \in X$ are arbitrary, therefore we have

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq min_{1\leq r<s\leq n} f\left(\prod_{i=1, i\neq r,s}^{n} x_i^{p_i} \left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right)$$

□

Note. There is equality sign in (2.2) for $n = 2$.

Theorem 2.3. For a monotonically decreasing function f defined on interval I which satisfies the relation $f(xy) = f(x) + f(y)$, we have

$$0 \leq \max_{1\leq r<s\leq n} \left(f\left(x_r^{p_r}\right) + f\left(x_s^{p_s}\right) - f\left(\left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right)\right)$$

$$\leq f\left(\prod_{i=1}^{n} x_i^{p_i}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right) \tag{2.3}$$

where $1 \leq r < s \leq n$.

Proof. Let $x_r, x_s \in X$ where $1 \leq r < s \leq n$ and let p_r, p_s be their corresponding probabilities. Since $x_r, x_s \in I$, therefore $\frac{p_r x_r + p_s x_s}{p_r + p_s} \in I$. So, Using theorem (2.2), we have

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &= f\left(\sum_{i=1, i\neq r,s}^{n} p_i x_i + (p_r + p_s)\left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)\right) \\ &\leq f\left(\prod_{i=1, i\neq r,s}^{n} x_i^{p_i} \cdot \left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right) \\ &= f\left(\prod_{i=1}^{n} x_i^{p_i}\right) + f\left(\left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right) - f\left(x_r^{p_r}\right) - f\left(x_s^{p_s}\right) \end{aligned}$$

that is,

$$f\left(x_r^{p_r}\right) + f\left(x_s^{p_s}\right) - f\left(\left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right) \leq f\left(\prod_{i=1}^{n} x_i^{p_i}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

Since $x_r, x_s \in X$ are arbitrary, the assertion of theorem (2.3) follows. □

A more general form of Theorem 2.4 may be elaborated as follows:

Theorem 2.4. For a monotonically decreasing function f defined on interval I which satisfies the relation $f(xy) = f(x) + f(y)$, we have

$$\begin{aligned} 0 &\leq \max_{1\leq r<s\leq n}\left(f\left(x_r^{p_r}\right) + f\left(x_s^{p_s}\right) - f\left(\left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right)\right) \\ &\leq \max_{1\leq r<s<t\leq n}\left(f\left(x_r^{p_r}\right) + f\left(x_s^{p_s}\right) + f\left(x_t^{p_t}\right)\right. \\ &\quad \left. -f\left(\left(\frac{p_r x_r + p_s x_s + p_t x_t}{p_r + p_s + p_t}\right)^{(p_r+p_s+p_t)}\right)\right) \\ &\leq f\left(\prod_{i=1}^{n} x_i^{p_i}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right) \end{aligned}$$

(2.4)

where $1 \leq r < s < t \leq n$.

Proof. Let us suppose that maximum of the expression $f\left(x_r^{p_r}\right) + f\left(x_s^{p_s}\right) - f\left(\left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right)$ is obtained at $r = u, s = v, u, v \in (1, 2, .., n)$. We have to show that for any $w \in (1, 2, ..., n) - (u, v)$,

$$f\left(x_u^{p_u}\right) + f\left(x_v^{p_v}\right) - f\left(\left(\frac{p_u x_u + p_v x_v}{p_u + p_v}\right)^{(p_u+p_v)}\right)$$

$$\leq \left(f\left(x_u^{p_u}\right) + f\left(x_v^{p_v}\right) + f\left(x_w^{p_w}\right) - f\left(\left(\frac{p_u x_u + p_v x_v + p_w x_w}{p_u + p_v + p_w}\right)^{(p_u+p_v+p_w)}\right)\right)$$

that is,

$$f\left(\left(\frac{p_u x_u + p_v x_v + p_w x_w}{p_u + p_v + p_w}\right)^{(p_u+p_v+p_w)}\right) \leq f\left(x_w^{p_w}\right) + f\left(\left(\frac{p_u x_u + p_v x_v}{p_u + p_v}\right)^{(p_u+p_v)}\right)$$

Now

$$f\left(\left(\frac{p_u x_u + p_v x_v + p_w x_w}{p_u + p_v + p_w}\right)^{(p_u+p_v+p_w)}\right) \leq f\left(x_w^{p_w}\right) + f\left(\left(\frac{p_u x_u + p_v x_v}{p_u + p_v}\right)^{(p_u+p_v)}\right)$$

$$\text{iff } f\left(\left(\frac{p_u x_u + p_v x_v + p_w x_w}{p_u + p_v + p_w}\right)^{(p_u+p_v+p_w)}\right) \leq f\left(x_w^{p_w}\left(\frac{p_u x_u + p_v x_v}{p_u + p_v}\right)^{(p_u+p_v)}\right)$$

$$\text{iff } x_w^{\frac{p_w}{p_u+p_v+p_w}}\left(\frac{p_u x_u + p_v x_v}{p_u + p_v}\right)^{\frac{(p_u+p_v)}{p_u+p_v+p_w}} \leq \left(\frac{p_u x_u + p_v x_v + p_w x_w}{p_u + p_v + p_w}\right)$$

that is,

$$\text{iff } x_w^{\frac{p_w}{p_u+p_v+p_w}}\left(\frac{p_u x_u + p_v x_v}{p_u + p_v}\right)^{\frac{(p_u+p_v)}{p_u+p_v+p_w}}$$

$$\leq \frac{p_u + p_v}{p_u + p_v + p_w}\left(\frac{p_u x_u + p_v x_v}{p_u + p_v}\right) + \frac{p_w}{p_u + p_v + p_w} x_w$$

which is always true.

So, the first part of the theorem is proved as maximum of the expression

$$\left(f\left(x_r^{p_r}\right)+f\left(x_s^{p_s}\right)+f\left(x_t^{p_t}\right)-f\left(\left(\frac{p_r x_r+p_s x_s+p_t x_t}{p_r+p_s+p_t}\right)^{(p_r+p_s+p_t)}\right)\right)$$

is greater than or equal to

$$\left(f\left(x_u^{p_u}\right)+f\left(x_v^{p_v}\right)+f\left(x_w^{p_w}\right)-f\left(\left(\frac{p_u x_u+p_v x_v+p_w x_w}{p_u+p_v+p_w}\right)^{(p_u+p_v+p_w)}\right)\right)$$

Next, we have to show that

$$\max_{1\le r<s<t\le n}\left(f\left(x_r^{p_r}\right)+f\left(x_s^{p_s}\right)+f\left(x_t^{p_t}\right)-f\left(\left(\frac{p_r x_r+p_s x_s+p_t x_t}{p_r+p_s+p_t}\right)^{(p_r+p_s+p_t)}\right)\right)$$
$$\le f\left(\prod_{i=1}^{n} x_i^{p_i}\right)-f\left(\sum_{i=1}^{n} p_i x_i\right)$$

Choose arbitrary $x_u, x_v, x_w \in X$ and we will show that

$$f\left(x_u^{p_u}\right)+f\left(x_v^{p_v}\right)+f\left(x_w^{p_w}\right)-f\left(\left(\frac{p_u x_u+p_v x_v+p_w x_w}{p_u+p_v+p_w}\right)^{(p_u+p_v+p_w)}\right)$$
$$\le f\left(\prod_{i=1}^{n} x_i^{p_i}\right)-f\left(\sum_{i=1}^{n} p_i x_i\right)$$

that is,

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &\le f\left(\prod_{i=1}^{n} x_i^{p_i}\right)-f\left(x_u^{p_u}\right)-f\left(x_v^{p_v}\right)-f\left(x_w^{p_w}\right) \\ &+ f\left(\left(\frac{p_u x_u+p_v x_v+p_w x_w}{p_u+p_v+p_w}\right)^{(p_u+p_v+p_w)}\right) \end{aligned}$$

Now

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &\le f\left(\prod_{i=1}^{n} x_i^{p_i}\right)-f\left(x_u^{p_u}\right)-f\left(x_v^{p_v}\right)-f\left(x_w^{p_w}\right) \\ &+ f\left(\left(\frac{p_u x_u+p_v x_v+p_w x_w}{p_u+p_v+p_w}\right)^{(p_u+p_v+p_w)}\right) \end{aligned}$$

$$\text{iff } f\left(\sum_{i=1}^{n} p_i x_i\right) \le f\left(\prod_{i=1, i\ne u,v,w}^{n} x_i^{p_i} \cdot \left(\frac{p_u x_u + p_v x_v + p_w x_w}{p_u + p_v + p_w}\right)^{(p_u+p_v+p_w)}\right)$$

which is true as proved in Theorem 2.2.

Since the inequality is true for any $u, v, w \in \{1, 2, ..., n\}$,

$$\max_{1\le r<s<t\le n}\left(f\left(x_r^{p_r}\right) + f\left(x_s^{p_s}\right) + f\left(x_t^{p_t}\right) - f\left(\left(\frac{p_r x_r + p_s x_s + p_t x_t}{p_r + p_s + p_t}\right)^{(p_r+p_s+p_t)}\right)\right)$$

$$\le f\left(\prod_{i=1}^{n} x_i^{p_i}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

Hence the theorem. □

We now give the generalization of Theorem 2.4 in the following theorem:

Theorem 2.5. For a monotonically decreasing function f defined on interval I which satisfies the relation $f(xy) = f(x) + f(y)$, we have

$$0 \le S_2 \le S_3 \le ... \le S_{n-1} \le f\left(\prod_{i=1}^{n} x_i^{p_i}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

where

$$S_2 = \max_{1\le r_1<r_2\le n}\left(f\left(x_{r_1}^{p_{r_1}}\right) + f\left(x_{r_2}^{p_{r_2}}\right) - f\left(\left(\frac{p_{r_1} x_{r_1} + p_{r_2} x_{r_2}}{p_{r_1} + p_{r_2}}\right)^{(p_{r_1}+p_{r_2})}\right)\right)$$

....

$$S_{n-1} = \max_{1\le r_1<r_2<...<r_{n-1}\le n}\left(\sum_{i=1}^{n-1} f\left(x_{r_i}^{p_{r_i}}\right) - f\left(\left(\frac{\sum_{i=1}^{n-1} p_{r_i} x_{r_i}}{\sum_{i=1}^{n-1} p_{r_i}}\right)^{(\sum_{i=1}^{n-1} p_{r_i})}\right)\right)$$

Proof. The proof is similar as proved in Theorem 2.5. □

Theorem 2.6. Let f be a monotonically decreasing function on interval $I = [a, b]$, then

$$f(A_n) = f\left(\sum_{i=1}^{n} p_i x_i\right) \le f\left(\prod_{i=1}^{n} x_i^{p_i}\right) = f(G_n) \le f\left(\sum_{i=1}^{n} \frac{p_i}{x_i}\right)^{-1} = f(H_n) \tag{2.5}$$

with equality sign if and only if all the members of $X = (x_1, x_2, ..., x_n)$ are equal.

Proof. By using the definition of monotonic decreasing function, the proof follows from the inequality (1.1). □

3. Applications of Inequalities in Entropy and Coding Theory

We will prove the standard results already existing in information theory as well as new bounds for the entropy by applying above inequalities.

Theorem 3.1. For any probability distribution $P = \{p_1, p_2, ..., p_n\}$, we have

$$H(P) \leq \log n \tag{3.1}$$

where $H(P) = -\sum_{i=1}^{n} p_i \log p_i$ is Shannon's [2] measure of entropy for n variables with equality if and only if all $p_i = \frac{1}{n}$.

Proof. Since $f(x) = -\log x$ is a monotonically decreasing function in the interval $[1, \infty[$, theorem (2.1) gives

$$-\log \sum_{i=1}^{n} p_i x_i \leq -\log \left(\prod_{i=1}^{n} x_i^{p_i} \right)$$

that is,

$$-\log \sum_{i=1}^{n} p_i x_i \leq -\sum_{i=1}^{n} p_i \log x_i \tag{3.2}$$

Again substituting $x_i = \frac{1}{p_i}, 1 \leq i \leq n$ in (3.2), we get the desired result. □

Remark. On substituting the monotonically decreasing functions $f(x) = \exp(-x), -x, x^{-1}$ where $x > 0$ in theorem (2.1), we again arrive at the standard result $\sum_{i=1}^{n} p_i x_i \geq \prod_{i=1}^{n} x_i^{p_i}$, that is, arithmetic mean is greater than or equal to geometric mean.

In the next theorem, we will further improve the inequality (3.1).

Theorem 3.2. For any probability distribution $P = \{p_1, p_2, ..., p_n\}$, we have

$$0 \leq \max_{1 \leq r < s \leq n} \left(p_r \log \left(\frac{2p_r}{p_r + p_s} \right) + p_s \log \left(\frac{2p_s}{p_r + p_s} \right) \right) \leq \log n - H(P) \tag{3.3}$$

Proof. Since $f(x) = -\log x$ is a monotonically decreasing function in the interval $[1,\infty[$ which satisfies the relation $f(xy) = f(x) + f(y)$, substituting it in the theorem (2.3), we get

$$\begin{aligned} 0 &\leq \max_{1\leq r<s\leq n}\left(-\log\left(x_r^{p_r}\right) - \log\left(x_s^{p_s}\right) + \log\left(\left(\frac{p_r x_r + p_s x_s}{p_r + p_s}\right)^{(p_r+p_s)}\right)\right) \\ &\leq -\log\left(\prod_{i=1}^{n} x_i^{p_i}\right) + \log\left(\sum_{i=1}^{n} p_i x_i\right) \end{aligned} \tag{3.4}$$

Again, substituting $x_i = \frac{1}{p_i}, 1 \leq i \leq n$ in (3.4), we have

$$\begin{aligned} 0 &\leq \max_{1\leq r<s\leq n}\left(p_r \log p_r + p_s \log p_s + (p_r + p_s)\log\left(\frac{2}{p_r + p_s}\right)\right) \\ &\leq \log n - \sum_{i=1}^{n} p_i \log p_i \end{aligned}$$

that is,

$$0 \leq \max_{1\leq r<s\leq n}\left(p_r \log\left(\frac{2p_r}{p_r + p_s}\right) + p_s \log\left(\frac{2p_s}{p_r + p_s}\right)\right) \leq \log n - H(P)$$

□

Note. 1. Proceeding as above, the result (3.3) can also be achieved using theorem 2.2.

2. The result (3.3) have been achieved by Simic [9] but by using Jensen's [5] inequality.

The inequality (3.3) can further be improved by means of the following theorem:

Theorem 3.3. For any probability distribution $P = \{p_1, p_2, ..., p_n\}$, we have

$$0 \leq H(P) \leq \log n - \max_{1\leq r_1<r_2<\ldots<r_{n-1}\leq n} \log\left(\left(\frac{n-1}{\sum_{i=1}^{n-1} p_{r_i}}\right)^{\sum_{i=1}^{n} p_{r_i}} \prod_{i=1}^{n-1} p_{r_i}^{p_{r_i}}\right) \tag{3.5}$$

Proof. Applying theorem (2.5) with $f(x) = -\log x$, $x_i = \frac{1}{p_i}, i = 1, 2, ..., n$, we get the desired result. □

Note. The result (3.4) have been achieved by Tapus and Popescu [7] but by using Jensen's [5] inequality.

Theorem 3.4. For any probability distribution $P = \{p_1, p_2, ..., p_n\}$, we have

$$R_2(P) \leq H(P) \leq \log n \tag{3.6}$$

with equality if and only if all $p_i = \frac{1}{n}$ where $R_2(P) = -\log \sum_{i=1}^{n} p_i^2$ is the Renyi's [1] entropy of order 2 .

Proof. Applying theorem (2.6) with $f(x) = -\log x$, $x_i = \frac{1}{p_i}, i = 1, 2, ..., n$, and after simplification, we get the desired result. □

Theorem 3.5. For any probability distribution $P = \{p_1, p_2, ..., p_n\}$, we have

$$(i)\ L_\alpha = \frac{\alpha}{1-\alpha} \log_D \left(\sum_{i=1}^{n} p_i D^{\frac{1-\alpha}{\alpha} l_i} \right) \geq \sum_{i=1}^{n} p_i l_i = L,\ \alpha < 1, \alpha \neq 1 \tag{3.7}$$

$$(ii)\ L^\alpha = \frac{1}{\alpha - 1} \log_D \left(\frac{\sum_{i=1}^{n} p_i^\alpha D^{(\alpha-1)l_i}}{\sum_{i=1}^{n} p_i^\alpha} \right) \geq \frac{\sum_{i=1}^{n} p_i^\alpha l_i}{\sum_{i=1}^{n} p_i^\alpha} = L(\alpha), \alpha > 1, \alpha \neq 1 \tag{3.8}$$

where L_α is the Campbell's [6] mean codeword length, L^α is Kapur's [4] mean codeword length, $L(\alpha)$ is Parkash and Kakkar's [8] mean codeword length.

Proof. (i) Let f be a monotonically decreasing function , then using theorem (2.1), we have

$$f\left(\sum_{i=1}^{n} p_i D^{\frac{1-\alpha}{\alpha} l_i} \right) \leq f\left(\prod_{i=1}^{n} D^{\frac{1-\alpha}{\alpha} l_i p_i} \right),\ \alpha < 1 \tag{3.9}$$

Letting $f(x) = -\log_D x$ in (3.9), we get the desired result (3.7).

Note. The inequality sign reverses for $\alpha > 1$.

(ii) Let f be a monotonically decreasing function. Again, using theorem (2.1), we have

$$f\left(\frac{\sum_{i=1}^{n} p_i^\alpha D^{(\alpha-1)l_i}}{\sum_{i=1}^{n} p_i^\alpha} \right) \leq f\left(\prod_{i=1}^{n} D^{\frac{(\alpha-1) l_i p_i^\alpha}{\sum_{i=1}^{n} p_i^\alpha}} \right),\ \alpha > 1 \tag{3.10}$$

Letting $f(x) = -\log_D x$ in (3.10), we get

$$-\log_D \left(\frac{\sum_{i=1}^{n} p_i^\alpha D^{(\alpha-1)l_i}}{\sum_{i=1}^{n} p_i^\alpha} \right) \leq -\log_D \left(\prod_{i=1}^{n} D^{\frac{(\alpha-1) l_i p_i^\alpha}{\sum_{i=1}^{n} p_i^\alpha}} \right)$$

After simplification, we get the desired result (3.8).
Note. The inequality sign reverses for $\alpha < 1$. □

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