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ON DECOMPOSITIONS OF ORDERED SEMIGROUPS

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Abstract: The purpose of this paper is to introduce and describe a-connected ordered semigroups and weakly extremely commutative ordered semigroups. The results obtained generalize the results on semigroups without order.

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1. Preliminaries

Decompositions of semigroups without order has been studied by Protič and Stevanovič in [3]. The authors introduced and studied a-connected semigroups and weakly extremely commutative semigroups. The main result is as follows: a weakly extremely commutative semigroup is a semilattice of a-connected semigroups. This paper deals with ordered semigroups, the results obtained extended the result mentioned before.

We now recall some definitions and results used throughout this paper.

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A semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that, for $x, y, z \in S$,

$$x \leq y \Rightarrow zx \leq zy,\ xz \leq yz,$$

is called an *ordered semigroup* ([1], [2]). In this paper, we write $(S, \cdot, \leq)$ for an ordered semigroup $(S, \cdot)$ with a partial order $\leq$ and write xy $(x, y \in S)$ for $x \cdot y$.

An element 0 of an ordered semigroup $(S, \cdot, \leq)$ is called a *zero element* [1] if $0x = x0 = 0$ and $0 \leq x$ for all $x \in S$.

A mapping $\varphi : S \to T$ from an ordered semigroup $(S, \cdot, \leq_S)$ into an ordered semigroup $(T, \cdot, \leq_T)$ is called a *homomorphism* [6] if, for all $x, y \in S$,

(i) $\varphi(xy) = \varphi(x)\varphi(y)$;

(ii) $x \leq_S y \Rightarrow \varphi(x) \leq_T \varphi(y)$.

If A and B are nonempty subsets of an ordered semigroup $(S, \cdot, \leq)$, we write subsets AB and $(A]$ of S as follows:

$$AB = \{xy \mid x \in A, y \in B\},$$

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

For $x \in S$, we write Ax and xA for $A\{x\}$ and $\{x\}A$, respectively.

In [5], the following statements hold for any nonempty subsets A, B of an ordered semigroup $(S, \cdot, \leq)$:

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB]$.

Definition 1. ([4]) Let $(S, \cdot, \leq)$ be an ordered semigroup. An equivalence relation ρ on S is called a *congruence* on S if, for $x, y, z \in S$, if $x\rho y$ then $zx\rho zy$ and $xz\rho yz$.

Definition 2. ([4]) Let $(S, \cdot, \leq)$ be an ordered semigroup. A congruence ρ on S is called a *semilattice congruence* if, for $x, y \in S$,

(i) $x\rho x^2$;

(ii) $xy\rho yx$.

2. Decompositions of Ordered Semigroups

We begin with the concept of a-connected as follows:

Definition 3. Let $(S, \cdot, \leq)$ be an ordered semigroup and let $a \in S$. Then two elements $x, y \in S$ are said to be *a-connected* if

$$(xa)^n \in (yaS] \text{ and } (ya)^m \in (xaS]$$

for some positive integers m, n. If every element of S is a-connected then S is said to be an *a-connected ordered semigroup.*

Note from the definition above that if $(xa)^n \in (yaS]$ and $(ya)^m \in (xaS]$ for some positive integers m and n, then

$$(xa)^p \in (yaS] \text{ and } (ya)^p \in (xaS]$$

where $p = \max\{m, n\}$.

Definition 4. An ordered semigroup $(S, \cdot, \leq)$ is called an *extremely commutative ordered semigroup* if

$$xyz = zyx$$

for all $x, y, z \in S$.

Definition 5. An ordered semigroup $(S, \cdot, \leq)$ is called a *weakly extremely commutative ordered semigroup* if there exists $a \in S$ such that

$$xay = yax$$

for all $x, y \in S$.

It is clear that every extremely commutative ordered semigroup is a weakly extremely commutative ordered semigroup.

Example 6. Let $(K, \cdot, \leq_K)$ be a commutative ordered semigroup and let $(T, \cdot, \leq_T)$ is an ordered semigroup with zero. Let $\varphi : T \setminus \{0\} \to K$ be a homomorphism. Let $S = K \cup T \setminus \{0\}$. Define a multiplication on S, denoted by $\circ$, as follows:

(1) For $A, B \in T$,

$$A \circ B = AB \text{ if } AB \in T \setminus \{0\};\ A \circ B = \varphi(A)\varphi(B) \text{ if } AB = 0;$$

(2) For $c, d \in K$, $c \circ d = cd$;

(3) For $A \in T, c \in K,\ A \circ c = \varphi(A)c, c \circ A = c\varphi(A)$.

Define a partial order on S, denoted by $\leq_S$, as follows:

(4) For $A, B \in T,\ A \leq_S B \Leftrightarrow A \leq_T B$;

(5) For $c, d \in K,\ c \leq_S d \Leftrightarrow c \leq_K d$;

(6) For $A \in T, c \in K$,

$$A \leq_S c \Leftrightarrow \varphi(A) \leq_K c \text{ and } c \leq_S A \Leftrightarrow c \leq_K \varphi(A).$$

It is a routine matter to verify that $(S, \circ, \leq_S)$ is an ordered semigroup. If $A, B \in T \setminus \{0\}$ and $s \in K$, then

$$\begin{aligned} A \circ s \circ B = (\varphi(A)s) \circ B &= (\varphi(A)s)\varphi(B) \\ &= \varphi(A)s\varphi(B) \\ &= \varphi(B)s\varphi(A) \\ &= (B \circ s)\varphi(A) \\ &= B \circ s \circ A \end{aligned}$$

Hence S is a weakly extremely commutative ordered semigroup.

The following two lemmas can be proved in the same manner as the proofs of Lemma 1 and Lemma 2 in [3].

Lemma 7. *If $(S, \cdot, \leq)$ is a weakly extremely commutative ordered semigroup and if*

$$B = \{a \in S \mid \forall x, y \in S, xay = yax\},$$

then B is nonempty and $SBS \subseteq B$.

Proof. By assumption we have B is nonempty. Let $a \in B$ and $u, v \in S$. To show that $uav \in B$, we let $x, y \in S$. Since $xu, vy \in S$, it follows that

$$x(uav)y = (xu)a(vy) = (vy)a(xu) = v(xay)u = y(uav)x,$$

and hence $uav \in B$. □

Lemma 8. *Let $(S, \cdot, \leq)$ be a weakly extremely commutative ordered semigroup, $a \in B$ and $x, y \in S$. Then*

$$(xay)^{2k} = (xa)^{2k-1}y^{2k-1}xay \text{ and } (xay)^{2k+1} = (xa)^{2k+1}y^{2k+1}$$

for all positive integers k.

Proof. We shall prove the assertion by induction. By Lemma 7, $xay \in B$. If $k = 1$, then

$$(xay)^2 = (xay)(xay)$$

and

$$(xay)^3 = xay(xay)xay = xaxa(xay)yy = (xa)^3y^3.$$

We have

$$(xay)^{2k+2} = (xay)^{2k+1}(xay) = (xa)^{2k+1}y^{2k+1}(xay)$$

and

$$\begin{aligned}(xay)^{2k+3} &= (xay)^{2k+2}(xay) = (xa)^{2k+1}y^{2k+1}(xay)xay \\ &= (xa)^{2k+1}xa(xay)y^{2k+1}y = (xa)^{2k+3}y^{2k+3}.\end{aligned}$$

Therefore, the assertion holds. □

Corollary 9. *Let $(S, \cdot, \leq)$ be a weakly extremely commutative ordered semigroup, $a \in B$ and $x, y \in S$. Then*

$$(xay)^m \in (xa)^{m-1}S$$

for all positive integers m.

Proof. This is a consequence of Lemma 8. □

We now prove the main theorem of this paper:

Theorem 10. *Let $(S, \cdot, \leq)$ be a weakly extremely commutative ordered semigroup and let $a \in B$. Then S is a semilattice of a-connected ordered semigroups.*

Proof. We define a relation ρ on S by, for $x, y \in S$,

$$x\rho y$$

if and only if

$$(xa)^n \in (yaS] \text{ and } (ya)^n \in (xaS] \text{ for some positive integer } n.$$

To prove the theorem, it suffices to show that ρ is a semilattice congruence on S.

If $x \in S$, then $(xa)^2 \in (xaS]$, and so $x\rho x$. It is clear that, for $x, y \in S$, if $x\rho y$ then $y\rho x$. Let $x, y, z \in S$ be such that $x\rho y$ and $y\rho z$; thus

$$(xa)^n \in (yaS], (ya)^n \in (xaS] \text{ and } (ya)^m \in (zaS], (za)^n \in (yaS]$$

for some positive integers m, n. Let $s, s' \in S$ such that

$$(xa)^n \leq yas \text{ and } (za)^m \leq yas'.$$

By Corollary 9, we obtain

$$\begin{aligned} (xa)^{(n+1)(m+1)} &= (xa)^{n(m+1)}(xa)^{m+1} \\ &\leq (yas)^{m+1}(xa)^{m+1} \\ &\in (ya)^m S(xa)^{m+1} \\ &\subseteq (zaS] \end{aligned}$$

and

$$\begin{aligned} (za)^{(n+1)(m+1)} &= (za)^{m(n+1)}(za)^{n+1} \\ &\leq (yas')^{n+1}(za)^{n+1} \\ &\in (ya)^n S(za)^{n+1} \\ &\subseteq (xaS]. \end{aligned}$$

Thus $x\rho z$. Therefore, ρ is an equivalent relation on S.

To show that ρ is a congruence on S, let $x, y, z \in S$ be such that $x\rho y$. Then

$$(xa)^n \in (yaS], (ya)^n \in (xaS]$$

for some positive integer n; thus $(xa)^{2n+1} \in (yaS]$. Since $az \in B$, it follows that

$$\begin{aligned} (xza)^{2n+2} &= x(zax)^{2n+1}za = x(xaz)^{2n+1}za \\ &= x(xa)^{2n+1}z^{2n+1}za = x(xa)^{2n+1}zz^{2n+1}a \\ &\in x(yaS]zz^{2n+1}a \subseteq (xyaSz]z^{2n+1}a \\ &\subseteq (xzaSy]z^{2n+1}a \subseteq (yzaSx]z^{2n+1}a \\ &\subseteq (yzaS] \end{aligned}$$

Similarly, $(zya)^{2n+2} \in (xzaS]$, and so $xz\rho yz$. We have

$$\begin{aligned} (zxa)^{2n+2} &= z(xaz)^{2n+1}xa = z(xa)^{2n+1}z^{2n+1}xa \\ &\in (zyaS]z^{2n+1}xa \subseteq (zyaS] \end{aligned}$$

and, similarly, $(zya)^{2n+2} \in (zxaS]$. Hence ρ is a congruence on S.

Finally, we shall show that ρ is a semilattice congruence. Let $x, y \in S$. Since $xa, x^2a \in B$, so

$$(x^2a)^3 = xx(ax^2)ax^2a = xa(ax^2)xx^2a \in axS$$

and

$$(xa)^3 = xaxaxa = xxa(xa)a \in x^2aS$$

That is $x\rho x^2$. We have

$$(xya)^2 = x(yax)ya = y(yax)xa = yy(ax)xa = yxa(ax)y \in yxaS.$$

Similarly, $(yxa)^2 \in yxaS$. Thus $xy\rho yx$. □

Corollary 11. *An extremely commutative ordered semigroup $(S, \cdot, \leq)$ is a semilattice of a-connected ordered semigroups for every $a \in B$.*

Proof. Obvious. □

Definition 12. An ordered semigroup $(S, \cdot, \leq)$ is said to be *medial ordered semigroup* if

$$xyzt = xzyt$$

for all $x, z, y, t \in S$.

Theorem 13. *A medial ordered semigroup $(S, \cdot, \leq)$ is a band of a-connected ordered semigroups for every $a \in S$, that is there exists a congruence ρ on S such that $x\rho x^2$ for all $x \in S$ and each ρ-class is an a-connected ordered semigroup.*

Proof. Let $a \in S$. Define a relation ρ on S by

$$x\rho y \text{ if and only if } (xa)^n \in (yaS] \text{ and } (ya)^n \in (xaS].$$

for some positive integer n. As in the Proof of Theorem 10, ρ is an equivalence relation.

To show that ρ is a congruence on S, we let $x, y, z \in S$ be such that $x\rho y$. Then

$$(xa)^n \in (yaS] \text{ and } (ya)^n \in (xaS].$$

for some positive integer n. Since S is a medial ordered semigroup, we have

$$(xza)^{n+1} = (xa)^n z^{n+1} xa \in (yaS]zz^n xa \subseteq (yzaS]z^n xa \subseteq (yzaS].$$

Similarly, $(yza)^{n+1} \in (xzaS]$. In the same manner, we have

$$(zxa)^{n+1} \in (zyaS] \text{ and } (zya)^{n+1} \in (zxaS].$$

Thus ρ is a congruence on S.

To show that $x\rho x^2$ for all $x \in S$, we let $x \in S$. Then

$$(x^2a)^2 = x^2ax^2a = xax^3a \in xaS \subseteq (xaS]$$

and

$$(xa)^2 = xaxa = x^2aa \in x^2aS \subseteq (x^2aS],$$

and thus $x\rho x^2$. It is clear by the definition of ρ that each ρ-class is an a-connected. $\square$

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