FUZZY DOT $\beta-$SUBALGEBRAS OF $\beta-$ALGEBRAS

M. Abu Ayub Ansari$^1$, M. Chandramouleeswaran$^2$§

$^1$M.S.S. Wakf Board College
Madurai, 625 020, Tamilnadu, INDIA

$^2$Saiva Bhanu Kshatriya College
Aruppukottai, 626101, Tamilnadu, INDIA

Abstract: In this paper, we introduce the notion of fuzzy dot $\beta-$subalgebras on $\beta-$algebras and investigate some of their properties.

AMS Subject Classification: 03E72, 06F35, 03G25

Key Words: BCK/BCI-algebras, fuzzy dot subalgebra, $\beta-$algebras

1. Introduction

In 1966, Y. Imai and K. Iseki (see [5], [6], [7]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK algebras is a proper subclass of the class of BCI algebras. In 2002, J. Neggers and H.S. Kim [12] introduced the notion of $B-$algebras which is another generalization of BCK algebras. Also they introduced the notion of $\beta-$algebra [13] where two operations are coupled in such a way as to reflect the natural coupling, which exists between the usual group operation and its associated B-algebra. In 2012, Y.H. Kim [10] investigated some properties of $\beta-$algebras.

The important point in the evaluation of the modern concept of uncertainty

In their paper [9], the authors introduced the notion of fuzzy dot subalgebras of BCK/BCI-algebras as a generalization of a fuzzy subalgebra, and then investigated several basic properties which are related to fuzzy dot subalgebras. In [2], Al-Shehrie introduced the notion of fuzzy dot d-ideals of d-algebras. In [4], the authors introduced the notion of fuzzy dot SU-subalgebras. In [11], K.H. Kim introduced the notion of fuzzy dot subalgebras of d-algebras. In [3], fuzzy dot BCK/BCI algebras were discussed.

This motivated us to study the fuzzy dot algebraic structures on $\beta$-algebras. In this paper, we introduce the notion of fuzzy dot $\beta$-subalgebras of a $\beta$-algebra and investigate some of their properties.

2. Preliminaries

In this section we recall some basic definitions that are required in the sequel.

**Definition 2.1.** [5] A BCK-algebra $(X, \ast, 0)$ is a non-empty set $X$ with a constant $0$ and a binary operation $\ast$ satisfying the following axioms:

1. $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0$.
2. $(x \ast (x \ast y)) \ast y = 0$.
3. $x \ast x = 0$.
4. $x \ast y = 0$ and $y \ast x = 0 \Rightarrow x = y$.
5. $0 \ast x = 0 \forall x, y, z \in X$.

**Definition 2.2.** [6] A BCI-algebra $(X, \ast, 0)$ is a non-empty set $X$ with a constant $0$ and a binary operation $\ast$ satisfying the following axioms:

1. $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0$.
2. $(x \ast (x \ast y)) \ast y = 0$.
3. $x \ast x = 0$.
4. $x \ast y = 0$ and $y \ast x = 0 \Rightarrow x = y \forall x, y, z \in X$. 

Definition 2.3. [12] A B-algebra \((X, *, 0)\) is a non-empty set \(X\) with a constant 0 and a binary operations \(*\) satisfying the following axioms:

1. \(x * x = 0\)
2. \(x * 0 = x\)
3. \((x * y) * z = x * (z * (0 * y))\) \(\forall x, y, z \in X.\)

Definition 2.4. [13] \([10]\) A \(\beta\)-algebra is a non-empty set \(X\) with a constant 0 and two binary operations \(+\) and \(−\) satisfying the following axioms:

1. \(x - 0 = x.\)
2. \((0 - x) + x = 0.\)
3. \((x - y) - z = x - (z + y)\) \(\forall x, y, z \in X.\)

Example 2.5. Let \(X = \{0, 1, 2, 3\}\) be a set with constant 0 and two binary operations \(+\) and \(-\) are defined on \(X\) with the Cayley’s table

\[
\begin{array}{cccc}
+ & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 2 \\
\end{array}
\quad
\begin{array}{cccc}
− & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 1 & 0 & 3 \\
3 & 3 & 2 & 1 & 0 \\
\end{array}
\]

Then \((X, +, −, 0)\) is a \(\beta\)-algebra.

Definition 2.6. Let \((X, +, −, 0)\) and \((Y, +, −, 0')\) be two \(\beta\)-algebras. A mapping \(f : X \to Y\) is said to be a \(\beta\)-homomorphism if \(f(x + y) = f(x) + f(y)\) and \(f(x - y) = f(x) - f(y)\), \(\forall x, y \in X.\)

Note: In a \(\beta\)-homomorphism \(f(0) = f(0').\)

Definition 2.7. Let \(X\) be a set of universal discourse. A fuzzy set \(\mu\) in \(X\) is defined as a function \(\mu : X \to [0, 1]\). For each element \(x\) in \(X\), \(\mu(x)\) is called the membership value of \(x\) in \(X\).

Definition 2.8. If \(\mu_1\) and \(\mu_2\) are two fuzzy sets of \(X\) then intersection \(\mu_1 \cap \mu_2\) of \(\mu_1\) and \(\mu_2\) is defined as \((\mu_1 \cap \mu_2)(x) = \min \{\mu_1(x), \mu_2(x)\}\).

In general \((\cap \mu_i)(x) = \min \{\mu_i(x)/i = 1, 2, 3, \ldots\}\)

Definition 2.9. If \(\mu_1\) and \(\mu_2\) are two fuzzy sets of \(X\) then union \(\mu_1 \cup \mu_2\) of \(\mu_1\) and \(\mu_2\) is defined as \((\mu_1 \cup \mu_2)(x) = \max \{\mu_1(x), \mu_2(x)\}\).
**Definition 2.10.** If \( \mu_1 \) and \( \mu_2 \) are two fuzzy sets of \( X \) then \( \mu_1 \subseteq \mu_2 \) if \( \mu_1(x) \leq \mu_2(x) \).

**Definition 2.11.** If \( \mu \) is a fuzzy set of \( X \) then the complement of \( \mu \) is \( \mu^c \) and defined as \( \mu^c(x) = 1 - \mu(x) \).

**Definition 2.12.** Let \( \mu_1 \) and \( \mu_2 \) be two fuzzy sets of \( X_1 \) and \( X_2 \) respectively. Then the direct product \( \mu_1 \times \mu_2 \) of \( \mu_1 \) and \( \mu_2 \) is defined as the fuzzy set of \( X_1 \times X_2 \)

\[
(\mu_1 \times \mu_2)(x_1, x_2) = \min \{\mu_1(x_1), \mu_2(x_2)\} \quad \forall \ (x_1, x_2) \in X_1 \times X_2.
\]

**Definition 2.13.** Let \( \mu \) be a fuzzy set in a set \( X \). For \( t \in [0, 1] \), the set \( \mu_t = \{x \in X / \mu(x) \geq t\} \) is called a level subset of \( \mu \).

**Proposition 2.14.** If \( t_1 \leq t_2 \), then \( \mu_{t_2} \subseteq \mu_{t_1} \) where \( \mu_{t_2} \) and \( \mu_{t_1} \) are any two level subsets of \( \mu \) where \( \mu \) be a fuzzy set on a set \( X \).

**Definition 2.15.** Let \( \mu \) be a fuzzy set of \( X \). \( \mu \) is said to have the supremum property if, for any subset \( A \) of \( X \), there exist a \( a_0 \in A \) such that \( \mu(a_0) = \sup_{a \in A} \mu(a) \).

3. Fuzzy Dot \( \beta \)-Subalgebras of \( \beta \)-Algebra

In section we introduce the notion of fuzzy dot \( \beta \)-subalgebras of \( \beta \)-algebras and prove some simple theorem.

**Definition 3.1.** Let \( \mu \) be a fuzzy set in a \( \beta \)-algebra \( X \). Then \( \mu \) is called a fuzzy dot \( \beta \)-subalgebra of \( X \) if

1. \( \mu(x + y) \geq \mu(x) \cdot \mu(y) \), \( \forall x, y \in X \).
2. \( \mu(x - y) \geq \mu(x) \cdot \mu(y) \), \( \forall x, y \in X \).

**Example 3.2.** Consider the \( \beta \)-algebra \((X, +, -, 0)\) in Example:2.5

Define \( \mu : X \rightarrow [0, 1] \) such that

\[
\mu(x) = \begin{cases} 
0.6 & \text{if } x = 0 \\
0.7 & \text{if } x = 1 \\
0.3 & \text{if } x = 2, 3 
\end{cases}
\]

then \( \mu \) is a fuzzy dot \( \beta \)-subalgebra of \( X \).
Theorem 3.3. Every fuzzy $\beta$-subalgebra of $X$ is a fuzzy dot $\beta$-subalgebra of $X$. The converse need not be true.

Proof. Let $\mu$ be a fuzzy $\beta$-subalgebra of $X$. Then

$$
\mu(x + y) \geq \min \{\mu(x), \mu(y)\} \geq \mu(x) \cdot \mu(y)
$$

and

$$
\mu(x - y) \geq \min \{\mu(x), \mu(y)\} \geq \mu(x) \cdot \mu(y)
$$

Therefore $\mu$ is a fuzzy dot $\beta$-subalgebra of $X$.

Note: In the example 3.2, $\mu$ is a fuzzy dot $\beta$-subalgebra of $X$ but $\mu$ is not a fuzzy $\beta$-subalgebra of $X$, since $\mu(1 - 1) = \mu(0) = 0.6 < 0.7 = \min \{0.7, 0.7\} = \min \{\mu(1), \mu(1)\}$.

Theorem 3.4. If $\mu_1$ and $\mu_2$ be two fuzzy dot $\beta$-subalgebras of $X$ then $\mu_1 \cap \mu_2$ is also a fuzzy dot $\beta$-subalgebra of $X$.

Proof. For $x, y \in X$,

$$
(\mu_1 \cap \mu_2)(x + y) = \min \{\mu_1(x + y), \mu_2(x + y)\} \\
\geq \min \{\mu_1(x) \cdot \mu_1(y), \mu_2(x) \cdot \mu_2(y)\} \\
\geq (\min \{\mu_1(x), \mu_2(x)\}) \cdot (\min \{\mu_1(y), \mu_2(y)\}) \\
= (\mu_1 \cap \mu_2)(x) \cdot (\mu_1 \cap \mu_2)(y)
$$

Similarly we can prove that $(\mu_1 \cap \mu_2)(x - y) \geq (\mu_1 \cap \mu_2)(x) \cdot (\mu_1 \cap \mu_2)(y)$

Therefore $\mu_1 \cap \mu_2$ is fuzzy dot $\beta$-subalgebra of $X$.

The above theorem can be generalized as follows.

Corollary 3.5. If $\{\mu_i/i = 1, 2, 3, \cdots\}$ be a family of dot-fuzzy $\beta$-subalgebra of $X$, then $\cap \mu_i$ is also a dot-fuzzy $\beta$-subalgebra of $X$.

Notation: Hereafter by a $\beta$-algebra $X$ we mean a $\beta$-algebra $(X, +, -, 0)$ derived from a group or a $\beta$-algebra $(X, +, -, 0)$ from a $B$-algebra $(X, -, 0)$.

Proposition 3.6. Let $X$ be a $\beta$-algebra and let $\mu$ be a dot-fuzzy $\beta$-subalgebra of $X$ then

1. $\{\mu(x)\}^6 \leq \{\mu(x)\}^2 \leq \mu(0), \forall x \in X$.
2. $\{\mu(x)\}^3 \leq \mu(x^*), \forall x \in X$, where $x^* = 0 - x$.

In general $\{\mu(x)\}^{2n-1} \leq \mu(0^n - x), \forall x \in X$, where $0^n - x = 0 - (0 - (0 - \cdots (0 - x)))$, such that 0 occurs $n$ times.

Proof.

1. For any $x \in X, \mu(0) = \mu(x - x) \geq \mu(x) \cdot \mu(x) = \{\mu(x)\}^2$. 
We can prove the general case by induction.

**Theorem 3.7.** Let \( X \) be a \( \beta \)-algebra and let \( \mu \) be a fuzzy set of \( X \). If \( \mu(x + y) \geq \mu(x) \cdot \mu(y) \) and \( \mu(x^*) \geq \mu(x) \), \( \forall x, y \in X \), then \( \mu \) be a fuzzy dot \( \beta \)-subalgebra of \( X \).

**Proof.** Given \( \mu(x + y) \geq \mu(x) \cdot \mu(y) \), \( \forall x, y \in X \).
Hence it is enough to prove \( \mu(x - y) \geq \mu(x) \cdot \mu(y) \), \( \forall x, y \in X \). Now

\[
\mu(x - y) = \mu(x - (y^*)^*) = \mu(x + y^*) \geq \mu(x) \cdot \mu(y^*) \geq \mu(x) \cdot \mu(y),
\]

since \( y = (y^*)^* \) and \( x + y = x - y^* \).
Therefore \( \mu \) is a fuzzy dot \( \beta \)-subalgebra of \( X \).

**Theorem 3.8.** If \( A \) and \( B \) a \( \beta \)-subalgebra of \( X \), then the characteristic function \( \chi_A \) is a fuzzy dot \( \beta \)-subalgebra of \( X \).

**Proof.** Let \( x, y \in X \).

**Case 1** If \( x, y \in A \), then \( x + y, x - y \in A \) since \( A \) is a \( \beta \)-subalgebra of \( X \).
This implies that \( \chi_A(x) = 1, \chi_A(y) = 1, \chi_A(x + y) = 1 \) and \( \chi_A(x - y) = 1 \).
which implies

1. \( \chi_A(x + y) = 1 = 1 \cdot 1 = \chi_A(x) \cdot \chi_A(y), \forall x, y \in X. \)
2. \( \chi_A(x - y) = 1 = 1 \cdot 1 = \chi_A(x) \cdot \chi_A(y), \forall x, y \in X. \)

**Case 2** If \( x, y \notin A \), then \( \chi_A(x) = 0, \chi_A(y) = 0 \).
which implies,

1. \( \chi_A(x + y) \geq 0 = 0 \cdot 0 = \chi_A(x) \cdot \chi_A(y), \forall x, y \in X. \)
2. \( \chi_A(x - y) \geq 0 = 0 \cdot 0 = \chi_A(x) \cdot \chi_A(y), \forall x, y \in X. \)

**Case 3** If \( x \in A, y \notin A \), then \( \chi_A(x) = 1, \chi_A(y) = 0 \).
which implies,

1. \( \chi_A(x + y) \geq 0 = 1 \cdot 0 = \chi_A(x) \cdot \chi_A(y), \forall x, y \in X. \)
2. \( \chi_A(x - y) \geq 0 = 1 \cdot 0 = \chi_A(x) \cdot \chi_A(y), \forall x, y \in X. \)

**Case 4** When \( x \notin A, y \in A \), by interchanging the roles of \( x \) and \( y \) in case 3) we can prove \( \mu \) is a fuzzy dot \( \beta \)-subalgebra of \( X \). This completes the proof.
The converse of this result is also true.

**Theorem 3.9.** Let \( A \) be any subset of a \( \beta \)-algebra \( X \). If any characteristic function \( \chi_A \) of \( A \) is a fuzzy dot \( \beta \)-subalgebra of \( X \) then \( A \) is a \( \beta \)-subalgebra of \( X \).
Proof. Let \( \chi_A \) is a fuzzy dot/\( \beta \)-subalgebra of \( X \) then
\[
\chi_A(x + y) \geq \chi_A(x) \cdot \chi_A(y), \quad \forall x, y \in X.
\]
And \( \chi_A(x - y) \geq \chi_A(x) \cdot \chi_A(y), \quad \forall x, y \in X. \)
Let \( x, y \in A \) which implies \( \chi_A(x) = 1, \chi_A(y) = 1 \). Therefore

1. \( \chi_A(x + y) \geq \chi_A(x) \cdot \chi_A(y) = 1 \cdot 1 = 1 \Rightarrow \chi_A(x + y) = 1 \Rightarrow x + y \in A. \)
2. \( \chi_A(x - y) \geq \chi_A(x) \cdot \chi_A(y) = 1 \cdot 1 = 1 \Rightarrow \chi_A(x + y) = 1 \Rightarrow x - y \in A. \)

Hence \( A \) is a \( \beta \)-subalgebra of \( X \).

Theorem 3.10. Let \( \mu_1 \) and \( \mu_2 \) be two fuzzy dot /\( \beta \)-subalgebras of \( \beta \)-algebra \( X \). Then the direct product \( \mu_1 \times \mu_2 \) of \( \mu_1 \) and \( \mu_2 \) is defined by \( (\mu_1 \times \mu_2)(x, y) = \mu_1(x) \cdot \mu_2(y) \) is also a fuzzy dot /\( \beta \)-subalgebra of \( X \times X \).

Proof. Let \( X = X \times X \) and let \( \mu = \mu_1 \times \mu_2 \).
Let \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) be two elements of \( X \). Now
\[
\mu(x + y) = \mu(((x_1, x_2) + (y_1, y_2))
= \mu(x_1 + y_1, x_2 + y_2)
= (\mu_1 \times \mu_2)(x_1 + y_1, x_2 + y_2)
= \mu_1(x_1 + y_1) \cdot \mu_2(x_2 + y_2)
\geq \mu_1(x_1) \cdot \mu_1(y_1) \cdot \mu_2(x_2) \cdot \mu_2(y_2)
= \mu_1(x_1) \cdot \mu_2(x_2) \cdot \mu_1(y_1) \cdot \mu_2(y_2)
= (\mu_1 \times \mu_2)(x_1, x_2) \cdot (\mu_1 \times \mu_2)(y_1, y_2)
= \mu(x) \cdot \mu(y).
\]

Similarly we can prove \( \mu(x - y) \geq \mu(x) \cdot \mu(y), \quad \forall x, y \in X. \)
Hence \( \mu_1 \times \mu_2 \) is a fuzzy dot /\( \beta \)-subalgebra of \( X \times X \).

Theorem 3.11. Let \( \mu_1 \) and \( \mu_2 \) be two fuzzy dot /\( \beta \)-subalgebras of \( \beta \)-algebra \( X_1 \) and \( X_2 \) respectively. Then the direct product \( \mu_1 \times \mu_2 \) of \( \mu_1 \) and \( \mu_2 \)'s defined by
\[
(\mu_1 \times \mu_2)(x, y) = \mu_1(x) \cdot \mu_2(y), \quad \forall x, y \in X_1 \times X_2 \text{ is a fuzzy dot } /\( \beta \)-subalgebra of \( X_1 \times X_2 \).
\]

Proof. The Proof is straightforward.

Theorem 3.12. Let \( f : X \rightarrow Y \) be a homomorphism of a \( \beta \)-algebra \( X \) into a \( \beta \)-algebra \( Y \). If \( \mu \) is a fuzzy dot /\( \beta \)-algebra of \( Y \), then the pre-image of \( \mu \), denoted by \( f^{-1}(\mu) \) is defined as \( \{ f^{-1}(\mu) \} (x) = \mu(f(x)), \quad \forall x \in X \), is a fuzzy dot /\( \beta \)-subalgebra of \( X \).
Proof. Let $\mu$ be a fuzzy dot $\beta$–subalgebra of $Y$ and let $x, y \in X$. Then
\[
\{f^{-1}(\mu)\}(x + y) = \mu(f(x + y)) \\
= \mu(f(x) + f(y)) \\
\geq \mu(f(x)) \cdot \mu(f(y)) \\
= \{f^{-1}(\mu)(x)\} \cdot \{f^{-1}(\mu)\}(y).
\]
Also
\[
\{f^{-1}(\mu)\}(x - y) = \mu(f(x - y)) \\
= \mu(f(x) - f(y)) \\
\geq \mu(f(x)) \cdot \mu(f(y)) \\
= \{f^{-1}(\mu)\}(x) \cdot \{f^{-1}(\mu)\}(y).
\]
Hence $f^{-1}(\mu)$ is a fuzzy dot $\beta$–subalgebra of $X$.

**Theorem 3.13.** Let $f : X \to X$ be an endomorphism on a $\beta$–algebra $X$. If $\mu$ be a fuzzy dot $\beta$–subalgebra of $X$. Define a fuzzy set $\mu_f : X \to [0, 1]$ by $\mu_f(x) = \mu(f(x)), \forall x \in X$. Then $\mu_f$ is a fuzzy dot $\beta$–subalgebra of $X$.

Proof. Let $x, y \in X$. Then
\[
\mu_f(x + y) = \mu(f(x + y)) \\
= \mu(f(x) + f(y)) \\
\geq \mu(f(x)) \cdot \mu(f(y)) \\
= \mu_f(x) \cdot \mu_f(y)
\]
Also,
\[
\mu_f(x - y) = \mu(f(x - y)) \\
= \mu[f(x) - f(y)] \\
\geq \mu(f(x)) \cdot \mu(f(y)) \\
= \mu_f(x) \cdot \mu_f(y)
\]
Hence $\mu_f$ is a fuzzy dot $\beta$–subalgebra of $X$.

**Theorem 3.14.** For a fuzzy set $\theta$ of a $\beta$–algebra $X$. Let $\mu_\theta$ be a fuzzy relation defined by $\mu_\theta(x + y) = \theta(x) \cdot \theta(y)$. Then $\theta$ is a fuzzy dot $\beta$–subalgebra of $X$ if and only if $\mu_\theta$ is a fuzzy dot $\beta$–subalgebra of $X \times X$.

Proof. Assume that $\theta$ is a fuzzy dot $\beta$–subalgebra of $X$.
Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two elements of $X \times X$. We have
\[
\mu_\theta(x + y) = \mu_\theta((x_1, x_2) + (y_1, y_2))
\]
\[ \begin{align*}
\mu_{\theta}(x_1 + x_2, y_1 + y_2) &= \mu_{\theta}(x_1 + x_2, 0) + \mu_{\theta}(0, y_1 + y_2) \\
&= \mu(x_1 + x_2, y_1 + y_2) \\
&\geq \mu(x_1) \cdot \mu(x_2) \cdot \mu(y_1) \cdot \mu(y_2) \\
&= \mu(x_1, x_2) \cdot \mu(y_1, y_2) \\
&= \mu(x) \cdot \mu(y)
\end{align*} \]

Similarly we can prove \( \mu_{\theta}(x - y) \geq \mu_{\theta}(x) \cdot \mu_{\theta}(y) \). Hence \( \mu_{\theta} \) is a fuzzy dot \( \beta \)-subalgebra of \( X \times X \).

Conversely, \( \mu_{\theta} \) is a fuzzy dot \( \beta \)-subalgebra of \( X \times X \). Let \( x, y \in X \). Then
\[
[\theta(x + y)]^2 = \theta(x + y) \cdot \theta(x + y) \\
= \mu_{\theta}(x + y, x + y) \\
= \mu_{\theta}([x, x] + [y, y]) \\
\geq \mu_{\theta}(x, x) \cdot \mu_{\theta}(y, y) \\
= \theta(x) \cdot \theta(x) \cdot \theta(y) \cdot \theta(y) \\
= \theta(x)^2 \cdot \theta(y)^2
\]

Hence \( \theta(x + y) \geq \theta(x) \cdot \theta(y) \). Similarly we can prove \( \theta(x - y) \geq \theta(x) \cdot \theta(y) \). Therefore \( \theta \) is a fuzzy dot \( \beta \)-subalgebra of \( X \).

**Theorem 3.15.** Let \( X \) and \( Y \) be \( \beta \)-algebras. Let \( \mu \) be a fuzzy dot \( \beta \)-subalgebra of \( X \times Y \). Define a fuzzy set \( p_x(\mu) \) of \( X \) such that \( p_x(\mu)(x) = \mu(x, 0) \), \( \forall x \in X \). Then \( p_x(\mu) \) is a fuzzy dot \( \beta \)-subalgebra of \( X \). Also define a fuzzy set \( p_y(\mu) \) of \( Y \) such that \( p_y(\mu)(y) = \mu(0, y) \), \( \forall y \in Y \). Then \( p_y(\mu) \) is a fuzzy dot \( \beta \)-subalgebra of \( Y \).

**Proof.** For any \( x, y \in X \), we have
\[
\mu_\theta(x + y) = \mu(x + y, 0) \\
= \mu(x + y, 0 + 0) \\
= \mu([x, 0] + [y, 0]) \\
\geq \mu(x, 0) \cdot \mu(y, 0) \\
= p_x(\mu)(x) \cdot p_x(\mu)(y)
\]

Similarly we can prove \( p_x(\mu)(x - y) \geq p_x(\mu)(x) \cdot p_x(\mu)(y) \). Hence \( p_x(\mu) \) is a fuzzy dot \( \beta \)-subalgebra of \( X \).

Also For any \( x, y \in Y \), we have
\[
p_y(\mu)(x + y) = \mu(0, x + y)
\[= \mu(0 + 0, x + y)\]
\[= \mu[(0, x) + (0, y)]\]
\[\geq \mu(0, x) \cdot \mu(0, y)\]
\[= p_y(\mu)(x) \cdot p_y(\mu)(y)\]

Similarly we can prove \( p_y(\mu)(x - y) \geq p_y(\mu)(x) \cdot p_y(\mu)(y) \). Hence \( p_y(\mu) \) is a fuzzy dot \( \beta \)-subalgebra of \( X \).

References

[1] M. Abu Ayub Ansari, M. Chandramouleeswaran, Fuzzy \( \beta \)-subalgebra of \( \beta \)-algebra, Accepted.


