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ON THE DIOPHANTINE EQUATION $8^x + 13^y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $8^x+13^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 3)$.

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1. Introduction

In 2007, Acu [1] proved that the Diophantine equation $2^x + 5^y = z^2$ has only two non-negative integer solutions where x, y and z are non-negative integers. The solutions (x, y, z) are $(3, 0, 3)$ and $(2, 1, 3)$. In 2011, Suvarnamani, Singta and Chotchaisthit [11] proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers. In 2012, Chotchaisthit [3] solved the Diophantine equation $4^x + p^y = z^2$ for all positive prime number p, where x, y and z are non-negative integers. In the same year, Sroysang [5] proved that the Diophantine equation $3^x + 5^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 2)$. In the same year, Sroysang [6] proved that the Diophantine equation $8^x + 19^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 3)$. Moreover, he [7] proved that the

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Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. In 2013, Chotchaisthit [2] proved that the Diophantine equation $2^x + 11^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(3, 0, 3)$. In the same year, Sroysang [8] proved that the Diophantine equation $7^x + 8^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(0, 1, 3)$. In the same year, Sroysang [9] proved that the Diophantine equation $2^x + 3^y = z^2$ has only three non-negative integer solutions where x, y and z are non-negative integers. The solutions (x, y, z) are $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$. Moreover, he [10] proved that the Diophantine equation $23^x + 32^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. In this paper, we prove that the Diophantine equation $8^x + 13^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 3)$.

2. Preliminaries

Proposition 2.1. [4] (**the Catalan's conjecture**) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [6] $(1, 3)$ *is a unique solution* (x, z) *for the Diophantine equation* $8^x + 1 = z^2$ *where* x *and* z *are non-negative integers.*

Lemma 2.3. *The Diophantine equation* $1 + 13^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 13^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 1 + 13^y \geq 1 + 13^1 = 14$. Then $z \geq 4$. Now, we consider on the equation $z^2 - 13^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z^2 = 14$. This is a contradiction. □

3. Results

Theorem 3.1. $(1, 0, 3)$ *is a unique solution* (x, y, z) *for the Diophantine equation* $8^x + 13^y = z^2$ *where* x, y *and* z *are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $8^x + 13^y = z^2$. By Lemma 2.3, we have $x \geq 1$. This implies that z is odd. Then $z = 2t + 1$ for some a non-negative integer t. Thus, $8^x + 13^y = 4t(t+1) + 1$. We note that $t(t+1)$ is even, so $8^x + 13^y = 8m + 1$ for some a non-negative integer m. It follows that $13^y \equiv 1 \pmod 8$. This implies that y is even. Now, we will divide the number y into two cases.

Case $y = 0$. By Lemma 2.2, we have $x = 1$ and $z = 3$.

Case $y \geq 2$. Let $y = 2n$ where n is a positive integer. Then $z^2 - 13^{2n} = 8^x$. Then $(z - 13^n)(z + 13^n) = 2^{3x}$. Thus, $z - 13^n = 2^k$ where k is a non-negative integer. Then $z + 13^n = 2^{3x-k}$. Thus, $2(13^n) = 2^{3x-k} - 2^k = 2^k(2^{3x-2k} - 1)$. Next, we will divide the number k into two subcases.

Subcase $k = 0$. Then $z - 13^n = 1$. This implies that z is even. This is a contradiction.

Subcase $k = 1$. Then $2^{3x-2} - 1 = 13^n$. Then $2^{3x-2} - 13^n = 1$. If $x = 1$, then $n = 0$ so $y = 0$. Thus, $x \geq 2$. By Proposition 2.1, we have $n = 1$. Then $2^{3x-2} = 14$. This is impossible.

Therefore, $(1, 0, 3)$ is a unique solution (x, y, z) for the equation $8^x + 13^y = z^2$. □

Corollary 3.2. *The Diophantine equation $8^x + 13^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $8^x + 13^y = w^4$. Let $z = w^2$. Then $8^x + 13^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 3)$. Then $w^2 = z = 3$. This is a contradiction. □

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