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NUMERICAL SOLUTION OF FRACTIONAL DELAY DIFFERENTIAL EQUATIONS USING SPLINE FUNCTIONS

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Abstract: In this paper, we investigate the use of suitable spline functions of polynomial form to approximate the solution of fractional delay differential equations. The description of the proposed approximation method is first introduced. The error analysis and stability of the method are theoretically investigated. The proposed spline function is an extension and generalization to the polynomial spline functions used in [15]. Numerical example is given to illustrate the applicability, accuracy and stability of the proposed method.

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1. Introduction

In the last few years there has been increasing interest in the use of various types of spline function in the numerical treatment of ordinary differential equations [10],[11],[12],[19] and delay differential equations [15],[16],[17]. Recently, fractional order differential equation has found interesting applications in the area of mathematical biology [1],[9] and [2] due to the relation of such equations with memory that is inherit in corresponding biological systems. Also, ordinary or partial differential equations of fractional order have useful applications in numerous engineering and physical phenomena.

The analysis of fractional differential equations (Existence, Uniqueness and Stability) of the form

$$y^{(\alpha)}(x) = f(x, y(x), y(g(x))), \qquad y(0) = y_0 \tag{1}$$

is studied by Kia Dithelm and Neville J. Ford, see [4]. A number of approximate solutions of equation (1) has been proposed in the literature, where the Adams-Bashforth-Moulton method is introduced in [6] and [7]. An alternative method is the backward differentiation formula of [5]. Notice that this method is based on the idea of discrediting the differential operator in the given equation (1) by certain finite difference. The main result of [5] was that under suitable assumption we can expect and $o\left(h^{\alpha-2}\right)$ convergence behavior. In [8] an improvement of the performance of the method presented in [5] is achieved by applying extrapolation principles. Kia Dithelm et. al. in [9] are considered. A fast algorithm for the numerical solution of initial value problems of the form equation (1) in the sense of caputo identify and discuss potential problems in the development of generally applicable schemes. More recently, Lagrange multiplier method and the homotopy perturbation method are used to solve numerically multi-order fractional differential equation see [14]. Micula, et al. [13] considered the problem

$$\begin{aligned} \acute{y}(x) &= f_1(x, y, z), \qquad & y(x_0) = y_0, \\ \acute{z}(x) &= f_2(x, y, z), \qquad & z(x_0) = z_0 \end{aligned} \tag{2}$$

where $f_1, f_2 \in C^r\,[0,1] \times R \times R, \qquad (x, y, z) \in [0,1] \times R \times R.$

They assumed that the functions $f_i^{(q)}, i = 1,2$ and $q = 0,1,2,...,r$ satisfy the Lipschitz conditions of the form:

$$\left| f_i^{(q)}(x, y_1, z_1) - f_i^{(q)}(x, y_2, z_2) \right| \leq L_i \{ |y_1 - y_2| + |z_1 - z_2| \}$$

with constant L_i for all $(x, y_i, z_i) \in [a, b] \times R \times R$.

An investigation to the extension of the spline functions form defined in [13] for approximating the solution of system of ordinary differential equations, namely, for the system (2) with unique solution $y = y(x), z = z(x)$ is also considered. The spline functions $S_\triangle(x)$ and $\check{S}_\triangle(x)$ to approximate $y = y(x), z = z(x)$are defined in polynomial form as:

$$S_\triangle(x) = S_k(x) = S_{k-1}(x_k) + \sum_{i=0}^{r} f_1^{(i)}(x_k, S_{k-1}(x_k), \check{S}_{k-1}(x_k)) \frac{(x - x_k)^{i+1}}{(i+1)!}$$

$$\check{S}_\triangle(x) = \check{S}_k(x) = \check{S}_{k-1}(x_k) + \sum_{i=0}^{r} f_2^{(i)}(x_k, S_{k-1}(x_k), \check{S}_{k-1}(x_k)) \frac{(x - x_k)^{i+1}}{(i+1)!}$$

for $x \in [x_k, x_{k-1}]$, $k = 0, 1, ..., n-1$, $S_{-1}(x_0) = y_0$, $\check{S}_{-1}(x_0) = z_0$.

Ramadan, M. A. introduced in [15] the solution of the first order delay differential equation of the form:

$$y^{\prime}(x) = f(x.y(x), y(g(x))), \qquad a \leq x \leq b,$$

$$y(a) = y_0, \ y(x) = \phi(x), \quad x \in [a^*, a], \ a^* < 0, \ a^* = \inf\{g(x) : x \in [a, b]\}$$

using the spline function of the polynomial form which defined as:

$$S_\triangle(x) = S_k(x) = S_{k-1}(x_k) + \sum_{i=0}^{r} M_k^{(i)} \frac{(x - x_k)^{i+1}}{(i+1)!}$$

where $M_k^{(i)} = f^{(i)}(x_k, S_{k-1}(x_k), S_{k-1}(g(x_k)))$ with $S_{-1}(x_0) = y_0$, $S_{-1}(g(x_0)) = \phi(g(x_0))$.

Ramadan, Z. in [18] considered the system of the initial value problem

$$y^{\prime\prime\prime}(x) = f_1(x, y, y^{\prime}, z, z^{\prime}), \qquad y^{(i)}(x_0) = y_0^{(i)},$$

$$z^{\prime\prime\prime}(x) = f_2(x, y, y^{\prime}, z, z^{\prime}), \qquad z^{(i)}(x_0) = z_0^{(i)},$$

where $f_1, f_2 \in C^r[0,1] \times R^4, i = 0,1,2$. A method is presented which uses polynomial splines to approximate the solutions of the system.

The purpose of this paper is to extend and generalize the polynomial spline functions used in the case of [15] to solve the fractional delay ordinary differential equation.

2. Description of the Proposed Spline Approximation Method

Consider the fractional ordinary delay differential equation of the form

$$y^{(\alpha)}(x) = f(x, y(x), y(g(x))), \qquad a \le x \le b \tag{3}$$

$$y^{(\alpha)}(x_0) = y_0^{(\alpha)}, \; y(x) = \phi(x), \;\; x \in [a^*, a], \; \alpha \in (0,1)$$

where the function g is called the delay function and it is assumed to be continuous on the interval $[a, b]$ and satisfies the inequality $a^* \le g(x) \le x, \; x \in [a, b]$ and $\phi \in C[a^*, a]$.

Suppose that $f : [a, b] \times R^2 \to R$ is continuous and satisfies the Lipschitz conditions

$$|f(x, y_1, v_1) - f(x, y_2, v_2)| \le L\{|y_1 - y_2| + |v_1 - v_2|\}$$

and the following estimate holds:

$$\left|f^{(\alpha)}(x, y_1, v_1) - f^{(\alpha)}(x, y_2, v_2)\right| \le L\{|y_1 - y_2| + |v_1 - v_2|\} \tag{4}$$

for all (x, y_1, v_1) and $(x, y_2, v_2) \in [a, b] \times R^2$ all Moreover, there exists a constant B so that

$$|v_1 - v_2| \le B \; \left|f^{(\alpha)}(x, y_1, v_1) - f^{(\alpha)}(x, y_2, v_2)\right|$$

with $LB < 1$ for all (x, y_1, v_1) and (x, y_2, v_2) in the domain of definition of the function f.

These conditions assure the existence of unique solution of y of equation (3). Let $\triangle$ be a uniform partition to the interval $[a, b]$ defined by the nodes

$$\triangle : a = x_0 < x_1 < ... < x_k < x_{k+1} < ... < x_n = b, x_k = x_0 + kh, h = \frac{b-a}{n} < 1$$

and $k = 0, 1, ..., n-1$.

Define the new form of fractional spline function of polynomial form approximating the exact solution y by:

$$S_\triangle(x) = S_k(x) = S_{k-1}(x_k) + \sum_{i=0}^{r} M_k^{(i\alpha)} \frac{(x - x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha + 1)}, \tag{5}$$

where $M_k^{(\alpha)} = f^{(\alpha)}(x_k, S_{k-1}(x_k), S_{k-1}(g(x_k)))$ with $S_{-1}(x_0) = y_0$, $S_{-1}(g(x_0)) = \phi(g(x_0))$, for $x \in [x_k, x_{k+1}]$. Such $S_k(x)$ exists and is unique.

3. Error Estimation and Convergence Analysis

To estimate the error of the approximate solution, we write the exact solution $y(x)$ in the following Taylor form [3]:

$$y(x) = \sum_{i=0}^{r} y_k^{(i\alpha)} \frac{(x - x_k)^{i\alpha}}{\Gamma(i\alpha + 1)} + y^{((r+1)\alpha)}(\zeta_k) \frac{(x - x_k)^{(r+1)\alpha}}{\Gamma((r+1)\alpha + 1)} \tag{6}$$

where $\zeta_k \in (x_k, x_{k+1})$and $y_k = y(x_k)$. Moreover, we denote to the estimated error of $y(x)$ at any point $x \in [a, b]$ by

$$e_\alpha(x) = |y(x) - S_k(x)|$$

and at x_k by

$$e_{k,\alpha} = |y_k - S_{k-1,\alpha}(x_k)| \tag{7}$$

Define the modulus of continuity of $\omega\left(y^{((r+1)\alpha)}(\zeta_k, h)\right)$ as follows:

$$\omega\left(y^{((r+1)\alpha)}(\zeta_k, h)\right) = \max_{\zeta_k \in [a,b]} \left(\left|\left(y^{((r+1)\alpha)}(\zeta_k + h)\right) - y^{((r+1)\alpha)}(\zeta_k)\right|\right).$$

Next lemma gives an upper bound to the error.

Lemma 1. *Let $e_\alpha(x)$ be defined as in equation (7) then there exists a constant d_2 independent of h such that the following inequality:*

$$e_\alpha(x) \le (1 + d_2 h)\, e_k + \omega(h) \frac{h^{(r+1)\alpha}}{\Gamma((r+1)\alpha + 1)}$$

holds for all $x \in [a, b]$ where $d_2 = \sum\limits_{i=0}^{r} \frac{d_1}{\Gamma((i+1)\alpha+1)}$.

Proof. Using our estimate in equation (4), Taylor expansion, definition of error estimation and equation (7) we get, by dropping α :

$$\begin{aligned} e(x) &= |y(x) - S_k(x)| \\ &= \left|(y_k - S_{k-1}(x_k)) + \sum_{i=1}^{r} y_k^{(i\alpha)} \frac{(x - x_k)^{i\alpha}}{\Gamma(i\alpha + 1)} - \sum_{i=0}^{r} M_k^{(i\alpha)} \frac{(x - x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha + 1)} \right. \\ &\quad \left. + y^{((r+1)\alpha)}(\zeta_k) \frac{(x - x_k)^{(r+1)\alpha}}{\Gamma((r+1)\alpha + 1)} \right|, \end{aligned}$$

$$e\left(x\right)=\left|\left(y_k-S_{k-1}\left(x_k\right)\right)+\sum_{i=0}^{r-1}y_k^{((i+1)\alpha)}\frac{\left(x-x_k\right)^{(i+1)\alpha}}{\Gamma\left(\left(i+1\right)\alpha+1\right)}\right.$$
$$-\sum_{i=0}^{r-1}M_k^{(i\alpha)}\frac{\left(x-x_k\right)^{(i+1)\alpha}}{\Gamma\left(\left(i+1\right)\alpha+1\right)}$$
$$\left.+\,y^{((r+1)\alpha)}\left(\zeta_k\right)\frac{\left(x-x_k\right)^{(r+1)\alpha}}{\Gamma\left(\left(r+1\right)\alpha+1\right)}-M_k^{(r\alpha)}\frac{\left(x-x_k\right)^{(r+1)\alpha}}{\Gamma\left(\left(r+1\right)\alpha+1\right)}\right|,$$

hence

$$\begin{aligned} e\left(x\right) \quad \leq \quad & e_k+\sum_{i=0}^{r-1}\left|y_k^{((i+1)\alpha)}-M_k^{(i\alpha)}\right|\frac{\left|\left(x-x_k\right)^{(i+1)\alpha}\right|}{\Gamma\left(\left(i+1\right)\alpha+1\right)}+ \\ & \left|y^{((r+1)\alpha)}\left(\zeta_k\right)-M_k^{(r\alpha)}\right|\frac{\left|\left(x-x_k\right)\right|^{(r+1)\alpha}}{\Gamma\left(\left(r+1\right)\alpha+1\right)}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \left|y_k^{((i+1)\alpha)}-M_k^{(i\alpha)}\right| &=\left|f^{(i\alpha)}(x_k,y_k,y_k(g\left(x_k\right)))-f^{(i\alpha)}(x_k,S_{k-1}\left(x_k\right),S_{k-1}(g\left(x_k\right)))\right| \\ &\leq\frac{L}{1-LB}\left\{\left|y_k-S_{k-1}\left(x_k\right)\right|\right\}\leq d_1e_k \end{aligned}$$

where $d_1=\frac{L}{1-LB}$ .Similarly,

$$\begin{aligned} \left|y^{((r+1)\alpha)}\left(\zeta_k\right)-M_k^{(r\alpha)}\right| \quad &\leq \quad \left|y^{((r+1)\alpha)}\left(\zeta_k\right)-y_k^{((r+1)\alpha)}\right|+\left|y_k^{((r+1)\alpha)}-M_k^{(r\alpha)}\right| \\ &\leq \quad \omega\left(h\right)+d_1e_k \end{aligned}$$

where $d_1>0$ is constant independent of h, $\omega\left(h\right)$ is the modulus of continuity of $\omega\left(y^{((r+1)\alpha)}\left(\zeta_k,h\right)\right)$and $\left|\left(x-x_k\right)\right|<h<1$. The inequality (8) is then reduced to

$$e\left(x\right)\leq e_k+\sum_{i=0}^{r-1}d_1e_k\frac{\left|\left(x-x_k\right)^{(i+1)\alpha}\right|}{\Gamma\left(\left(i+1\right)\alpha+1\right)}+\left(\omega\left(h\right)+d_1e_k\right)\frac{\left|\left(x-x_k\right)\right|^{(r+1)\alpha}}{\Gamma\left(\left(r+1\right)\alpha+1\right)}$$

$$=\left(1+d_2h\right)e_k+\omega\left(h\right)\frac{h^{(r+1)\alpha}}{\Gamma\left(\left(r+1\right)\alpha+1\right)},$$

where $d_2=\sum\limits_{i=0}^{r}\frac{d_1}{\Gamma((i+1)\alpha+1)}$ is constant independent of h. The lemma is proved.

□

4. Stability Analysis of the Proposed Method

For analyzing the stability properties of the given method, we make a small change of the starting values and study the changes in the numerical solution produced by the method.

Now, we define the spline approximating function $\check{S}_{\triangle}$as:

$$\check{S}_{\triangle}(x) = \check{S}_k(x) = \check{S}_{k-1}(x_k) + \sum_{i=0}^{r} \tilde{M}_k^{(i\alpha)} \frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)}, \tag{9}$$

where

$$\tilde{M}_k^{(\alpha)} = f^{(\alpha)}(x_k, \check{S}_{k-1}(x_k), \check{S}_{k-1}(g(x_k)))$$

with $\check{S}_{-1}(x_0) = y_0^*$,

$$\check{S}_{-1}(g(x_0)) = \phi^*(g(x_0)),$$

for $x \in [x_k, x_{k+1}]$, $k = 0, 1, ..., n-1$, and use the notation

$$e_\alpha^*(x) = \left| S_k(x) - \check{S}_k^{(\alpha)}(x) \right|$$

and

$$e_{k,\alpha}^* = \left| S_{k-1,\alpha}(x_k) - \check{S}_{k-1,\alpha}(x_k) \right|. \tag{10}$$

Lemma 2. *Let $e_\alpha^*(x)$ be defined as in equation (10), then the inequality*

$$e_\alpha^*(x) \leq (1 + dh)\, e_k^*$$

holds where $d = \sum\limits_{i=0}^{r} \frac{d_1}{\Gamma((i+1)\alpha+1)}$ is a constant independent of h .

Proof. Using equations (5), (9) and (10) we get, by dropping α :

$$\begin{aligned} e^*(x) &= \left| S_k(x) - \check{S}_k(x) \right| \\ &= \Bigg| \left(S_{k-1}(x_k) - \check{S}_{k-1}(x_k) \right) + \sum_{i=0}^{r} M_k^{(i\alpha)} \frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \\ &\quad - \sum_{i=0}^{r} \tilde{M}_k^{(i\alpha)} \frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \Bigg| \\ &\leq e_k^* + \sum_{i=0}^{r} \left| M_k^{(i\alpha)} - \tilde{M}_k^{(i\alpha)} \right| \frac{\left| (x-x_k)^{(i+1)\alpha} \right|}{\Gamma((i+1)\alpha+1)}, \end{aligned}$$

α	x	*Appr. solution*	Absolute Error	*Absolute diff.*
0.1	0.01	$y = 0$	0	3.36474×10^{-8}
	0.02	$y = 0.0000755819$	3.2×10^{-4}	1.13155×10^{-6}
	0.03	$y = 0.000265263$	6.3×10^{-4}	1.74622×10^{-6}
	0.04	$y = 0.000557029$	10.4×10^{-4}	2.13142×10^{-6}
	0.05	$y = 0.000945615$	15.5×10^{-4}	2.33393×10^{-6}
0.2	0.01	$y = 0$	0	3.23815×10^{-8}
	0.02	$y = 0.0000807984$	2.7×10^{-4}	1.16448×10^{-6}
	0.03	$y = 0.000273582$	6.3×10^{-4}	1.61947×10^{-6}
	0.04	$y = 0.000560054$	10.4×10^{-4}	1.74571×10^{-6}
	0.05	$y = 0.000932151$	15.7×10^{-4}	1.61128×10^{-6}
0.3	0.01	$y = 0$	0	4.1998×10^{-8}
	0.02	$y = 0.0000896535$	3.1×10^{-4}	1.18278×10^{-6}
	0.03	$y = 0.000287934$	6.1×10^{-4}	1.41014×10^{-6}
	0.04	$y = 0.000570342$	10.3×10^{-4}	1.22713×10^{-6}
	0.05	$y = 0.000926655$	15.7×10^{-4}	7.28621×10^{-7}
0.4	0.01	$y = 0$	0	6.66285×10^{-8}
	0.02	$y = 0.000101184$	2.9×10^{-4}	1.17189×10^{-6}
	0.03	$y = 0.000305454$	6.0×10^{-4}	1.1172×10^{-6}
	0.04	$y = 0.000583095$	10.2×10^{-4}	5.9927×10^{-7}
	0.05	$y = 0.000922591$	15.7×10^{-4}	2.61135×10^{-7}

Table 1: Accuracy and stability of the proposed method using spline function of polynomial form

$$e^{*}\left(x\right) \leq e_{k}^{*} + \sum_{i=0}^{r} \left| M_{k}^{(i\alpha)} - \tilde{M}_{k}^{(i\alpha)} \right| \frac{h^{(i+1)\alpha}}{\Gamma\left(\left(i+1\right)\alpha+1\right)} \tag{11}$$

but

$$\left| M_{k}^{(i\alpha)} - \tilde{M}_{k}^{(i\alpha)} \right| \leq \frac{L}{1-LB} e_{k}^{*} \leq d_{1} e_{k}^{*} \tag{12}$$

Thus from inequalities (11) and (12) we obtain:

$$e^{*}\left(x\right) \leq e_{k}^{*} + d e_{k}^{*} h \leq \left(1+dh\right) e_{k}^{*}.$$

where $d = \sum\limits_{i=0}^{r} \frac{d_1}{\Gamma((i+1)\alpha+1)}$ is constant independent of h and the lemma is proved.

$\square$

5. Numerical Example

To demonstrate the applicability, accuracy and stability of the proposed method, we consider the following numerical

Example 3. Consider the fractional delay differential equation

$$D^{\alpha}y\left(x\right)=-y\left(x\right)+y\left(\frac{x}{2}\right)+\frac{3}{4}x^{2}+\frac{2}{\Gamma\left(3-\alpha\right)}x^{2-\alpha}.$$

By simple calculation, the exact solution is $y=x^{2}$.

On the other hand, the obtained numerical results of this example are summarized in Table 1. The accuracy and stability of the proposed spline method using spline function of polynomial form are illustrated where, the first column represents the different values of α, the second column represents the values of x, the third column gives the approximate solution at the corresponding points while the fourth column gives the absolute error between the exact solution and the obtained approximate numerical solution with the initial conditions $y(0)=0$. With small change in the initial conditions, $y^{*}(0)=0.0001$. To test the stability, the difference between the approximate solution and the perturbed problem is computed as shown in the fifth column.

From the obtained results for the test example, one can see that the proposed method gives an accepted accuracy as well as it is stable.

6. Conclusion

In this paper, we investigated the possibility of extending and generalizing the spline functions of polynomial form given in Ramadan M. A. [15] to be in fractional form with some additional assumptions and definitions for approximating the solution of fractional ordinary differential equation. The error analysis and stability are investigated. Numerical example is given to illustrate the applicability, accuracy and stability of the proposed method. The obtained numerical results are in good agreement with the exact analytical solutions. Also, they reveal that the method is stable.

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