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TWO CASES OF RECONSTRUCTION OF SEPARABLE GRAPHS

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Abstract: The author has shown that a separable graph with more than one block which is not an edge is reconstructible. Here reconstructibility is proved for a single such block, in two particular cases.

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1. Introduction

Say that a vertex v is indicated in a tree if the tree is equipped with a 0-1 labeling of the vertices which is 1 only at v. Define a twig of a graph G to be an edge incident to a vertex of degree 1. If G is not a tree, the trunk of G is defined to be the graph obtained by successively removing twigs until none remain. Define a limb of G to be a maximal subtree, with the vertex contained in the trunk indicated. Note that G is the trunk, with the limbs attached at distinct vertices.

In [4] it is shown that trees are reconstructible. In [1] it is shown that the limbs of a separable graph are reconstructible. In [3] it is shown that if the trunk of G has more than one block then G is reconstructible.

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From hereon, assume that G has a single block B which is not an edge, and at least one limb. By the size of a limb is meant its number of edges. Note that B can be determined, by considering any G_v where v is a degree 1 vertex.

Theorem 1. *If there is an $s > 1$ such that there are limbs with s vertices but none with $s-1$ vertices then G is reconstructible.*

Proof. This was observed in [1]. For a proof, let v be any vertex such that G_v has a limb of size $s-1$. G may be reconstructed by replacing the limb of size $s-1$ by the missing limb of size s. □

A limb of size 1 will be called a 1-limb. From hereon it will be assumed that there is at least one 1-limb. Let $\mathcal{T}_{ds}$ be the multiset of limbs of size s attached to vertices if degree d.

Theorem 2. *Except in the case of a single limb of size 1, the family of $\mathcal{T}_{ds}$ is reconstructible.*

Proof. If there is no 1-limb the result follows by theorem 1. Let v be a degree 1 vertex of a 1-limb; then each $\mathcal{T}_{ds}$ for $s > 1$ may be determined from G_v. When there is more that one 1-limb it is readily seen that the family of $\mathcal{T}_{d1}$ may be determined from this family in G_v as v ranges over the degree 1 vertices of 1-limbs. In there is a single 1-limb and some other limb, let v be a degree 1 vertex of a limb of least size greater than 1; then in G_v there are one or two 1-limbs, and the degree of the vertex where the 1-limb occurs in G may be determined in either case. □

2. The Case of a Cycle

In this section let B be a cycle with b vertices. Let k be the number of limbs of size greater than 1.

Lemma 3. *Suppose $k \geq 3$. Then G is reconstructible.*

Proof. Let v be the degree 1 vertex of a 1-limb. In G_v let P_1 be the path between roots of limbs of size greater than 1, which includes all such roots, and let P_2 be the other path between the endpoints of P_1. Choose an orientation of P_1, and let $T_1, \ldots, T_k$ be the limbs of size greater than 1 in order of their roots along P_1. Let p_t be the number of vertices in P_t for $t = 1, 2$. If $p_1 \neq p_2$ or $k \geq 4$ or T_2 has more than 2 vertices, a vertex w can be found in a T_i with $1 < i < k$, so that P_1 can be found in G_w. If there is no ambiguity in the orientation of P_1

in G_w, G can be reconstructed by replacing $T_i - w$ in G_w with T_i. Otherwise, the T_j must be symmetric, with k odd and $i = (k+1)/2$. Again the graph may be reconstructed by replacing $T_i - w$ in G_w with T_i.

Say that w is good if it is a degree 1 vertex of some T_t, and it can be determined in G_w which limb equals $T_t - w$. If w is good then G may be reconstructed by replacing $T_j - w$ in G_w by T_t.

Suppose $p_1 = p_2$, $k = 3$, and T_2 has 2 vertices. Let t_t be the size of T_t for $t = 1, 3$. Let j be such that $t_j \leq t_{4-j}$. If $t_j \geq 3$ then a good vertex may be found in T_j. If $t_{4-j} \geq 4$ then a good vertex may be found in T_{4-j}. If $t_j = 2$ and $t_{4-j} = 3$. a good vertex may be found in T_j.

In the remaining case, $t_j = 2$ and $t_{4-j} = 2$. If either T_1 or T_3 is different from T_2, a good vertex may be found in T_t for $t = 1$ or $t = 3$, by ensuring that T_{4-j} differs from T_2. Let p_{12} be the distance from the root of T_1 to the root of T_2, and similarly for p_{23}. If $p_{12} \neq p_{23}$ then a good vertex may be found in T_1 (or T_3).

If $p_{12} = p_{23}$, choose w in T_2. If only one path from the root of T_1 to the root of T_3 has an edge attached to its midpoint, replace the edge with T_2. Otherwise let w be the degree 1 vertex of the edge attached to the midpoint of T_2; w can be found, and an edge attached to the correct vertex in G_w. □

Lemma 4. *Suppose $k = 2$. Then G is reconstructible.*

Proof. Let T_1, T_2 be the limbs and t_1, t_2 their sizes. By theorem 1, it mat be assumed that either $t_1 = 2$ and $t_2 = 3$, or $t_1 = t_2 = 2$. Let $p_1 \leq p_2$ be the lengths of the paths between the roots of T_1 and T_2; these may be determined from G_v where v is the degree 1 vertex of any 1-limb.

If $p_1 = p_2$ let v be a degree 1 vertex of T_1. In G_v the 1-limb which needs to be replaced by T_1 may be determined, so G may be reconstructed by performing the replacement. By considering degree 1 vertices of 1-limbs it may be determined whether one of the paths has no 1-limbs attached. If so let v be the degree 1 vertex of T_1. In G_v, let r be the root of T_2 in G_v and let s be the root of a 1-limb, such that one of the paths between r and s contains all roots of 1-limbs. G may be reconstructed by replacing the 1-limb with root s by T_1.

Suppose $p_1 < p_2$ and both paths have 1-limbs attached. By considering the G_v with v a degree 1 vertex of a 1-limb, P_1 and P_2 may be determined, where these are the paths between the roots of T_1, T_2, with P_i of length p_i, with all limbs attached. If both P_1 and P_2 contain a single 1-limb, let v be a degree 1 vertex of T_1; in G_v, replace the "middle" 1-limb by T_1.

Otherwise, let P be one of P_1, P_2 with more than 1 1-limb. Let t_i be the 1-limb nearest T_i, and let d_i be the distance between the roots of t_i and T_i. If

$d_1 = d_2$ let v be the degree 1 vertex of either t_1 or t_2; G may be reconstructed from G_v by adding a 1-limb at the appropriate position in P. Otherwise, renumbering if needed, suppose $d_1 < d_2$. Let t_3 be the next 1-limb after t_1 starting at T_1. if $d_3 \neq d_2$ then v may be chosen as the degree vertex of t_1, and t_1 added to G_v. If $d_3 = d_2$ then v may be chosen as the degree 1 vertex of t_2 (even if $t_2 = t_3$), and t_2 added to G_v. □

Lemma 5. *Suppose $k = 1$. Then G is reconstructible.*

Proof. Let T be the limb of size greater than 1; by theorem 1 T may be assumed to have size 2. Let v be a vertex of degree 1 in T. If there is only 1 1-limb t, the two distances from t to T may be determined by considering G_v. Let w be the degree 1 vertex of t; then t may be added to G_w. If there is more than 1 1-limb, the two closest to T may be found, say t_1 and t_2, with distances $d_1 \leq d_2$. The rest of the argument is a variation of the argument given for P in the proof of lemma 4. □

Lemma 6. *Suppose $k = 0$. Then G is reconstructible.*

Proof. Let v range over the roots of 1-limbs. If for any G_v the path is of length $b-2$ or b, G may be reconstructed from such a G_v. Otherwise, G consists of the circular concatenation of paths of length 3, where the degree 1 vertices have no 1-limb and the degree 2 vertices have a 1-limb. □

Theorem 7. *If B is a cycle then G is reconstructible. If there are limbs of size greater than 1 then G is reconstructible from $\{G_v : v$ is a vertex of degree 1$\}$.*

Proof. The first claim follows by lemmas 3, 4, 5, and 6. The second claim follows by examining the proofs. □

3. The Case of 3 Paths

It is well-known (proposition 3.1.1 of [2]) that any 2-connected graph can be built up from a cycle by successively attaching paths. Thus, a natural case to consider for B is that of a cycle, with one path added. Such a graph may be characterized as a pair of vertices, with 3 paths between them; or as a graph with two vertices of degree 3 and all other vertices of degree 2.

Suppose B is such a graph. Let $a \leq b \leq c$ be the path lengths. Call a limb an end limb if it is attached to a degree 3 vertex, else an internal limb.

By theorem 2, except in the case of a single limb of size 1, the end limbs and internal limbs are determined.

Lemma 8. *If there is a single limb, of size 1, then G is reconstructible.*

Proof. Let v range over the degree 3 vertices. Let w be the degree 1 vertex of the limb. If the limb is an end limb, attach a 1-limb to a degree 3 vertex in G_w. Otherwise, choose one of the G_v, and let α, β, γ be the path lengths. There is exactly one of the values a, b, c which is not among $\alpha + 1, \beta + 1, \gamma + 1$; let x denote it. G may be reconstructed from G_v by adding a vertex, adding an edge to the neighbor of the end of a path of length x, and adding edges to the ends of the other paths. □

Lemma 9. *If there are any end limbs then G is reconstructible.*

Proof. Let e be the sum of the sizes of the end limbs. Suppose $e \geq 2$. Let v be a degree 1 vertex of a smaller of the two end limbs (or the only end limb). The end at which the missing end limb needs to be attached is readily determined.

Suppose $e = 1$; by lemma 8 it may be assumed that there are internal limbs. Letting s denote the sum of the sizes on a path, let P be such that s takes its least nonzero value. Let w be a degree 1 vertex of a limb of P, closest to the end with the limb attached; G is readily reconstructed from G_w. □

For the rest of the section, suppose all limbs are internal. Let s be the sum of the sizes of the limbs on a path. Writing a letter for $s \geq 2$, there are 14 possibilities for the size triples in nondecreasing order:
001, 00r, 011, 01r, 0rr, 0rs, 111, 11r, 1rr, 1rs, rrr, rrs, rss, rst

Each size triple gives rise to a list of size triples, of the G_v where v ranges over degree 1 vertices. By direct computation, these lists are distinct, and the sizes determined, except in the case 002 and 011. Which of these occurs is known since the limbs are. The case 001 has already been proved.

Lemma 10. *If the size triple contains two 0's then G is reconstructible.*

Proof. Let P denote the path containing limbs. If there are limbs of size greater than 1, let v be a degree 1 vertex in such a limb, and let w be the degree 1 vertex of a 1-limb. It is readily verified that G can be reconstructed from G_v by comparing P in G_v and G_w.

Suppose all limbs are 1-limbs. Since 001 is already proved, P may be located in G_v where v is an end vertex. If there is no missing 1-limb in G_v connect a new vertex to all end vertices of the paths in G_v. Otherwise, connect the new vertex to the neighbor of the end vertex of P instead. □

Lemma 11. *If the size triple contains one 0 then G is reconstructible.*

Proof. By letting v range over degree 1 vertices, the length l_0 of the path with no limbs may be determined.

Suppose the size triple is $01r$, and all limbs are 1-limbs. Let v range over the end vertices. If there is a v such that no limbs are missing in G_v, G may be reconstructed from G_v by adding a vertex and edges to the ends of all 3 paths. If there is a v such that two limbs are missing in G_v, G may be reconstructed from G_v by adding a vertex, an edge to a path of length $l_0 - 1$, and edges to the neighbors of the ends of the other 2 paths. If for both v one limb is missing in G_v, choose the one with the most 1-limbs on one of the paths; add an edge to the end of this path, and the neighbor of the end of the other path.

Suppose the size triple is 012, and there is a limb of size 2. Proceed as in the previous case, where the limb of size 2 is considered missing if it is either completely missing or has become a 1-limb.

Suppose the size triple is $01r$ where r is greater than 2, and there is a limb of size greater than 1. Let w be the degree 1 vertex of the 1-limb t on the path containing just this limb. Let v_1 be a degree 1 vertex of a limb T_1 of size greater than 1, of distance d from the closest end vertex. In G_w, the path P of size r may be determined. In G_{v_1}, if this may be laid over the path of size $a - 1$ in only one way, G has been reconstructed. Otherwise, there is a limb T_2 at distance d from the other end, and the remaining limbs are symmetric. If $T_1 = T_2$ or t is at the midpoint of its path then P may be overlaid either way. Otherwise the choice may be made by considering G_{v_1} and G_{v_2}.

Suppose the size triple is $0rr$ or $0rs$, and all limbs are 1-limbs. Proceed as in the case of $01r$. If for both v one limb is missing in G_v, choose the one with the smallest number of limbs on a path. Add an edge to the neighbor of the end vertex of the path of size $r - 1$, and to the end of the longer path.

Suppose the size triple is $0rr$ or $0rs$, and there are limbs of size greater than 1. Let C be the cycle of the two paths with limbs. C may be determined in any G_v where v is a degree 1 vertex. The argument of theorem 7 may be adapted. Examination of the proof shows that only the case of a single limb of size 2 and a single 1-limb requires further argument. This case has already been handled. □

Lemma 12. *If the size triple of G is 111 then G is reconstructible.*

Proof. Let v be an end vertex. In G_v connect an added vertex to the ends of paths of size 0, and the neighbor of the end in paths of size 1. □

Lemma 13. *If the size triple of G is $11r$ then G is reconstructible.*

Proof. Suppose $r \geq 3$. Let w be a degree 1 vertex of a 1-limb on a path of size 1. Let v be a degree 1 vertex of a limb on the path P of size r. P may be determined from G_w. The other two paths C may be determined from G_v. If C or P is symmetric G is readily reconstructed from G_v. In the remaining case the orientation of P overlaying P in G_v is determined.

Suppose $r = 2$. If there is a limb of size 2 and the length of its path is greater than 2, let v be an end vertex where there is a limb of size 2 in G_v; G is readily reconstructed from G_v. If there is a limb of size 2 and the length of its path is 2, let v be a degree 1 vertex of the limb of size 2 in G_v; G is readily reconstructed from G_v. Suppose all limbs are 1-limbs; let v range over the end vertices. If there is a v such that in G_v there is a path of size 2, G is readily reconstructed from G_v. Otherwise let v be such that the number of 1-limbs adjacent to an end vertex is as large as possible; G is readily reconstructed from G_v. □

Lemma 14. *If the size triple of G is $1rr$ then G is reconstructible.*

Proof. Suppose $r \geq 3$. Let w be the degree 1 vertex of the 1-limb t on the path P of size 1. Let v be a degree 1 vertex of a limb on a path of size r. The paths C of size r, with their ends marked, may be determined from G_w. The path P_1 of size 1 may be determined from G_v. If the limb is at the middle of P_1, G may be reconstructed from G_w.

Otherwise, there are 2 or 4 possible limbs in G_w which can cover the missing limb in G_v, according to whether it is at the middle of its path or not, in accordance with the possibilities for covering C in G_v. Some of these possibilities may be eliminated sine only some of the possible coverings are allowed. The remaining ones which occur in G_v may also be eliminated, leaving a single possibility.

Suppose $r = 2$. Let w be as above, and let l be the length of P_1. If $l = 2$ then G is readily reconstructed from G_v. If $l \geq 4$ there is a vertex x where G_x is a cycle with a limb of size at least 3; G is readily reconstructed from G_x.

If $l = 3$ there is an end vertex v where G_v has a path of length 2 with a 1-limb. Now, it is readily verified that if v_1 and v_2 are the end vertices of C. If C_{v_1} and C_{v_2} are isomorphic then C is symmetric. In this case a 1-limb may be added at either internal vertex of the path of size 0 in G_w. Otherwise, C_v in

G_v can be matched with the correct C_{v_t}, and G is readily reconstructed from G_v. □

Lemma 15. *If the size triple of G is $1rs$ then G is reconstructible.*

Proof. Let w be the degree 1 vertex of the 1-limb t on the path P of size 1. Let v be a degree 1 vertex of a limb on the path of length s. The paths of size greater than 1, with their ends marked, may be determined from G_w. The path P_1 of size 1 may be determined from G_v. If the limb is at the middle of P_1, G may be reconstructed from G_w.

If $r \geq 3$ let v_1 be a degree 1 vertex of a limb T_1 of the path of length r. If G_w may be laid over G_{v_1} then G is reconstructed. Otherwise, T_1 has an alternative T_t; which should be chosen can be determined, since t is not in the middle of T_1.

If $r = 2$ let x be the end vertex nearest t. By considering G_y for y a degree 1 vertex of a limb on the path of length r, the distance of these limb(s) from x may be determined. □

Lemma 16. *If the size triple is rrr, then G is reconstructible.*

Proof. Let v range over degree 1 vertices. In each G_v there is a cycle C with two paths of size r. Up to isomorphism, there are 1, 2, or 3 distinct cycles. Orient each of these, so that there is a top and bottom branch, and a left and right end vertex. Let P^r denote the reverse of the path P.

If there is only one cycle C, a second copy may be matched to the first as either $C_b = C_t$, $C_b = C_t^r$, $C_b = C_b^r$, or $C_b = C_b^r$. If $C_b = C_t$ then all 3 paths are equal. If $C_b = C_t^r$ then all 3 paths are equal and symmetric. If $C_b = C_b^r$ then all 3 paths are equal. If $C_b = C_b^r$ then all 3 paths are equal and symmetric.

If there are two cycles C_1 and C_2 the cases are as follows. In each case the pattern of the paths C_{1t}, C_{1b}, and the third path are given. $C_{1b} = C_{2t}$: ABB; $C_{1b} = C_{2t}^r$: ABB; $C_{1b} = C_{2b}$: ABB; $C_{1b} = C_{2b}^r$: ABB; $C_{1b} = C_{1t}$: impossible; $C_{1b} = C_{1t}^r$: AA^rA; $C_{1b} = C_{1b}$: ABA; $C_{1b} = C_{1b}^r$: Aa^rA. Note that additional equalities might hold, such as $B = A^r$, as long as there remain two distinct cycles.

If there are three cycles C_1, C_2, C_3, by relabeling it may be assumed that $C_{3t} = C_{1t} = A$, $C_{2t} = C_{1b} = B$, and $C_{3b} = C_{2b} = C$. The possibilities for C_{1b} are C_{2t}, C_{2t}^r, C_{2b}, C_{2b}^r, C_{1t}, C_{1t}^r, C_{1b}, and C_{1b}^r. These merely impose additional restrictions on A, B, C, some of which are impossible (such as equality of any two). □

Lemma 17. *If the size triple is rrs, then G is reconstructible.*

Proof. Let v range over degree 1 vertices of paths of size r. In each G_v the cycle C of the paths of size r in G may be identified. The argument of theorem 7 may be adapted. The limbs are not all 1-limbs, and the exceptional case of a single 1-limb and a single limb of size 2 does not occur. □

Lemma 18. *If the size triple is rss, then G is reconstructible.*

Proof. Let v be a degree 1 vertex in a limb attached to the path P of length r. In G_v the cycle C of the paths of length s, with their ends marked, may be found. The number n of limbs on P may also be determined.

Suppose $n \geq 3$. Letting T_1 and T_2 denote the limbs closest to the end vertices, with the root of T_i at distance d_i from the closest end, by considering G_v as v ranges over degree 1 vertices in limbs on P, T_1, d_1, T_2, and d_2 may be determined. Now let w be a degree 1 vertex in one of the T_i, and replace the limb $T_i - w$ (which may be empty) by T_i (noting that T_{3-1} is known).

Suppose $n = 2$. Unless both limbs are 1-limbs, the argument for $n \geq 3$ applies. Suppose both limbs are 1-limbs. If $s \geq 4$ than P with all limbs attached may be determined. This may be laid over P in G_v where v is a degree 1 vertex of a limb on P. The same argument applies if $s = 3$ and either path of length 3 has a limb of size greater than 1.

Thus, suppose $r = 2$, $s = 3$, and all limbs are 1-limbs. Let v be the root of the middle 1-limb on a path of size 3; it is readily verified that G is reconstructible from G_v.

Suppose $n = 1$. The case $r = 1$ has already been handled. Otherwise, let v be a degree 1 vertex in the limb T on P, and replace $T - v$ by T. □

Lemma 19. *If the size triple is rst, then G is reconstructible.*

Proof. Let P be the path of size r. Letting v be a degree 1 vertex of a limb attached to P, the other two paths, with the end vertices marked, can be determined from G_v. Letting w be a degree 1 vertex of a limb attached to the path of length t, P with its end vertices marked, may be determined. If either P or the other two paths are symmetric, P can be overlaid in G_v arbitrarily. Otherwise, the choice of orientation of P may be determined by an argument as in the proof of lemma 11. □

Theorem 20. *If B is a 3 path graph then G is reconstructible.*

Proof. This follows by lemmas 8 to 19. □

4. Concluding Remarks

It is recognizable whether G is a cycle, i.e., if all vertices are of degree 2. G is clearly reconstructible. It is recognizable whether G is a 3 path graph, i.e., if there are 2 vertices of degree 3 and other vertices have degree 2. G is reconstructible in this case also. Let v be a degree 3 vertex; G is reconstructible from G_v, unless there is a path of length 1. In this case the sum of the lengths of the other two paths P_1 and P_2 is known from G_v, and the length of one of them may be determined from G_w where w is on the other.

Theorem 21. *Suppose G is a 2-connected graph obtained by adding k paths. Then $k = e - v$ where e is the number of edges and v the number of vertices.*

Proof. The proof is an easy induction on k. □

Since the dimension of the cycle space over the two element field is $e-v+1$, this quantity equals $k+1$.

Adding another path to a 3 path graph, there are 3 possibilities, which may be characterized in terms of the degree sequence of the vertices of degree greater than 2. The possibilities are:

$$4{,}4; \quad 4{,}3{,}3; \quad 3{,}3{,}3{,}3$$

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