ON THE DIOPHANTINE EQUATION $8^x + 13^y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $8^x + 13^y = z^2$ has a unique non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(1, 0, 3)$.

AMS Subject Classification: 11D61
Key Words: exponential Diophantine equation

1. Introduction

In 2007, Acu [1] proved that the Diophantine equation $2^x + 5^y = z^2$ has only two non-negative integer solutions where $x$, $y$ and $z$ are non-negative integers. The solutions $(x, y, z)$ are $(3, 0, 3)$ and $(2, 1, 3)$. In 2011, Suvarnamani, Singta and Chotchaisthit [11] proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. In 2012, Chotchaisthit [3] solved the Diophantine equation $4^x + p^y = z^2$ for all positive prime number $p$, where $x$, $y$ and $z$ are non-negative integers. In the same year, Sroysang [5] proved that the Diophantine equation $3^x + 5^y = z^2$ has a unique non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(1, 0, 2)$. In the same year, Sroysang [6] proved that the Diophantine equation $8^x + 19^y = z^2$ has a unique non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(1, 0, 3)$. Moreover, he [7] proved that the
Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. In 2013, Chotchaisthit [2] proved that the Diophantine equation $2^x + 11^y = z^2$ has a unique non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(3, 0, 3)$. In the same year, Sroysang [8] proved that the Diophantine equation $7^x + 8^y = z^2$ has a unique non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(0, 1, 3)$. In the same year, Sroysang [9] proved that the Diophantine equation $2^x + 3^y = z^2$ has only three non-negative integer solutions where $x$, $y$ and $z$ are non-negative integers. The solutions $(x, y, z)$ are $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$. Moreover, he [10] proved that the Diophantine equation $23^x + 32^y = z^2$ has no non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. In this paper, we prove that the Diophantine equation $8^x + 13^y = z^2$ has a unique non-negative integer solution where $x$, $y$ and $z$ are non-negative integers. The solution $(x, y, z)$ is $(1, 0, 3)$.

2. Preliminaries

**Proposition 2.1.** [4] (the Catalan’s conjecture) $(3, 2, 2, 3)$ is a unique solution $(a, b, x, y)$ for the Diophantine equation $a^x - b^y = 1$ where $a, b, x$ and $y$ are integers with $\min\{a, b, x, y\} > 1$.

**Lemma 2.2.** [6] $(1, 3)$ is a unique solution $(x, z)$ for the Diophantine equation $8^x + 1 = z^2$ where $x$ and $z$ are non-negative integers.

**Lemma 2.3.** The Diophantine equation $1 + 13^y = z^2$ has no non-negative integer solution where $y$ and $z$ are non-negative integers.

*Proof.* Suppose that there are non-negative integers $y$ and $z$ such that $1 + 13^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 1 + 13^y \geq 1 + 13^1 = 14$. Then $z \geq 4$. Now, we consider on the equation $z^2 - 13^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z^2 = 14$. This is a contradiction. \qed

3. Results

**Theorem 3.1.** $(1, 0, 3)$ is a unique solution $(x, y, z)$ for the Diophantine equation $8^x + 13^y = z^2$ where $x$, $y$ and $z$ are non-negative integers.
Proof. Let \( x, y \) and \( z \) be non-negative integers such that \( 8^x + 13^y = z^2 \). By Lemma 2.3, we have \( x \geq 1 \). This implies that \( z \) is odd. Then \( z = 2t + 1 \) for some a non-negative integer \( t \). Thus, \( 8^x + 13^y = 4t(t + 1) + 1 \). We note that \( t(t + 1) \) is even, so \( 8^x + 13^y = 8m + 1 \) for some a non-negative integer \( m \). It follows that \( 13^y \equiv 1 \pmod{8} \). This implies that \( y \) is even. Now, we will divide the number \( y \) into two cases.

Case \( y = 0 \). By Lemma 2.2, we have \( x = 1 \) and \( z = 3 \).

Case \( y \geq 2 \). Let \( y = 2n \) where \( n \) is a positive integer. Then \( z^2 - 13^{2n} = 8^x \). Then \( (z - 13^n)(z + 13^n) = 2^{3x} \). Thus, \( z - 13^n = 2^k \) where \( k \) is a non-negative integer. Then \( z + 13^n = 2^{3x-k} \). Thus, \( 2(13^n) = 2^{3x-k} - 2^k = 2^k(2^{3x-2k} - 1) \). Next, we will divide the number \( k \) into two subcases.

Subcase \( k = 0 \). Then \( z - 13^n = 1 \). This implies that \( z \) is even. This is a contradiction.

Subcase \( k = 1 \). Then \( 2^{3x-2} - 1 = 13^n \). Then \( 2^{3x-2} - 13^n = 1 \). If \( x = 1 \), then \( n = 0 \) so \( y = 0 \). Thus, \( x \geq 2 \). By Proposition 2.1, we have \( n = 1 \). Then \( 2^{3x-2} = 14 \). This is impossible.

Therefore, \( (1, 0, 3) \) is a unique solution \( (x, y, z) \) for the equation \( 8^x + 13^y = z^2 \). \( \square \)

Corollary 3.2. The Diophantine equation \( 8^x + 13^y = w^4 \) has no non-negative integer solution where \( x, y \) and \( w \) are non-negative integers.

Proof. Suppose that there are non-negative integers \( x, y \) and \( w \) such that \( 8^x + 13^y = w^4 \). Let \( z = w^2 \). Then \( 8^x + 13^y = z^2 \). By Theorem 3.1, we have \( (x, y, z) = (1, 0, 3) \). Then \( w^2 = z = 3 \). This is a contradiction. \( \square \)

References


