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FUZZY DOT $\beta-$SUBALGEBRAS OF $\beta-$ALGEBRAS

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Abstract: In this paper, we intorduce the notion of fuzzy dot $\beta-$subalgebras on $\beta-$algebras and investigate some of their properties.

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1. Introduction

In 1966, Y. Imai and K. Iseki (see [5], [6], [7]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK algebras is a proper subclass of the class of BCI algebras. In 2002, J. Neggers and H.S. Kim [12] introduced the notion of $B-$algebras which is another generalization of BCK algebras. Also they introduced the notion of $\beta-$algebra [13] where two operations are coupled in such a way as to reflect the natural coupling, which exists between the usual group operation and its associated B-algebra. In 2012, Y.H. kim [10] investigated some properties of $\beta-$algebras.

The important point in the evaluation of the modern concept of uncertainty

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was the paper by Lofti A. Zadeh [16] that introduced the theory of fuzzy sets. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1977, by Rosenfeld [14]. O.G. Xi [15] applied the concept of fuzzy sets to BCK algebras and got some results in 1991. In 1993, Y.B. Jun [8] applied it to BCI-algebras.

In their paper [9], the authors introduced the notion of fuzzy dot subalgebras of BCK/BCI-algebras as a generalization of a fuzzy subalgebra, and then investigated several basic properties which are related to fuzzy dot subalgebras. In [2], Al-Shehrie introduced the notion of fuzzy dot d-ideals of d-algebras. In[4], the authors introduced the notion of fuzzy dot SU-subalgebras. In [11], K.H.Kim introduced the notion of fuzzy dot subalgebras of d-algebras. In [3], fuzzy dot BCK/BCI algebras were discussed.

This motivated us to study the fuzzy dot algebraic structures on $\beta-$algebras. In this paper, we intorduce the notion of fuzzy dot $\beta-$subalgebras of a $\beta-$algbera and investigate some of their properties.

2. Preliminaries

In this section we recall some basic definitions that are required in the sequel.

Definition 2.1. [5] A BCK-algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $((x * y) * (x * z)) * (z * y) = 0$.
2. $(x * (x * y)) * y = 0$.
3. $x * x = 0$
4. $x * y = 0 \textit{ and } y * x = 0 \Rightarrow x = y$
5. $0 * x = 0 \ \forall \ x, y, z \ \in \ X$.

Definition 2.2. [6] A BCI-algebra $(X, *, 0)$ is a non-empty set X with a consnant 0 and a binary operation $*$ satisfying the following axioms:

1. $((x * y) * (x * z)) * (z * y) = 0$.
2. $(x * (x * y)) * y = 0$.
3. $x * x = 0$
4. $x * y = 0 \textit{ and } y * x = 0 \Rightarrow x = y \ \forall \ x, y, z \ \in \ X$.

Definition 2.3. [12] A B-algebra $(X, *, 0)$ is a non-empty set X with a consnant 0 and a binary operations $*$ satisfying the following axioms:

1. $x * x = 0$
2. $x * 0 = x$
3. $(x * y) * z = x * (z * (0 * y)) \ \forall\ x, y, z \ \in \ X.$

Definition 2.4. [13] [10] A $\beta-$algebra is a non-empty set X with a consnant 0 and two binary operations $+$ and $-$ satisfying the following axioms:

1. $x - 0 = x.$
2. $(0 - x) + x = 0.$
3. $(x - y) - z = x - (z + y) \ \forall\ x, y, z \ \in \ X.$

Example 2.5. Let $X = \{0, 1, 2, \}$ be a set with constant 0 and two binary operations + and - are defined on X with the Cayley's table

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$-$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X, +, -, 0)$ is a β-algebra.

Definition 2.6. Let $(X, +, -, 0)$ and $(Y, +, -, 0')$ be two $\beta-$algebras. A mapping $f : X \to Y$ is said to be a $\beta-$homomorphism
if $f(x + y) = f(x) + f(y)$ and $f(x - y) = f(x) - f(y), \forall x, y \in X.$

Note: In a $\beta-$homomorshism $f(0) = f(0').$

Definition 2.7. Let X be a set of universal discourse. A fuzzy set μ in X is defined as a function $\mu : X \to [0, 1]$. For each element x in X, $\mu(x)$ is called the membership value of x in X.

Definition 2.8. If μ_1 and μ_2 are two fuzzy sets of X then intersection$\mu_1 \cap \mu_2$ of μ_1 and μ_2 is defined as $(\mu_1 \cap \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}$.

In general $(\cap \mu_i)(x) = min\{\mu_i(x)/i = 1, 2, 3, ...\}$

Definition 2.9. If μ_1 and μ_2 are two fuzzy sets of X then union $\mu_1 \cup \mu_2$ of μ_1 and μ_2 is defined as $(\mu_1 \cup \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}$.

Definition 2.10. If μ_1 and μ_2 are two fuzzy sets of X then $\mu_1 \subseteq \mu_2$ if $\mu_1(x) \leq \mu_2(x)$.

Definition 2.11. If μ is a fuzzy set of X then the complement of μ is μ^c and defined as $\mu^c(x) = 1 - \mu(x)$.

Definition 2.12. Let μ_1 and μ_2 be two fuzzy sets of X_1 and X_2 respectively. Then the direct product $\mu_1 \times \mu_2$ of μ_1 and μ_2 is defined as the fuzzy set of $X_1 \times X_2$

$$(\mu_1 \times \mu_2)(x_1, x_2) = min\,\{\mu_1(x_1), \mu_2(x_2)\}\ \forall\ (x_1, x_2) \in X_1 \times X_2.$$

Definition 2.13. Let μ be a fuzzy set in a set X. For $t \in [0, 1]$, the set $\mu_t = \{x \in X/\mu(x) \geq t\}$ is called a level subset of μ

Proposition 2.14. If $t_1 \leq t_2$, then $\mu_{t_2} \subseteq \mu_{t_1}$ where μ_{t_2} and μ_{t_1} are any two level subsets of μ where μ be a fuzzy set on a set X.

Definition 2.15. Let μ be a fuzzy set of X. μ is said to have the supremum property if, for any subset A of X, there exist a $a_0 \in A$ such that $\mu(a_0) = \sup\limits_{a \in A}\ \mu(a)$.

3. Fuzzy Dot $\beta-$Subalgebras of $\beta-$Algebra

In section we introduce the notion of fuzzy dot $\beta-$subalgebras of $\beta-$algebras and prove some simple theorem.

Definition 3.1. Let μ be a fuzzy set in a $\beta-$algebra X. Then μ is called a fuzzy dot $\beta-$subalgebra of X if

1. $\mu(x + y) \geq \mu(x) \cdot \mu(y),\ \forall x, y \in X.$
2. $\mu(x - y) \geq \mu(x) \cdot \mu(y),\ \forall x, y \in X.$

Example 3.2. Consider the $\beta-$algebra$(X, +, -, 0)$ in Example:2.5
Define $\mu : X \rightarrow [0, 1]$ such that

$$\mu(x) \;=\; \begin{cases} 0.6 & \text{if } x = 0 \\ 0.7 & \text{if } x = 1 \\ 0.3 & \text{if } x = 2, 3 \end{cases}$$

then μ is a fuzzy dot $\beta-$subalgebra of X.

Theorem 3.3. Every fuzzy $\beta-$subalgebra of X is a fuzzy dot $\beta-$subalgebra of X. The converse need not be true.

Proof. Let μ is a fuzzy $\beta-$subalgebra of X. Then
$\mu(x+y) \geq min\{\mu(x), \mu(y)\} \geq \mu(x) \cdot \mu(y)$ and
$\mu(x-y) \geq min\{\mu(x), \mu(y)\} \geq \mu(x) \cdot \mu(y)$
Therefore μ is a fuzzy dot $\beta-$subalgebra of X.
Note:In the example 3.2, μ is a fuzzy dot $\beta-$subalgebra of X but μ is not a fuzzy $\beta-$subalgebra of X, since $\mu(1-1) = \mu(0) = 0.6 < 0.7 = min\{0.7, 0.7\} = min\{\mu(1), \mu(1)\}$.

Theorem 3.4. If μ_1 and μ_2 be two fuzzy dot $\beta-$subalgebras of X then $\mu_1 \cap \mu_2$ is also a fuzzy dot $\beta-$subalgebra of X.

Proof. For $x, y \in X$,

$$\begin{aligned}(\mu_1 \cap \mu_2)(x+y) &= min\{\mu_1(x+y), \mu_2(x+y)\} \\ &\geq min\{\mu_1(x) \cdot \mu_1(y), \mu_2(x) \cdot \mu_2(y)\} \\ &\geq (min\{\mu_1(x), \mu_2(x))\} \cdot (min\{\mu_1(y), \mu_2(y)\}) \\ &= (\mu_1 \cap \mu_2)(x) \cdot (\mu_1 \cap \mu_2)(y)\end{aligned}$$

Similarly we can prove that $(\mu_1 \cap \mu_2)(x-y) \geq (\mu_1 \cap \mu_2)(x) \cdot (\mu_1 \cap \mu_2)(y)$
Therefore $\mu_1 \cap \mu_2$ is fuzzy dot $\beta-$subalgebra of X.
The above theorem can be generalized as follows.

Corollary 3.5. If $\{\mu_i / i = 1, 2, 3, \cdots\}$ be a family of dot-fuzzy $\beta-$subalgebra of X, then $\cap \mu_i$ is also a dot-fuzzy $\beta-$subalgebra of X.

Notation: Hereafter by a $\beta-$algebra X we mean a $\beta-$algebra $(X, +, -, 0)$ derived from a group or a $\beta-$algebra $(X, +, -, 0)$ from a $B-$algebra $(X, -, 0)$.

Proposition 3.6. Let X be a $\beta-$algebra and let μ be a dot-fuzzy $\beta-$subalgebra of X then

1. $\{\mu(x)\}^6 \leq \{\mu(x)\}^2 \leq \mu(0), \ \forall x \in X$.
2. $\{\mu(x)\}^3 \leq \mu(x^*), \ \forall x \in X$, where $x^* = 0 - x$.

In general $\{\mu(x)\}^{2n-1} \leq \mu(0^n - x), \ \forall x \in X$, where $0^n - x = 0 - (0 - (0 - \cdots (0 - x)))$, such that 0 occurs n times.

Proof.

1. For any $x \in X, \mu(0) = \mu(x-x) \geq \mu(x) \cdot \mu(x) = \{\mu(x)\}^2$.

2. $\mu(x^*) = \mu(0 - x) \geq \mu(0) \cdot \mu(x) = \{\mu(x)\}^2 \cdot \mu(x) = \{\mu(x)\}^3$.
   Also $\mu(0) = \mu(x^* - x^*) \geq \mu(x^*) \cdot \mu(x^*) \geq \{\mu(x)\}^3 \cdot \{\mu(x)\}^3 = \{\mu(x)\}^6$.

We can prove the general case by induction.

Theorem 3.7. Let X be a $\beta-$algebra and let μ be a fuzzy set of X. If $\mu(x+y) \geq \mu(x) \cdot \mu(y)$ and $\mu(x^*) \geq \mu(x)$, $\forall x, y \in X$, then μ be a fuzzy dot $\beta-$subalgebra of X.

Proof. Given $\mu(x+y) \geq \mu(x) \cdot \mu(y)$, $\forall x, y \in X$.
Hence it is enough to prove $\mu(x-y) \geq \mu(x) \cdot \mu(y)$, $\forall x, y \in X$. Now

$$\mu(x-y) = \mu(x-(y^*)^*) = \mu(x+y^*) \quad \geq \quad \mu(x) \cdot \mu(y^*) \geq \mu(x) \cdot \mu(y),$$

since $y = (y^*)^* and x + y = x - y^*$.
Therefore μ is a fuzzy dot $\beta-$subalgebra of X.

Theorem 3.8. If A and B b a $\beta-$subalgbra of X, then the characteristic function χ_A is a fuzzy dot $\beta-$subalgebra of X.

Proof. Let $x, y \in X$.
Case 1) If $x, y \in A$, then $x+y, x-y \in A$ since A is a $\beta-$subalgebra of X. This implies that $\chi_A(x) = 1$, $\chi_A(y) = 1$, $\chi_A(x+y) = 1$ and $\chi_A(x-y) = 1$. which implies

1. $\chi_A(x+y) = 1 = 1 \cdot 1 = \chi_A(x) \cdot \chi_A(y)$, $\forall x, y \in X$.
2. $\chi_A(x-y) = 1 = 1 \cdot 1 = \chi_A(x) \cdot \chi_A(y)$, $\forall x, y \in X$.

Case 2) If $x, y \notin A$, then $\chi_A(x) = 0, \chi_A(y) = 0$. which implies,

1. $\chi_A(x+y) \geq 0 = 0 \cdot 0 = \chi_A(x) \cdot \chi_A(y)$, $\forall x, y \in X$.
2. $\chi_A(x-y) \geq 0 = 0 \cdot 0 = \chi_A(x) \cdot \chi_A(y)$, $\forall x, y \in X$.

Case 3) If $x \in A, y \notin A$, then $\chi_A(x) = 1, \chi_A(y) = 0$. which implies,

1. $\chi_A(x+y) \geq 0 = 1 \cdot 0 = \chi_A(x) \cdot \chi_A(y)$, $\forall x, y \in X$.
2. $\chi_A(x-y) \geq 0 = 1 \cdot 0 = \chi_A(x) \cdot \chi_A(y)$, $\forall x, y \in X$.

Case 4) When $x \notin A, y \in A$. by interchanging the roles of x and y in case 3) we can prove μ is a fuzzy dot $\beta-$subalgebra of X. This completes the proof.
The converse of this result is also true.

Theorem 3.9. Let A be any subset of a $\beta-$algebra X. If any characteristic function χ_A of A is a fuzzy dot $\beta-$subalgebra of X then$A$ is a $\beta-$subalgebra of X.

Proof. Let χ_A is a fuzzy dot$\beta-$subalgebra of X then
$\chi_A(x+y) \geq \chi_A(x) \cdot \chi_A(y),\ \forall x, y \in X.$
And $\chi_A(x-y) \geq \chi_A(x) \cdot \chi_A(y),\ \forall x, y \in X.$
Let $x, y \in A$ which implies $\chi_A(x) = 1, \chi_A(y) = 1$. Therefore

1. $\chi_A(x+y) \geq \chi_A(x) \cdot \chi_A(y) = 1 \cdot 1 = 1 \Rightarrow \chi_A(x+y) = 1 \Rightarrow x + y \in A.$

2. $\chi_A(x-y) \geq \chi_A(x) \cdot \chi_A(y) = 1 \cdot 1 = 1 \Rightarrow \chi_A(x+y) = 1 \Rightarrow x - y \in A.$

Hence A is a $\beta-$subalgebra of X.

Theorem 3.10. Let μ_1 and μ_2 be two fuzzy dot $\beta-$subalgebras of $\beta-$algebra X. Then the direct product $\mu_1 \times \mu_2$ of μ_1 and μ_2 is defined by $(\mu_1 \times \mu_2)(x, y) = \mu_1(x) \cdot \mu_2(y)$ is also a fuzzy dot $\beta-$subalgebra of $X \times X$.

Proof. Let $X = X \times X and$ Let $\mu = \mu_1 \times \mu_2$.
Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two elements of X. Now

$$\begin{aligned}
\mu(x+y) &= \mu((x_1, x_2) + (y_1, y_2)) \\
&= \mu(x_1 + y_1, x_2 + y_2) \\
&= (\mu_1 \times \ \mu_2)(x_1 + y_1, x_2 + y_2) \\
&= \mu_1(x_1 + y_1) \cdot \mu_2(x_2 + y_2) \\
&\geq \mu_1(x_1) \cdot \mu_1(y_1) \cdot \mu_2(x_2) \cdot \mu_2(y_2) \\
&= \mu_1(x_1) \cdot \mu_2(x_2) \cdot \mu_1(y_1) \cdot \mu_2(y_2) \\
&= (\mu_1 \times \mu_2)(x_1, x_2) \cdot (\mu_1 \times \mu_2)(y_1, y_2) \\
&= \mu(x) \cdot \mu(y).
\end{aligned}$$

Similarly we can prove $\mu(x-y) \geq \mu(x) \cdot \mu(y),\ \forall x, y \in X.$
Hence $\mu_1 \times \mu_2$ is a fuzzy dot$\beta-$subalgebra of $X \times X$.

Theorem 3.11. Let μ_1 and μ_2 be two fuzzy dot$\beta-$subalgebras of $\beta-$algebra X_1 and X_2 respectively. Then the direct product $\mu_1 \times \mu_2$ of μ_1 and μ_2s defined by

$(\mu_1 \times \mu_2)(x, y) = \mu_1(x) \cdot \mu_2(y),\ \forall x, y \in X_1 \times X_2$ is a fuzzy dot $\beta-$subalgebra of $X_1 \times X_2$.

Proof. The Proof is straightforward.

Theorem 3.12. Let $f : X \to Y$ be a homomorphism of a $\beta-$algebra X into a $\beta-$algebra$Y$. If μ is a fuzzy dot $\beta-$algebra of Y, then the pre-image of μ, denoted by $f^{-1}(\mu)$ is defined as $\left\{f^{-1}(\mu)\right\}(x) = \mu(f(x)),\ \forall x \in X$, is a fuzzy dot $\beta-$subalgebra of X.

Proof. Let μ be a fuzzy dot $\beta-$subalgebra of Y and let $x, y \in X$. Then

$$\begin{aligned}
\left\{f^{-1}(\mu)\right\}(x+y) &= \mu(f(x+y)) \\
&= \mu(f(x)+f(y)) \\
&\geq \mu(f(x)) \cdot \mu(f(y)) \\
&= \left\{f^{-1}(\mu)(x)\right\} \cdot \left\{f^{-1}(\mu)\right\}(y). \\
Also \left\{f^{-1}(\mu)\right\}(x-y) &= \mu(f(x-y)) \\
&= \mu(f(x)-f(y)) \\
&\geq \mu(f(x)) \cdot \mu(f(y)) \\
&= \left\{f^{-1}(\mu)\right\}(x) \cdot \left\{f^{-1}(\mu)\right\}(y).
\end{aligned}$$

Hence $f^{-1}(\mu)$ is a fuzzy dot $\beta-$subalgebra of X.

Theorem 3.13. Let $f : X \to X$ be an endomorphism on a $\beta-$algebra X. If μ be a fuzzy dot$\beta-$subalgebra of X. Define a fuzzy set $\mu_f : X \to [0, 1]$ by $\mu_f(x) = \mu(f(x)),\ \forall x \in X$. Then μ_f is a fuzzy dot$\beta-$subalgebra of X.

Proof. Let $x, y \in X$. Then

$$\begin{aligned}
\mu_f(x+y) &= \mu(f(x+y)) \\
&= \mu(f(x)+f(y)) \\
&\geq \mu(f(x)) \cdot \mu(f(y)) \\
&= \mu_f(x) \cdot \mu_f(y)
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_f(x-y) &= \mu(f(x-y)) \\
&= \mu[f(x)-f(y)] \\
&\geq \mu(f(x)) \cdot \mu(f(y)) \\
&= \mu_f(x) \cdot \mu_f(y)
\end{aligned}$$

Hence μ_f is a fuzzy dot $\beta-$subalgebra of X.

Theorem 3.14. For a fuzzy set θ of a $\beta-$algebra X. Let μ_θ be a fuzzy relation defined by $\mu_\theta(x+y) = \theta(x) \cdot \theta(y)$. Then θ is a fuzzy dot $\beta-$subalgebra of X if and only if μ_θ is a fuzzy dot $\beta-$subalgebra of $X \times X$.

Proof. Assume that θ is a fuzzy dot $\beta-$subalgebra of X.
Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two elements of $X \times X$. We have

$$\mu_\theta(x+y) = \mu_\theta((x_1, x_2) + (y_1, y_2))$$

$$\begin{aligned} &= \mu_\theta(x_1+x_2, y_1+y_2) \\ &= \theta(x_1+x_2)\cdot\theta(y_1+y_2) \\ &\geq \theta(x_1)\cdot\theta(x_2)\cdot\theta(y_1)\cdot\theta(y_2) \\ &= \theta(x_1)\cdot\theta(y_1)\cdot\theta(x_2)\cdot\theta(y_2) \\ &= \mu_\theta(x_1,x_2)\cdot\mu_\theta(y_1,y_2) \\ &= \mu_\theta(x)\cdot\mu_\theta(y) \end{aligned}$$

Similarly we can prove $\mu_\theta(x-y) \geq \mu_\theta(x)\cdot\mu_\theta(y)$. Hence μ_θ s a fuzzy dot $\beta-$subalgebra of $X\times X$.

Conversely, μ_θ is a fuzzy dot $\beta-$subalgebra of $X\times X$. Let $x,y\in X$. Then

$$\begin{aligned} [\theta(x+y)]^2 &= \theta(x+y)\cdot\theta(x+y) \\ &= \mu_\theta(x+y, x+y) \\ &= \mu_\theta[(x,x)+(y,y)] \\ &\geq \mu_\theta(x,x)\cdot\mu_\theta(y,y) \\ &= \theta(x)\cdot\theta(x)\cdot\theta(y)\cdot\theta(y) \\ &= \theta(x)^2\cdot\theta(y)^2 \end{aligned}$$

Hence $\theta(x+y)\geq\theta(x)\cdot\theta(y)$. Similarly we can prove $\theta(x-y)\geq\theta(x)\cdot\theta(y)$. Therefore θ is a fuzzy dot $\beta-$subalgebra of X.

Theorem 3.15. Let X and Y be $\beta-$ algebras. Let μ be a fuzzy dot$\beta-$ subalgebra of $X\times Y$. Define a fuzzy set $p_x(\mu)$ of X such that $p_x(\mu)(x) = \mu(x,0)$, $\forall x\in X$. Then $p_x(\mu)$ is a fuzzy dot $\beta-$subalgebra of X. Also define a fuzzy set $p_y(\mu)$ of Y such that $p_y(\mu)(y) = \mu(0,y)$, $\forall y\in Y$. Then $p_y(\mu)$ is a fuzzy dot $\beta-$subalgebra of Y.

Proof. For any $x,y\in X$, we have

$$\begin{aligned} p_x(\mu)(x+y) &= \mu(x+y,0) \\ &= \mu(x+y.0+0) \\ &= \mu[(x,0)+(y,0)] \\ &\geq \mu(x,0)\cdot\mu(y,0) \\ &= p_x(\mu)(x)\cdot p_x(\mu)(y) \end{aligned}$$

Similarly we can prove $p_x(\mu)(x-y)\geq p_x(\mu)(x)\cdot p_x(\mu)(y)$. Hence $p_x(\mu)$ is a fuzzy dot $\beta-$subalgebra of X.
Also For any $x,y\in Y$, we have

$$p_y(\mu)(x+y) = \mu(0,x+y)$$

$$\begin{aligned} &= \mu(0+0, x+y) \\ &= \mu[(0,x)+(0,y)] \\ &\geq \mu(0,x)\cdot\mu(0,y) \\ &= p_y(\mu)(x)\cdot p_y(\mu)(y) \end{aligned}$$

Similarly we can prove $p_y(\mu)(x-y) \geq p_y(\mu)(x)\cdot p_y(\mu)(y)$. Hence $p_y(\mu)$ is a fuzzy dot $\beta-$subalgebra of X.

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