ON DECOMPOSITIONS OF ORDERED SEMIGROUPS

Thawhat Changphasr
Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen 40002, THAILAND

Abstract: The purpose of this paper is to introduce and describe $a$-connected ordered semigroups and weakly extremely commutative ordered semigroups. The results obtained generalize the results on semigroups without order.

AMS Subject Classification: 06F05
Key Words: semigroup, ordered semigroup, external commutative, weakly extremely commutative, $a$-connected, semilattice, congruence, semilattice congruence

1. Preliminaries

Decompositions of semigroups without order has been studied by Protić and Stevanović in [3]. The authors introduced and studied $a$-connected semigroups and weakly extremely commutative semigroups. The main result is as follows: a weakly extremely commutative semigroup is a semilattice of $a$-connected semigroups. This paper deals with ordered semigroups, the results obtained extended the result mentioned before.

We now recall some definitions and results used throughout this paper.

Received: September 25, 2013
A semigroup \((S, \cdot)\) together with a partial order \(\leq\) that is \textit{compatible} with the semigroup operation, meaning that, for \(x, y, z \in S\),

\[ x \leq y \Rightarrow zx \leq zy, \quad xz \leq yz, \]

is called an \textit{ordered semigroup} ([1], [2]). In this paper, we write \((S, \cdot, \leq)\) for an ordered semigroup \((S, \cdot, \leq)\) with a partial order \(\leq\) and write \(xy (x, y \in S)\) for \(x \cdot y\).

An element 0 of an ordered semigroup \((S, \cdot, \leq)\) is called a \textit{zero element} [1] if \(0x = x0 = 0\) and \(0 \leq x\) for all \(x \in S\).

A mapping \(\varphi : S \to T\) from an ordered semigroup \((S, \cdot, \leq)\) into an ordered semigroup \((T, \cdot, \leq_T)\) is called a \textit{homomorphism} [6] if, for all \(x, y \in S\),

\[ (i) \quad \varphi(xy) = \varphi(x)\varphi(y); \]

\[ (ii) \quad x \leq_S y \Rightarrow \varphi(x) \leq_T \varphi(y). \]

If \(A\) and \(B\) are nonempty subsets of an ordered semigroup \((S, \cdot, \leq)\), we write subsets \(AB\) and \((A)\) of \(S\) as follows:

\[ AB = \{xy \mid x \in A, y \in B\}, \]

\[ (A) = \{x \in S \mid x \leq a \text{ for some } a \in A\}. \]

For \(x \in S\), we write \(Ax\) and \(xA\) for \(A\{x\}\) and \(\{x\}A\), respectively.

In [5], the following statements hold for any nonempty subsets \(A, B\) of an ordered semigroup \((S, \cdot, \leq)\):

\[ (1) \quad A \subseteq (A); \]

\[ (2) \quad A \subseteq B \Rightarrow (A) \subseteq (B); \]

\[ (3) \quad (A)(B) \subseteq (AB). \]

\textbf{Definition 1.} ([4]) Let \((S, \cdot, \leq)\) be an ordered semigroup. An equivalence relation \(\rho\) on \(S\) is called a \textit{congruence} on \(S\) if, for \(x, y, z \in S\), if \(x \rho y\) then \(z \rho xy\) and \(z \rho yz\).

\textbf{Definition 2.} ([4]) Let \((S, \cdot, \leq)\) be an ordered semigroup. A congruence \(\rho\) on \(S\) is called a \textit{semilattice congruence} if, for \(x, y \in S\),

\[ (i) \quad x \rho x^2; \]

\[ (ii) \quad xy \rho yx. \]
2. Decompositions of Ordered Semigroups

We begin with the concept of $a$-connected as follows:

**Definition 3.** Let $(S, \cdot, \leq)$ be an ordered semigroup and let $a \in S$. Then two elements $x, y \in S$ are said to be $a$-connected if

$$(xa)^n \in (yaS) \text{ and } (ya)^m \in (xaS)$$

for some positive integers $m, n$. If every element of $S$ is $a$-connected then $S$ is said to be an $a$-connected ordered semigroup.

Note from the definition above that if $(xa)^n \in (yaS)$ and $(ya)^m \in (xaS)$ for some positive integers $m$ and $n$, then

$$(xa)^p \in (yaS) \text{ and } (ya)^p \in (xaS)$$

where $p = \max\{m, n\}$.

**Definition 4.** An ordered semigroup $(S, \cdot, \leq)$ is called an extremely commutative ordered semigroup if

$$xyz = zyx$$

for all $x, y, z \in S$.

**Definition 5.** An ordered semigroup $(S, \cdot, \leq)$ is called a weakly extremely commutative ordered semigroup if there exists $a \in S$ such that

$$xay = yax$$

for all $x, y \in S$.

It is clear that every extremely commutative ordered semigroup is a weakly extremely commutative ordered semigroup.

**Example 6.** Let $(K, \cdot, \leq_K)$ be a commutative ordered semigroup and let $(T, \cdot, \leq_T)$ is an ordered semigroup with zero. Let $\varphi : T \setminus \{0\} \to K$ be a homomorphism. Let $S = K \cup T \setminus \{0\}$. Define a multiplication on $S$, denoted by $\circ$, as follows:

1. For $A, B \in T$,

$$A \circ B = AB \text{ if } AB \in T \setminus \{0\}; \quad A \circ B = \varphi(A)\varphi(B) \text{ if } AB = 0;$$

2. For $c, d \in K$, $c \circ d = cd$;
(3) For \( A \in T, c \in K \), \( A \circ c = \varphi(A)c, c \circ A = c\varphi(A) \).

Define a partial order on \( S \), denoted by \( \leq_S \), as follows:

(4) For \( A, B \in T \), \( A \leq_S B \iff A \leq_T B \);

(5) For \( c, d \in K \), \( c \leq_S d \iff c \leq_K d \);

(6) For \( A \in T, c \in K \),

\[
A \leq_S c \iff \varphi(A) \leq_K c \quad \text{and} \quad c \leq_S A \iff c \leq_K \varphi(A).
\]

It is a routine matter to verify that \((S, \circ, \leq_S)\) is an ordered semigroup. If \( A, B \in T \setminus \{0\} \) and \( s \in K \), then

\[
A \circ s \circ B = (\varphi(A)s) \circ B = (\varphi(A)s)\varphi(B) = \varphi(A)s\varphi(B) = \varphi(B)s\varphi(A) = (B \circ s)\varphi(A) = B \circ s \circ A
\]

Hence \( S \) is a weakly extremely commutative ordered semigroup.

The following two lemmas can be proved in the same manner as the proofs of Lemma 1 and Lemma 2 in [3].

**Lemma 7.** If \((S, \cdot, \leq)\) is a weakly extremely commutative ordered semigroup and if

\[
B = \{a \in S \mid \forall x, y \in S, xay = yax\},
\]

then \( B \) is nonempty and \( SBS \subseteq B \).

**Proof.** By assumption we have \( B \) is nonempty. Let \( a \in B \) and \( u, v \in S \). To show that \( uav \in B \), we let \( x, y \in S \). Since \( xu, vy \in S \), it follows that

\[
x(uav)y = (xu)a(vy) = (vy)a(xu) = v(xay)u = y(uav)x,
\]

and hence \( uav \in B \). \( \square \)

**Lemma 8.** Let \((S, \cdot, \leq)\) be a weakly extremely commutative ordered semigroup, \( a \in B \) and \( x, y \in S \). Then

\[
(xay)^{2k} = (xa)^{2k-1}y^{2k-1}xay \quad \text{and} \quad (xay)^{2k+1} = (xa)^{2k+1}y^{2k+1}
\]
for all positive integers $k$.

**Proof.** We shall prove the assertion by induction. By Lemma 7, $xay \in B$. If $k = 1$, then

$$(xay)^2 = (xay)(xay)$$

and

$$(xay)^3 = xay(xay)xay = xaxa(xay)yy = (xa)^3y^3.$$ We have

$$(xay)^{2k+2} = (xay)^{2k+1}(xay) = (xa)^{2k+1}y^{2k+1}(xay)$$

and

$$(xay)^{2k+3} = (xay)^{2k+2}(xay) = (xa)^{2k+1}y^{2k+1}(xay)xay$$

$$= (xa)^{2k+1}xa(xay)y^{2k+1}y = (xa)^{2k+3}y^{2k+3}. $$

Therefore, the assertion holds. \qed

**Corollary 9.** Let $(S, \cdot, \leq)$ be a weakly extremely commutative ordered semigroup, $a \in B$ and $x, y \in S$. Then

$$(xay)^m \in (xa)^{m-1}S$$

for all positive integers $m$.

**Proof.** This is a consequence of Lemma 8. \qed

We now prove the main theorem of this paper:

**Theorem 10.** Let $(S, \cdot, \leq)$ be a weakly extremely commutative ordered semigroup and let $a \in B$. Then $S$ is a semilattice of $a$-connected ordered semigroups.

**Proof.** We define a relation $\rho$ on $S$ by, for $x, y \in S$,

$$x \rho y$$

if and only if

$$(xa)^n \in (yaS] \text{ and } (ya)^n \in (xaS]$$

for some positive integer $n$. To prove the theorem, it suffices to show that $\rho$ is a semilattice congruence on $S$.

If $x \in S$, then $(xa)^2 \in (xaS]$, and so $x \rho x$. It is clear that, for $x, y \in S$, if $x \rho y$ then $y \rho x$. Let $x, y, z \in S$ be such that $x \rho y$ and $y \rho z$; thus
$(xa)^n \in (yaS), (ya)^n \in (xaS)$ and $(ya)^m \in (zaS), (za)^n \in (yaS)$ for some positive integers $m, n$. Let $s, s' \in S$ such that $(xa)^n \leq yas$ and $(za)^m \leq yas'$.

By Corollary 9, we obtain

$$(xa)^{(n+1)(m+1)} = (xa)^{(n+1)(m+1)} \leq (yas)^{(m+1)(xa)^{m+1}}$$

$$\subseteq (ya)^m S(xa)^{m+1}$$

$$\subseteq (zaS)$$

and

$$(za)^{(n+1)(m+1)} = (za)^{(m+1)(n+1)} \leq (yas')^{n+1}(za)^{n+1}$$

$$\subseteq (ya)^n S(za)^{n+1}$$

$$\subseteq (xaS).$$

Thus $x \rho z$. Therefore, $\rho$ is an equivalent relation on $S$.

To show that $\rho$ is a congruence on $S$, let $x, y, z \in S$ be such that $x \rho y$. Then

$$(xa)^n \in (yaS), (ya)^n \in (xaS)$$

for some positive integer $n$; thus $(xa)^{2n+1} \in (yaS)$. Since $az \in B$, it follows that

$$(xza)^{2n+2} = x(za)^{2n+1}za = x(xaz)^{2n+1}za$$

$$= x(xa)^{2n+1}z^{2n+1}za = x(xa)^{2n+1}z^2z^{2n+1}a$$

$$\subseteq x(yaS)z^2z^{2n+1}a \subseteq (xyaSz)z^{2n+1}a$$

$$\subseteq (xzaSy)z^{2n+1}a \subseteq (yzaSx)z^{2n+1}a$$

$$\subseteq (yzaS)$$

Similarly, $(zya)^{2n+2} \in (xzaS)$, and so $xz \rho yz$. We have

$$(zxa)^{2n+2} = z(xaz)^{2n+1}xa = z(xa)^{2n+1}z^{2n+1}xa$$

$$\in (zyaS)z^2z^{2n+1}xa \subseteq (zyaS)$$

and, similarly, $(zya)^{2n+2} \in (zxaS)$. Hence $\rho$ is a congruence on $S$.

Finally, we shall show that $\rho$ is a semilattice congruence. Let $x, y \in S$. Since $xa, x^2a \in B$, so
\[(x^2a)^3 = xx(ax^2)ax^2a = xa(ax^2)xax^2a \in axS\]

and

\[(xa)^3 = xaxaxaxa = xaxa(xa)a \in x^2aS\]

That is \(x \rho x^2\). We have

\[(xya)^2 = x(yax)ya = y(yax)xa = yy(ax)xa = yxa(ax)y \in yxaS.\]

Similarly, \((yxa)^2 \in yxaS\). Thus \(xy \rho yx\).

\[\text{Corollary 11.} \text{ An extremely commutative ordered semigroup } (S, \cdot, \leq) \text{ is a semilattice of } a \text{-connected ordered semigroups for every } a \in B.\]

\[\text{Proof.} \text{ Obvious.} \]

\[\text{Definition 12.} \text{ An ordered semigroup } (S, \cdot, \leq) \text{ is said to be medial ordered semigroup if }\]

\[xyzt = xzyt\]

for all \(x, z, y, t \in S\).

\[\text{Theorem 13.} \text{ A medial ordered semigroup } (S, \cdot, \leq) \text{ is a band of } a \text{-connected ordered semigroups for every } a \in S, \text{ that is there exists a congruence } \rho \text{ on } S \text{ such that } x \rho x^2 \text{ for all } x \in S \text{ and each } \rho \text{-class is an } a \text{-connected ordered semigroup.} \]

\[\text{Proof.} \text{ Let } a \in S. \text{ Define a relation } \rho \text{ on } S \text{ by }\]

\[x \rho y \text{ if and only if } (xa)^n \in (yaS) \text{ and } (ya)^n \in (xaS).\]

for some positive integer \(n\). As in the Proof of Theorem 10, \(\rho\) is an equivalence relation.

To show that \(\rho\) is a congruence on \(S\), we let \(x, y, z \in S\) be such that \(x \rho y\). Then

\[(xa)^n \in (yaS) \text{ and } (ya)^n \in (xaS).\]

for some positive integer \(n\). Since \(S\) is a medial ordered semigroup, we have

\[(xza)^{n+1} = (xa)^n z^{n+1} xa \in (yaS)zz^n xa \subseteq (yzaS)z^n xa \subseteq (yzaS).\]

Similarly, \((yza)^{n+1} \in (xzaS)\). In the same manner, we have

\[(zxa)^{n+1} \in (zyaS) \text{ and } (zya)^{n+1} \in (zxaS).\]
Thus $\rho$ is a congruence on $S$.

To show that $x\rho x^2$ for all $x \in S$, we let $x \in S$. Then

$$(x^2a)^2 = x^2ax^2a = xax^3a \in xaS \subseteq (xaS)$$

and

$$(xa)^2 = xaxa = x^2aa \in x^2aS \subseteq (x^2aS),$$

and thus $x\rho x^2$. It is clear by the definition of $\rho$ that each $\rho$-class is an $a$-connected.

\[\square\]

References


